PARTIAL DIFFERENTIAL EQUATIONS OF APPLIED MATHEMATICS
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by

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Answers to Selected Exercises

Chapter 1

Section 1.1

1.1.5. $v(x,y,t+\tau) = [v(x-\delta,y,t)+v(x+\delta,y,t)+v(x,y-\delta,t)+v(x,y+\delta,t)]/4.$

1.1.7.

$$\frac{\partial v(x,y,t)}{\partial t}+\frac{D_1(x,y)}{2}\frac{\partial^2 v(x,y,t)}{\partial x^2}+c_1(x,y)\frac{\partial v(x,y,t)}{\partial x}+\frac{D_2(x,y)}{2}\frac{\partial^2 v(x,y,t)}{\partial y^2}+c_2(x,y)\frac{\partial v(x,y,t)}{\partial y}=0.$$

1.1.11. (a): $v(x,t+\tau) = pv(x-\delta,t)+qv(x+\delta,t)\,.$

(c): $\sum_{k=-\infty}^{\infty} v(k\delta,n\tau) = (q+p)^n = 1.$

1.1.12. (c): $E(x) = 0, \qquad V(x) = D/2\omega - De^{-2\,\omega\, t}/2\omega.$

Section 1.2

1.2.7. (b): $\sum_{k=-\infty}^{\infty}\alpha(k\delta,n\tau) = \sum_{k=-\infty}^{\infty}\beta(k\delta,n\tau) = \frac{1}{2}\,(q+p)^{n-1} = \frac{1}{2}.$

1.2.10. (b): $v(n\delta,n\tau) = v(-n\delta,n\tau) = \frac{1}{2}p^{n-1}.$

Section 1.3

1.3.6. $v(1,1) = 7/24, \quad v(2,1) = 1/12, \quad v(1,2) = 1/12, \quad v(2,2) = 1/24.$

1.3.15. (a): $u(x) = -x^2/D + x/D.$

(b): $u(x) = -x^2/D + 2\,x/D.$

Section 1.4

1.4.1. The θ scheme:

$$v\left(x,t+\tau\right)=\left(1-\frac{\theta\ |c|\,\tau}{\delta}\right)v\left(x,t\right)+\left(\frac{\theta\ |c|\,\tau}{2\delta}+\frac{c\tau}{2\delta}\right)v\left(x-\delta,t\right)+\left(\frac{\theta\ |c|\,\tau}{2\delta}-\frac{c\tau}{2\delta}\right)v\left(x+\delta,t\right).$$

If we put $\theta=1$ in the difference equation we obtain the backward and forward schemes.

$$v\left(x,t+\tau\right)=\left(1-\frac{|c|\,\tau}{\delta}\right)v\left(x,t\right)+\left(\frac{|c|\,\tau}{2\delta}+\frac{c\tau}{2\delta}\right)v\left(x-\delta,t\right)+\left(\frac{|c|\,\tau}{2\delta}-\frac{c\tau}{2\delta}\right)v\left(x+\delta,t\right).$$

With $|c|=c$ we have the forward- backward scheme.

$$v\left(x,t+\tau\right)=\left(1-\frac{c\tau}{\delta}\right)v\left(x,t\right)+\frac{c\tau\,v\left(x-\delta,t\right)}{\delta}.$$

With $|c|=-c$ we have the forward- forward scheme.

$$v\left(x,t+\tau\right)=\left(1+\frac{c\tau}{\delta}\right)v\left(x,t\right)-\frac{c\tau\,v\left(x+\delta,t\right)}{\delta}.$$

If we put $\theta=\delta/(|c|\tau)$ in the difference equation, we obtain the Lax-Friedrichs scheme.

$$v\left(x,t+\tau\right)=\left(\frac{1}{2}+\frac{c\tau}{2\delta}\right)v\left(x-\delta,t\right)+\left(\frac{1}{2}-\frac{c\tau}{2\delta}\right)v\left(x+\delta,t\right).$$

1.4.5.

$$v\left(x,t\right)=\left(1-\frac{\theta\ |c|\,\tau}{\delta}\right)v\left(x,t+\tau\right)+\left(\frac{\theta\ |c|\,\tau}{2\delta}-\frac{c\tau}{2\delta}\right)v\left(x-\delta,t+\tau\right)+\left(\frac{\theta\ |c|\,\tau}{2\delta}+\frac{c\tau}{2\delta}\right)v\left(x+\delta,t+\tau\right).$$

Section 1.5

1.5.1. $v\left(x,t\right)=\frac{1}{\sqrt{4\pi\,t}}\,\exp\left(-\frac{(x-10\,t)^2}{4t}\right).$

Chapter 2

Section 2.2

2.2.1. $v(x,t) = F(x-ct)\, e^{-\lambda t}$

2.2.2. (b): $v(x,t) = -ct^3/6 + xt^2/2 + \sin(x-ct)$

2.2.4. (b):

$$v(x,t) = \int_0^t \int_{x-\gamma(t-s)}^{x+\gamma(t-s)} \frac{F(\sigma,s)}{2\gamma} d\sigma\, ds + \frac{f(x-\gamma t) + f(x+\gamma t)}{2} + \int_{x-\gamma t}^{x+\gamma t} \frac{g(\sigma)}{2\gamma} d\sigma.$$

2.2.5. $v(x,t) = G\left(\frac{-x+ct}{c}\right).$

2.2.6. $v(x,t) = (x-t)/2 - \sqrt{-(x-t)^2 + 2}\,/2.$

2.2.7. $v(x,t) = -\ln\left(t + e^{-x}\right)$

2.2.8. (a): $v(x,t) = G\left(\frac{-x+ct}{c}\right), \quad x < ct, \qquad v(x,t) = F(x-ct), \quad x > ct.$

2.2.12. (a): $v(x,t) = \frac{H(x)x}{c} - \frac{(x-ct)H(x-ct)}{c}, \qquad c > 0.$

2.2.15.

$$v(x,t) = \begin{cases} f(x-c_1 t), & x < 0, \\ (c_1/c_2) f(c_1(x-c_2 t)/c_2), & x < c_2 t, \\ f(x-c_2 t), & x > c_2 t. \end{cases}$$

