2.4.2 Proofs Involving Boxy Things

The Arithmetic-Geometric Mean Inequality

Definition: Let $a,b \geq 0$

The arithmetic mean of a and b is $(a+b)/2$

The geometric mean of a and b is $\sqrt{(ab)}$

When given a rectangle R with side lengths a and b:

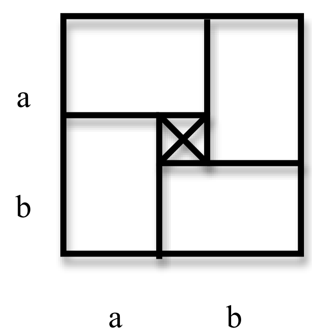
The arithmetic mean is the side length of a square with the same perimeter as R.

The geometric mean is the side length of a square with the same area as R.

Theorem: $\sqrt{(ab)} \leq (a+b)/2$, and $\sqrt{(ab)} = (a+b)/2$ if and only if $a=b$ (square).

The geometric mean can never be larger than the arithmetic mean, but they can be equal to each other if it's a square.

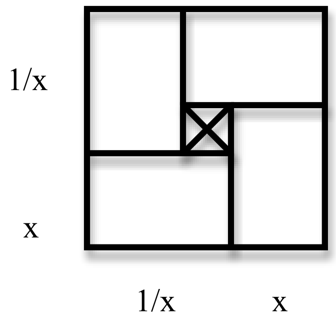
Proof: The square below has side length a+b and the area of the entire square is at least as big as the combined area of the 4 rectangles.



1. From the picture you know that $4ab \leq (a+b)^2$
2. Take the square root of both sides to get $\sqrt{4ab} \leq a+b$
3. $\sqrt{4ab}$ can also be written as $\sqrt{4} \times \sqrt{ab} \leq a+b$ which can be written as $2\sqrt{ab} \leq a+b$
4. Divide both sides by 2 to get $\sqrt{(ab)} \leq (a+b)/2$.

By following the above steps we see that the picture proves the Arithmetic-Geometric Mean Inequality.

Practice Problem (#19 on page 74): Explain how the following picture "proves" that the sum of a number x and its inverse $1/x$ is at least 2.



Each of the four rectangles has an area of 1.

1. From the picture you know that $4(x \cdot 1/x) \leq (x + 1/x)^2$
2. The simplified version of this is $4 \leq (x + 1/x)^2$
3. Take the square root of both sides to get $2 \leq x + 1/x$

By following the steps above we see that the picture proves that the sum of a positive number x and its inverse $1/x$ is at least 2.