2.2.24. $v(x,t) = f\left(x - 3t^{2/3}/2\right) \exp\left(3t^{2/3}/2\right)$

Section 2.3

2.3.2. (a): $u = f\left[x - u\left(e^{ct} - 1\right)/c\right] e^{-ct}.$

2.3.3. $u(x,t) = \frac{axce^{-ct}}{c+a-ae^{-ct}}.$

2.3.5. (a): $u(x,t) = \frac{x}{1+t}, \quad \rho(x,t) = \frac{1}{1+t}.$

2.3.6. $u(x,t) = \frac{x-ct}{xt-ct^2+1}.$

2.3.9. $u(x,t) = \frac{-1+\sqrt{1+4xt}}{2t}.$

2.3.10. $f(x) = 1,\ u(x,t) = x\tanh(t) + \operatorname{sech}(t); \quad f(x) = x,\ u(x,t) = x.$

2.3.14. $u(x,t) = t + \sqrt{t^2 - 2x}, \qquad u(x,t) = t - \sqrt{t^2 - 2x}.$

2.3.19.

$$\text{Shock Wave:} \quad \begin{cases} A, & x < (A+B)t/2 + a, \\ B, & x > (A+B)t/2 + a. \end{cases}$$

2.3.21.

$$u(x,t)\Big|_{t<1} = \begin{cases} 1, & x \le t-1, \\ x/(t-1), & x \le 0, \\ 0, & \text{otherwise.} \end{cases}; \quad u(x,t)\Big|_{t>1} = \begin{cases} 1, & x \le (t-1)/2, \\ 0, & \text{otherwise.} \end{cases}$$

2.3.28. $u(x,y,z) = \frac{x-2z+y}{2z^2-zx-zy+1}$.

Section 2.4

2.4.1. $u(x,t) = t + x$.

2.4.2. $u(x,t) = t^2/2$.

2.4.3. $u(x,t) = 1; \qquad u(x,t) = x + 1 - t$.

2.4.5. $u(x,t) = ix + t; \qquad u(x,t) = -ix + t$.

2.4.6. $u(x,t) = e^{-2t} - e^{-t} + e^{-t}x$.

2.4.9. $u(x,t) = \frac{ix-ct}{c}; \quad u(x,t) = -\frac{ix+ct}{c}$.

2.4.10. $A(x,y) = (x^2+y^2)^{-1/4}$.

2.4.15. $u(x,t) = \frac{x^2}{4t-1}$.

2.4.16. $2v(x,t)\,v_x(x,t) + v_t(x,t) = 0$.

Section 2.5

2.5.9. (a): $u(x,y) = \_F1\left(-\frac{-y+x\ln(x)}{x}\right)x^c$.

(b): $u(x,y) = x^c f\left(-\frac{-y+x\ln(x)}{x}\right)$.

2.5.11. $u(x,y,z) = f\left(x - z^2/2, ye^{-z}\right)$.

2.5.14 $u(x,t) = \frac{2t^3a+3t^2+6ax}{6(at+1)}$.

2.5.17 $u(x,y,z) = -\frac{x-2z+y}{zx-2z^2+zy-1}$.

Chapter 3

Section 3.1

3.1.4.

$$\xi = y - x,\ \ \eta = y - 3\,x, \quad \frac{\partial^2 u\left(\xi,\eta\right)}{\partial\eta\partial\xi} + \frac{\partial u\left(\xi,\eta\right)}{\partial\xi} + \frac{5}{2}\,\frac{\partial u\left(\xi,\eta\right)}{\partial\eta} - \frac{1}{2}\,u\left(\xi,\eta\right) = 0,$$

$$u\left(\xi,\eta\right) = e^{-5\xi/2-\eta} v\left(\xi,\eta\right), \qquad \frac{\partial^2 v\left(\xi,\eta\right)}{\partial\eta\partial\xi} - 3\,v\left(\xi,\eta\right) = 0.$$

3.1.5.

$$\xi = y - x,\ \ \eta = y, \quad \frac{\partial^2 u\left(\xi,\eta\right)}{\partial\eta^2} - 2\,\frac{\partial u\left(\xi,\eta\right)}{\partial\xi} + 3\,\frac{\partial u\left(\xi,\eta\right)}{\partial\eta} + u\left(\xi,\eta\right) = 0,$$

$$u\left(\xi,\eta\right) = e^{-5\xi/8-3\eta/2} v\left(\xi,\eta\right), \qquad \frac{\partial^2 v\left(\xi,\eta\right)}{\partial\eta^2} - 2\,\frac{\partial v\left(\xi,\eta\right)}{\partial\xi} = 0.$$

3.1.6. $\alpha = y + 3\,x, \quad \beta = \sqrt{3}x,$

$$3\frac{\partial^2 u\left(\alpha,\beta\right)}{\partial\alpha^2} + 3\,\frac{\partial^2 u\left(\alpha,\beta\right)}{\partial\beta^2} + 12\,\frac{\partial u\left(\alpha,\beta\right)}{\partial\alpha} + 4\sqrt{3}\frac{\partial u\left(\alpha,\beta\right)}{\partial\beta} - u\left(\alpha,\beta\right) = \sin\left[\frac{\beta\,\left(\alpha-\beta\sqrt{3}\right)}{\sqrt{3}}\right],$$

$$u\left(\xi,\eta\right) = e^{-2\,\alpha-2\beta/\sqrt{3}} v\left(\xi,\eta\right), \quad \frac{\partial^2 v\left(\alpha,\beta\right)}{\partial\beta^2} + \frac{\partial^2 v\left(\alpha,\beta\right)}{\partial\alpha^2} - \frac{17}{3}\,v\left(\alpha,\beta\right) = \frac{1}{3}\,e^{2\,\alpha+2\beta/\sqrt{3}} \sin\left[\frac{\beta\alpha}{\sqrt{3}} - \beta^2\right].$$

3.1.8. The characteristic coordinates in the hyperbolic region $y < 0$ are $\xi = -x + 2\sqrt{-y},\ \ \eta = -x - 2\sqrt{-y}$ and the canonical form is $\partial^2 u\left(\xi,\eta\right)/\partial\eta\partial\xi = 0$.

3.1.9. (a). Hyperbolic. (b). Parabolic. (c). Elliptic.

Section 3.2

3.2.5. $[u_{\xi\xi}] = 2\,\beta_2 - 2\ \beta_1, \qquad \partial[u_{\xi\xi}]/\partial\eta = 0.$

Section 3.3

3.3.1.

$$\frac{\partial^2 v\left(\rho_1,\rho_2,\rho_3\right)}{\partial{\rho_1}^2} + \frac{\partial^2 v\left(\rho_1,\rho_2,\rho_3\right)}{\partial{\rho_2}^2} + \frac{\partial^2 v\left(\rho_1,\rho_2,\rho_3\right)}{\partial{\rho_3}^2} + \frac{35}{6}\,v\left(\rho_1,\rho_2,\rho_3\right) = 0.$$

3.3.3. (a). Hyperbolic. (b). Parabolic. (c). Hyperbolic. (d). Parabolic.

3.3.6. $|x| < 1$, Elliptic; $\ |x| = 1$, Parabolic; $\ |x| > 1$, Hyperbolic.

3.3.11. $|\phi_x a + \phi_y B| = -\left(h'\left(x\right)\right)^2 - 1$ so that $h'\left(x\right) = \pm\,i$.

3.3.12. $\phi_1 = y - x, \quad \phi_2 = y - x/2, \quad \phi_3 = y + x.$

$$\frac{d\mathbf{v}}{dy} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{d\mathbf{v}}{dx} = \begin{bmatrix} 0 & 7 & 1 \\ 3/2 & -15/2 & -1/2 \\ -1/2 & -5/2 & -3/2 \end{bmatrix} \mathbf{v},$$

$$\mathbf{v} = \begin{bmatrix} u_2 + u_3 \\ -u_3/2 \\ u_1 + u_3/2 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} v_2 + v_3 \\ v_1 + 2\,v_2 \\ -2\,v_2 \end{bmatrix}.$$

3.3.14. $\mathbf{u}(x,y) = [\sin(x)/2 + \sin(x - 2\,y)/2, 1 + \sin(x - 2\,y)/2 - \sin(x)/2, e^{x-2\,y}]^T$.

3.3.21. Riemann Invariants:

$$u + 2\sqrt{gh} = \text{constant} \quad \text{on} \quad \frac{dx(t)}{dt} = u + \sqrt{gh},$$

$$u - 2\sqrt{gh} = \text{constant} \quad \text{on} \quad \frac{dx(t)}{dt} = u - \sqrt{gh}.$$

3.3.22.

$$c^2\frac{\partial \rho}{\partial t} + c^2 u\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial t} - u\frac{\partial p}{\partial x} = 0,$$

$$c\rho\,\frac{\partial u}{\partial t} + c^2\rho\,\frac{\partial u}{\partial x} + c\rho u\,\frac{\partial u}{\partial x} + \frac{\partial p}{\partial t} + c\frac{\partial p}{\partial x} + u\frac{\partial p}{\partial x} = 0,$$

$$-c\rho\,\frac{\partial u}{\partial t} + c^2\rho\,\frac{\partial u}{\partial x} - c\rho u\,\frac{\partial u}{\partial x} + \frac{\partial p}{\partial t} - c\frac{\partial p}{\partial x} + u\frac{\partial p}{\partial x} = 0.$$

3.3.23. $\phi_1 = x, \quad \phi_2 = ct + x, \quad \phi_3 = -ct + x.$

Section 3.4

3.4.2. $u(x,t) = \sum_{n=0}^{\infty} \frac{c^{2n}}{n!} f^{(2n)}(x) t^n.$

3.4.4. $u(x,t) = e^{-\lambda t}(\lambda t + 1)\cos(x), \quad \lambda = \gamma.$

Section 3.5

3.5.1. $\lambda(k) = \pm\sqrt{-\gamma^2 k^2 + c^2}.$

3.5.2. One root is $\lambda_1(k) = i\alpha k + (\alpha^2 - c^2)k^2 + O(k^3)$ and if $c^2 < \alpha^2$ then $0 < \mathrm{Re}[\lambda_1(k)]$.

3.5.3. (a): $\omega(k) = \pm\frac{\gamma\,k}{\sqrt{1+\alpha^2 k^2}}.$

(c): $\omega(k) = \alpha\,k - \beta\,k^3.$

3.5.5. $[\lambda(k)]^3 + [\lambda(k)]^2 + \gamma^2 k^2 \lambda(k) + c^2 k^2 = 0.$

Section 3.6

3.6.2. (a): $L^{*} w(x,y) = [e^x w(x,y)]_{xx} + [x^2 w(x,y)]_{xy} - [yw(x,y)]_y - 10\,w(x,y).$

3.6.11.

$$w(x,t)\left(\frac{\partial^2 u(x,t)}{\partial t^2}+\frac{\partial^4 u(x,t)}{\partial x^4}\right)-u(x,t)\left(\frac{\partial^2 w(x,t)}{\partial t^2}+\frac{\partial^4 w(x,t)}{\partial x^4}\right)=$$

$$\frac{\partial}{\partial x}\left(w(x,t)\,\frac{\partial^3 u(x,t)}{\partial x^3}-\frac{\partial w(x,t)}{\partial x}\,\frac{\partial^2 u(x,t)}{\partial x^2}+\frac{\partial^2 w(x,t)}{\partial x^2}\,\frac{\partial u(x,t)}{\partial x}-u(x,t)\,\frac{\partial^3 w(x,t)}{\partial x^3}\right)$$

$$+\frac{\partial}{\partial t}\left(w(x,t)\,\frac{\partial u(x,t)}{\partial t}-u(x,t)\,\frac{\partial w(x,t)}{\partial t}\right).$$

Section 3.7

3.7.4.

$$-18\,\frac{\partial^2 u(\xi_1,\xi_2,\xi_3)}{\partial {\xi_3}^2}-2\,\frac{\partial^2 u(\xi_1,\xi_2,\xi_3)}{\partial {\xi_1}^2}+9\,\frac{\partial^2 u(\xi_1,\xi_2,\xi_3)}{\partial {\xi_2}^2}$$

$$+\frac{5\sqrt{2}}{3}\,\frac{\partial u(\xi_1,\xi_2,\xi_3)}{\partial \xi_3}-\frac{2}{3}\,\frac{\partial u(\xi_1,\xi_2,\xi_3)}{\partial \xi_2}-2\,\sqrt{2}\frac{\partial u(\xi_1,\xi_2,\xi_3)}{\partial \xi_1}+5\,u(\xi_1,\xi_2,\xi_3)=0.$$

Chapter 4

Section 4.1

4.1.2. $\alpha = -c/2.$

4.1.3. $u(x,t) = e^{-\hat{\lambda} t} v(x,t), \quad \partial^2 v(x,t)/\partial t^2 - \gamma^2 \partial^2 v(x,t)/\partial x^2 - \hat{\lambda}^2 v(x,t) = 0.$

Section 4.2

4.2.6. Compatibility condition: $\int_G \rho(x) f(x)\, dx = \int_G \rho(x) g(x)\, dx.$

4.2.8. The operator is not positive in this case.

4.2.14. (a): If $u = 0$ then $w = 0$.
(b): If $du/dn = 0$ then $pdw/dn + wb_1 n_1 + wb_2 n_2 = 0$.
(c): If $du/dn + hu = 0$ then $pdw/dn + pwh + wb_1 n_1 + wb_2 n_2 = 0$.

Section 4.3

4.3.5. $x = \sum_{k=1}^{\infty} \frac{2k\pi\left(1-2\,(-1)^k\right)}{k^2\pi^2+(\ln(2))^2} \sin\left(\frac{k\pi\,\ln(x)}{\ln(2)}\right).$

4.3.10. Eigenfunction: $v_0(x) = 1$.

4.3.13. Eigenvalues: $\lambda_k = \frac{\pi^2 (k-1/2)^2}{l^2},\ k = 1, 2, \ldots.$

Normalized Eigenfunctions: $v_k(x) = \sqrt{\frac{2}{l}} \cos\left(\frac{\pi\,(k-1/2)x}{l}\right).$

4.3.15. Eigenvalues: $\lambda_k = 1/4 + \frac{k^2\pi^2}{(\ln(1+l))^2},\ k = 1, 2, \ldots.$

Normalized Eigenfunctions: $v_k(x) = \frac{\sqrt{2}}{\sqrt{(1+x)\ln(1+l)}} \sin\left(\frac{k\pi\,\ln(1+x)}{\ln(1+l)}\right).$

4.3.16. (a): $x = \sum_{k=1}^{\infty} \frac{2l}{k\pi}\,(-1)^{1+k} \sin\left(\frac{k\pi\,x}{l}\right); \quad x = \frac{l}{2} + \sum_{k=1}^{\infty} \frac{2l}{k^2\pi^2} \left[(-1)^k - 1\right] \cos\left(\frac{k\pi\,x}{l}\right).$

4.3.22. $l^2 - x^2 = \sum_{k=1}^{\infty} \frac{8\,l^2 \text{BesselJ}(0,\ \text{BesselJZeros}(0,k)x/l)}{\texttt{BesselJ}(1,\ \text{BesselJZeros}(0,k))\text{BesselJZeros}(0,k)^3}.$

4.3.25. (c): $x^4 = \frac{1}{5}\,P_0 + \frac{4}{7}\,P_2 + \frac{8}{35}\,P_4.$

4.3.27. Single Negative Eigenvalue: $\lambda_0 = -1$; Eigenfunction: $y_0(x) = 1$.

Section 4.4

4.4.1. $u(x,t) = \sum_{k=1}^{\infty} \frac{2\,l^2 h}{a(l-a)k^2\pi^2} \sin\left(\frac{ak\pi}{l}\right) \cos\left(\frac{k\pi\,ct}{l}\right) \sin\left(\frac{k\pi\,x}{l}\right).$

4.4.3. $u(x,t) = \sum_{k=1}^{\infty} \frac{4\,l^3}{ck^4\pi^4} \left[(-1)^k - 1\right] \sin\left(\frac{k\pi\,ct}{l}\right) \sin\left(\frac{k\pi\,x}{l}\right).$

4.4.6. $u\left(x,t\right)=\frac{1}{l}\left(\int_0^l f\left(x\right)dx+t\int_0^l g\left(x\right)dx\right)+$

$\sum_{k=1}^{\infty}\frac{2}{clk\pi}\int_0^l \cos\left(\frac{k\pi\,x}{l}\right)\left[f\left(x\right)\cos\left(\frac{k\pi\,ct}{l}\right)\pi\,kc+g\left(x\right)\sin\left(\frac{k\pi\,ct}{l}\right)l\right]dx\ \cos\left(\frac{k\pi\,x}{l}\right).$

4.4.7. $u(x,t)=t.$

4.4.12. $u\left(x,t\right)=\sum_{k=1}^{\infty}\left[\frac{4\pi^2\sqrt{\lambda_k}\left(\cos\left(\sqrt{\lambda_k}l\right)-1\right)\cos\left(c\sqrt{\lambda_k}t\right)\beta\,\sin\left(\sqrt{\lambda_k}x\right)}{\left(\lambda_k l+2\sqrt{\lambda_k}\pi\right)\left(\lambda_k l-2\sqrt{\lambda_k}\pi\right)\left(\left(\cos\left(\sqrt{\lambda_k}l\right)\right)^2+l\beta\right)}\right].$

4.4.18. $u\left(x,t\right)=\frac{2}{l}\sum_{k=1}^{\infty}\sin\left(\frac{ak\pi}{l}\right)e^{-c^2\pi^2k^2t/l^2}\sin\left(\frac{k\pi\,x}{l}\right).$

4.4.20. (b): $u\left(x,t\right)=\frac{1}{2}-\frac{1}{2}\cos\left(\frac{2\pi\,x}{l}\right)e^{-4\,c^2\pi^2t/l^2}.$

4.4.24. $u\left(x,t\right)=\frac{2}{l}\sum_{k=1}^{\infty}e^{-\frac{\left(c^2\pi^2k^2+a^2l^2\right)t}{l^2}}\sin\left(\frac{k\pi\,x}{l}\right)\int_0^l \sin\left(\frac{k\pi\,x}{l}\right)f\left(x\right)dx.$

4.4.28. (b): The problem has no solution since the data are not compatible.

4.4.36. $u\left(r,\phi\right)=\sum_{k=0}^{\infty}\frac{(1+2\,k)P_k[\cos(\phi)]r^k\int_{-1}^{1}F(x)P_k(x)dx}{2\,l^k}.$

Section 4.5

4.5.1. (a): $u\left(x,t\right)=\frac{e^{-t}\sin(x)}{1+c^2}+\frac{\sin(x)\sin(ct)}{\left(1+c^2\right)c}-\frac{\sin(x)\cos(ct)}{1+c^2}.$

(d): $u\left(x,t\right)=\begin{cases}1/2c, & |x-a|<c\,|t-b|\,,\\ 0, & c\,|t-b|<|x-a|\,.\end{cases}$

4.5.3. (a): $u\left(x,t\right)=\sum_{k=1}^{\infty}\frac{4\,l^4}{\pi^5k^5c^2}\left[(-1)^k-1\right]\left[1-\cos\left(\frac{k\pi\,ct}{l}\right)\right]\sin\left(\frac{k\pi\,x}{l}\right).$

4.5.5. $\lim_{t\to\infty}u\left(x,t\right)=\sum_{k=1}^{\infty}\frac{2\,l}{c^2\pi^2k^2}\cos\left(\frac{k\pi\,x}{l}\right)\int_0^l \cos\left(\frac{k\pi\,x}{l}\right)g\left(x\right)dx.$
The steady state limit exists if $\int_0^l g(x)\ dx=0.$

4.5.6. $u\left(x,t\right)=\frac{l^2}{c^2\pi^2}\left[1-e^{-c^2\pi^2t/l^2}\right]\cos\left(\frac{\pi\,x}{l}\right).$

Section 4.6

4.6.6. If $\omega=\pi kc/l$ for some positive integer k, the corresponding term $u_k(x,t)$ in the series solution must be replaced by the following expression. The term t that multiplies the cosine term gives rise to the resonance effect.

$$u_k\left(x,t\right)=\frac{1}{c^2\pi^2k^2}\left[l\sin\left(\frac{k\pi\,ct}{l}\right)-c\pi\,k\,t\cos\left(\frac{k\pi\,ct}{l}\right)\right]\sin\left(\frac{k\pi\,x}{l}\right)\int_0^l \sin\left(\frac{k\pi\,x}{l}\right)F\left(x\right)dx.$$

4.6.10. $u\left(r,\theta\right)=c+\frac{3}{4}\,r\sin\left(\theta\right)-\frac{1}{12}\,\frac{r^3\sin(3\,\theta)}{R^2}.$

4.6.14. (b): $u\left(x,t\right)=\sum_{k=1}^{\infty}\frac{2(-1)^k}{k\pi}\sin\left(\frac{k\pi\,x}{l}\right)\left[-1+e^{-c^2\pi^2k^2t/l^2}\right].$

Section 4.7

4.7.4. $U(t) = \frac{\epsilon}{\epsilon + e^{-\hat{\lambda} t} - \epsilon e^{-\hat{\lambda} t}}$.

4.7.6. $\lambda_c = 1$.

Section 4.8

4.8.4. The first 10 eigenvalues:

$$[6.6071,\ 28.666,\ 68.939,\ 128.48,\ 207.59,\ 306.37,\ 424.86,\ 563.06,\ 721.00,\ 898.67].$$

4.8.8. The first 10 terms in the Fourier-Bessel series:

$$1-x^2 \approx 1.108022262\,\mathrm{BesselJ}\left(0, 2.404825558\,x\right) - 0.1397775052\,\mathrm{BesselJ}\left(0, 5.52007811\,x\right)$$
$$+\,0.04547647110\,\mathrm{BesselJ}\left(0, 8.653727913\,x\right) - 0.02099090194\,\mathrm{BesselJ}\left(0, 11.79153444\,x\right)$$
$$+0.01163624312\,\mathrm{BesselJ}\left(0, 14.93091771\,x\right) - 0.007221175738\,\mathrm{BesselJ}\left(0, 18.07106397\,x\right)$$
$$+0.004837871926\,\mathrm{BesselJ}\left(0, 21.21163663\,x\right) - 0.003425679000\,\mathrm{BesselJ}\left(0, 24.35247153\,x\right)$$
$$+0.002529529970\,\mathrm{BesselJ}\left(0, 27.49347913\,x\right) - 0.001930146609\,\mathrm{BesselJ}\left(0, 30.63460647\,x\right).$$

Chapter 5

Section 5.2

5.2.9. $F(\lambda) = \frac{\sqrt{2}\ k}{\sqrt{\pi}\left(k^2+\lambda^2\right)}.$

5.2.12. $y(x) = \begin{cases} \frac{1-e^{-ak}\cosh(kx)}{k^2}, & |x| < a, \\ \frac{e^{-k|x|}\sinh(ak)}{k^2}, & |x| > a. \end{cases}$

5.2.16. $u(x,t) = \frac{1}{2}\, u_0 \left[\operatorname{erf}\left(\frac{x+a}{2\sqrt{c^2 t}}\right) - \operatorname{erf}\left(\frac{x-a}{2\sqrt{c^2 t}}\right)\right].$

5.2.19. $u(x,t) = \frac{1}{2\sqrt{\pi\, t}} \int_{-\infty}^{\infty} f(x-s) \cos\left(\frac{s^2}{4t} - \frac{\pi}{4}\right)\ ds.$

5.2.22. $u(x,y) = \frac{b+a}{2} + \frac{b-a}{\pi}\ \arctan\left(\frac{x}{y}\right).$

5.2.24. $u(x,y) = \frac{2}{\pi}\ \int_0^{\infty} \frac{\sinh(\lambda\, L - \lambda\, y)\cos(x\lambda)}{\sinh(\lambda\, L)\left(\lambda^2+1\right)}\ d\lambda.$

Section 5.3

5.3.3. (b): $F_s(\lambda) = \frac{\sqrt{2}(1-\cos(\lambda\, a))}{\sqrt{\pi}\lambda}$; $F_c(\lambda) = \frac{\sqrt{2}\sin(\lambda\, a)}{\sqrt{\pi}\lambda}.$

5.3.5. $y(x) = \frac{1}{2k}\int_0^{\infty} f(t)\left(e^{-k|x-t|} - e^{-k(x+t)}\right)\ dt.$

5.3.7. $y(x) = \frac{e^{-x} - k^2 e^{-kx}}{1-k^2}.$

5.3.12. $u(x,t) = \begin{cases} 0, & t < \frac{x}{c}, \\ g\left(\frac{-x+ct}{c}\right), & \frac{x}{c} < t. \end{cases}$

5.3.13. $u(x,t) = c\int_0^{t-\frac{x}{c}} h(s)\, ds\left[H\left(\frac{x}{c} - t\right) - 1\right].$

5.3.17. $u(x,y) = \frac{1}{\pi}\left[\arctan\left(\frac{x+1}{y}\right) - \arctan\left(\frac{x-1}{y}\right)\right].$

5.3.18. $u(x,y) = \frac{1}{2\pi}\int_0^{\infty}\left[\ln\left((x-t)^2 + y^2\right) - \ln\left((x+t)^2 + y^2\right)\right] g(t)dt.$

Section 5.4

5.4.5. $u(r,t) = r^2 t + t^3 c^2 + 1.$

5.4.7. $u(x,y,t) = x.$

5.4.9. $v(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} f(\xi)\, I_0\left((a/c)\sqrt{c^2t^2 - (x-\xi)^2}\right)\ d\xi.$

Section 5.5

5.5.6. $u(r,z) = \frac{1}{4\pi}\int_0^{\infty}\int_{-\infty}^{\infty} e^{-\sqrt{\lambda^2+k^2}|-z+s|} f(s) J_0(\lambda\, r)\ ds\, d\lambda.$

Section 5.6

5.6.10. $F(\lambda) = \frac{1}{\sqrt{\lambda^2 + a^2}}.$

5.6.11. $u(x,t) = \frac{1}{2\sqrt{\pi\, c^2 t}} \int_0^\infty f(s) \left(e^{-\frac{(x-s)^2}{4c^2 t}} - e^{-\frac{(x+s)^2}{4c^2 t}} \right) ds.$

5.6.22. $u(x,t) = x + \sin\left(\frac{3\pi\, x}{l}\right) e^{-9\pi^2 c^2 t/l^2} - l \sum_{k=0}^\infty \left(\operatorname{erfc}\left[\frac{2lk+l-x}{2\sqrt{c^2 t}}\right] - \operatorname{erfc}\left[\frac{2\, lk+l+x}{2\sqrt{c^2 t}}\right] \right).$

5.6.26. $u(x,t) = 1 + \sum_{k=1}^\infty \frac{2}{\pi k} \left[(-1)^k - 1\right] \sin\left(\frac{\pi\, kx}{l}\right) e^{-\pi^2 k^2 c^2 t/l^2}.$

5.6.27. $u(x,t) = e^{-t} + 2\,t + \begin{cases} 0, & t < \frac{x}{c}, \\ \sin\left(t - \frac{x}{c}\right) + \frac{2x}{c} - 2\,t - e^{x/c-t}, & \frac{x}{c} < t. \end{cases}$

Section 5.7

5.7.3. $u(x,t) \approx \frac{2}{\sqrt{t}}\ \mathrm{Re}\left\{ H_+\left(\frac{x}{t}\right) e^{i\left[c\sqrt{\gamma^2 t^2 - x^2}/\gamma - 1/4\pi\right]} \right\}, \quad |x|, t \to \infty.$

5.7.6. $u(x,t) \approx \frac{\sqrt{2}}{4\sqrt{\pi\, a^2 t}}\ F(0) \left[e^{-\frac{(x+t)^2}{2a^2 t}} + e^{-\frac{(x-t)^2}{2a^2 t}} \right], \quad t \to \infty.$

Chapter 6

Section 6.2

6.2.4. $u(x,t) = \begin{cases} \frac{p_1 c_2 A + p_2 c_1 B}{p_1 c_2 + p_2 c_1} - \frac{p_2 c_1 (A-B) \mathtt{erf}\left(x/\sqrt{4 c_1{}^2 t}\right)}{p_1 c_2 + p_2 c_1}, & x < 0. \\ \frac{p_1 c_2 A + p_2 c_1 B}{p_1 c_2 + p_2 c_1} + \frac{p_1 c_2 (B-A) \mathtt{erf}\left(x/\sqrt{4 c_2{}^2 t}\right)}{p_1 c_2 + p_2 c_1}, & x > 0. \end{cases}$

6.2.8.

$$\text{Eigenvalue equation}: \ c_2 \rho_2 \cot\left(\frac{\sqrt{\lambda_k} x_0}{c_1}\right) + c_1 \rho_1 \cot\left(\frac{\sqrt{\lambda_k}\,(l - x_0)}{c_2}\right) = 0.$$

$$\text{Eigenfunctions}: \ v_k(x) = \begin{cases} \cos\left(\sqrt{\lambda_k} x / c_1\right) / \cos\left(\sqrt{\lambda_k} x_0 / c_1\right), & x < x_0, \\ \cos\left(\sqrt{\lambda_k}\,(l - x)/c_2\right) / \cos\left(\sqrt{\lambda_k}\,(l - x_0)/c_2\right), & x_0 < x. \end{cases}$$

6.2.13. $\hat{\theta} = \pi - \theta, \quad \phi = \arcsin\left(\frac{k_1 \sin(\theta)}{k_2}\right),$

$$T = \frac{2 k_1 \cos(\theta)}{k_1 \cos(\theta) + \sqrt{k_2{}^2 - k_1{}^2 (\sin(\theta))^2}}, \quad R = \frac{k_1 \cos(\theta) - \sqrt{k_2{}^2 - k_1{}^2 (\sin(\theta))^2}}{k_1 \cos(\theta) + \sqrt{k_2{}^2 - k_1{}^2 (\sin(\theta))^2}}.$$

Section 6.4

6.4.5. For example, we can set $u(x,0) = 0$, $u_t(x,0) = x$.

6.4.6. $u(x,t) = \begin{cases} 1, & t < t_0, \\ \mathtt{erf}\left(\frac{x}{2c\sqrt{t - t_0}}\right), & t_0 < t. \end{cases}$

Section 6.5

6.5.3. (b): $u_x(0,t) = -\sin(\omega t), \quad u(x,t) = \begin{cases} (c/\omega)\left[1 - \cos\left(\frac{(ct-x)\omega}{c}\right)\right], & x < ct, \\ 0, & ct < x. \end{cases}$

6.5.6. $u(x,t) = \begin{cases} A \cos\left(\frac{\omega(ct-x)}{c_0 + c}\right), & x < ct, \\ 0, & ct < x. \end{cases}$

6.5.8. $u(x,t) = \frac{1}{\omega^2}\left[e^{-\left(\frac{-x_0 + x + ct}{2c}\right)}\omega^2 + \cos\left[\frac{\omega(-x_0 + x + ct)}{2c}\right] + \cos\left[\frac{(-x + ct + x_0])\omega}{2c}\right] - \cos(\omega t) - 1\right].$

6.5.9. $u(x,t) = \frac{9}{10} \sin\left(\frac{x}{3} + \frac{t}{3}\right) + \frac{1}{10} \sin(x - t) - \frac{1}{\sqrt{5}}\ x + \frac{2}{\sqrt{5}}\ t.$

6.5.12. $u(x,t) = g\left(\frac{x + ct}{2c}\right) - g\left(\frac{x - ct}{2c}\right) + f(x - ct).$

Section 6.6

6.6.2.

$$\mathtt{Eigenvalues:} \quad m_1 \sqrt{\lambda_k} \sin\left(\sqrt{\lambda_k}\,(\pi - 1)\right) \sin\left(\sqrt{\lambda_k}\right) - \sin\left(\pi \sqrt{\lambda_k}\right) = 0, \ k \geq 1.$$

Eigenfunctions: $M_k(1) = 1,\ k \geq 1$: $\quad M_k(x) = \begin{cases} \sin\left(\sqrt{\lambda_k} x\right)/\sin\left(\sqrt{\lambda_k}\right), & x < 1, \\ \sin\left(\sqrt{\lambda_k}\,(\pi - x)\right)/\sin\left(\sqrt{\lambda_k}\,(\pi - 1)\right), & 1 < x. \end{cases}$

6.6.4.

$$\frac{d}{dx}\left(p(x)\,\frac{dM(x)}{dx}\right) + \lambda\,\rho(x)\,M(x) = 0, \quad 0 < x < l,$$

$$M(0) = 0, \quad p(l)\,\frac{dM(x)}{dx}\bigg|_{x=l} - \lambda\, m_1 M(l) = 0.$$

6.6.5. $\quad \text{Norm}[M_k(x)] = \sqrt{\frac{m_1}{2} + \frac{l}{2(\sin(\sqrt{\lambda_k} l))^2}}.$

6.6.6. $\quad u(x,t) = \begin{cases} 0, & ct < |x|, \\ \frac{cA}{2\omega}\sin\left(\frac{\omega\,(x+ct)}{c}\right), & x < 0, \\ -\frac{cA}{2\omega}\sin\left(\frac{\omega\,(x-ct)}{c}\right), & x > 0. \end{cases}$

6.6.8. $\quad u(x,t) = \sqrt{\frac{p}{4\pi\,\rho}}\int_0^t f_0(s)\, e^{-\left(\rho\, x^2/4p(t-s)\right)}\,\frac{1}{\sqrt{t-s}}\, ds.$

Section 6.7

6.7.10. $\quad u(x,y,z,t) = \frac{1}{8}\int_0^t e^{-\left(\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4c^2(t-\tau)}\right)} f(\tau)\,[\pi\, c\,(t-\tau)]^{-3/2}\, d\tau.$

Section 6.8

6.8.4. In two dimensions $u(x,y)$ must be bounded at infinity. In three dimensions $u(x,y,z)$ must vanish at infinity.

6.8.7. $E(t) = \frac{1}{2}\int_{-\infty}^{\infty}(u(x,t))^2\, dx, \quad E'(t) = -\int_{-\infty}^{\infty}[b(x,t) - a_x(x,t)/2]\,(u(x,t))^2\, dx.$
If $b - a_x/2 \geq 0$ uniqueness follows. Otherwise, the transformation given in the text must be used.

6.8.11. If u and u_x vanish at $x = 0$ and $x = l$ then $E(t) = \frac{1}{2}\int_0^l \left(\frac{\partial u(x,t)}{\partial t}\right)^2 + \left(\frac{\partial^2 u(x,t)}{\partial x^2}\right)^2 dx.$

Chapter 7

Section 7.1

7.1.4. $u(\xi) = -\int_{S_1} (p(x)/\alpha(x)) \frac{\partial K(x;\xi)}{\partial n} B(x)\ ds + \int_{S_2 \cup S_3} (p(x)/\beta(x)) K(x;\xi) B(x)\ ds.$

7.1.12. $\partial^2 K(x,t;\xi,\tau)/\partial t^2 + c^2\, \partial^4 K(x,t;\xi,\tau)/\partial x^4 = \delta(x-\xi)\,\delta(\tau - t),\ 0 < x < l,\ t < T,\quad K(x,T;\xi,\tau) = 0,\quad K_t(x,T;\xi,\tau) = 0,\ 0 < x < l,\quad K(0,t;\xi,\tau) = 0,\ K(l,t;\xi,\tau) = 0,\ K_x(0,t;\xi,\tau) = 0,\ K_x(l,t;\xi,\tau) = 0,\ t < T.$

7.1.15.

$$-p'(x)\frac{\partial K(x;\xi)}{\partial x} - p(x)\frac{\partial^2 K(x;\xi)}{\partial x^2} - K(x;\xi)\, b'(x) - \frac{\partial K(x;\xi)}{\partial x} b(x) + q(x)\, K(x;\xi) = \delta(-x+\xi),$$

$$\int_0^l \rho(x)\, K(x;\xi)\, F(x)\, dx - u(\xi) = -p(l)\, K(l;\xi)\, \mathrm{D}(u)(l) + u(l)\, p(l)\, D_1(K)(l;\xi)$$

$$+u(l)\, K(l;\xi)\, b(l) + p(0)\, K(0;\xi)\, \mathrm{D}(u)(0) - u(0)\, p(0)\, D_1(K)(0;\xi) - u(0)\, K(0;\xi)\, b(0).$$

If $\beta_1 = 0,\ \beta_2 = 0$, then $u(x)$ is specified at the endpoints, but its derivatives are not known, so we put $K(0;\xi) = 0,\ K(l;\xi) = 0$. The solution formula is given as $u(\xi) = \int_0^l \rho(x)\, K(x;\xi)\, F(x)\, dx - p(0)\, a_1 K_x(0;\xi) + p(l)\, a_2 K_x(l;\xi)$.

Section 7.2

7.2.9. (a): $\frac{d}{dx}(H(x-1)\sin(x)) = \sin(1)\,\delta(x-1) + H(x-1)\cos(x)$.

(b): $\frac{d}{dx}\left(x^2 \delta(x)\right) = 0$.

(c): $\frac{d}{dx}\left(e^x \delta'(x+3) - H(x)(\cos(x))^2\right) = -e^{-3}\delta'(x+3) + e^{-3}\delta''(x+3) - \delta(x) + 2\, H(x)\cos(x)\sin(x)$.

(d): $\frac{d}{dx}(\ln(x)\, H(x-1)) = \frac{H(x-1)}{x}$.

(e): $\frac{d}{dx} e^{|x|} = \mathtt{sgn}(x)\, e^{|x|}$.

7.2.11. With $a > 0$ we have $\delta\left(x^2 - a^2\right) = \delta(x+a)/2a + \delta(x-a)/2a$.

7.2.12. $\delta(\sin(x)) = \sum_{n=-\infty}^{\infty} \delta(x - n\pi)$.

7.2.22. $2\sqrt{c} - 2/\sqrt{c}$.

Section 7.3

7.3.1. $u(x) = (l - x/l)\int_0^x \xi\, f(\xi)\, d\xi + (x/l)\int_x^l (l-\xi)\, f(\xi)\, d\xi$.

7.3.3. $K(x;\xi) = \begin{cases} \frac{\cosh(c(l-\xi))\sinh(cx)}{c\cosh(cl)}, & 0 < x \le \xi, \\ \frac{\sinh(c\xi)\cosh(c(x-l))}{c\cosh(cl)}, & \xi < x < l. \end{cases}$

7.3.5. $K(x;\xi) = \begin{cases} \frac{\sin(cl - c\xi)\sin(cx)}{c\sin(cl)}, & 0 < x \le \xi, \\ \frac{\sin(cl - cx)\sin(c\xi)}{c\sin(cl)}, & \xi < x < l. \end{cases}$

7.3.6. $K(x;\xi) = \begin{cases} -\ln(\xi), & 0 < x \le \xi, \\ -\ln(x), & \xi < x < l. \end{cases}$

7.3.9.

$$u(\xi) = \int_0^l \left(\begin{cases} \frac{-\cosh(c(l-\xi))\sinh(cx)}{c\cosh(cl)}, & x < \xi, \\ \frac{-\sinh(c\xi)\cosh(c(l-x))}{c\cosh(cl)}, & \xi < x, \end{cases} \right) e^x dx + \frac{10\sinh(c\xi)}{c\cosh(cl)} + \frac{3\cosh(c(l-\xi))}{c\cosh(cl)}.$$

The integral can be evaluated in closed form but we do not exhibit the result.

7.3.13. Compatibility Condition: $1 = -p(l)B + p(0)A$.

$$\text{Nonunique Green's Function:} \quad K(x;\xi) = \begin{cases} p(0)\,A\int_\xi^x \frac{1}{p(s)}\,ds, & 0 < x < \xi, \\ p(l)\,B\int_\xi^x \frac{1}{p(s)}\,ds, & \xi < x < l. \end{cases}$$

7.3.23. There is an additional term given as $N_0(t)M_0(x) = (\tau - t)M_0(x)M_0(\xi)H(\tau - t)$.

7.3.26. (b):

$$K(x,t;\xi,\tau) = \left[\sum_{k=1}^{\infty} \frac{4\,\sin\left(\frac{\pi\,(2\,k-1)(\tau-t)}{2l}\right)\sin\left(\frac{\pi\,(2\,k-1)x}{2l}\right)\sin\left(\frac{\pi\,(2\,k-1)\xi}{2l}\right)}{\pi\,(2\,k-1)} \right] H(\tau - t)\,.$$

7.3.29. (b):

$$K(x,t;\xi,\tau) = \left(\frac{2}{l}\sum_{k=1}^{\infty} e^{\left(\frac{\pi^2(k-1/2)^2(t-\tau)}{l^2}\right)} \sin\left[\frac{\pi\,(k-1/2)\,x}{l}\right] \sin\left[\frac{\pi\,(k-1/2)\,\xi}{l}\right] \right) H(\tau - t)\,.$$

Section 7.4

7.4.4. $K(x,t;\xi,\tau) = \frac{H(\tau-t)}{\sqrt{4\pi\,c^2(t-\tau)}}\left[e^{\left(\frac{(x+\xi)^2}{4(t-\tau)c^2}\right)} + e^{\left(\frac{(x-\xi)^2}{4(t-\tau)c^2}\right)} \right].$

7.4.10. $K(x,y,z,t;\xi,\eta,\zeta,\tau) = \frac{\delta(\gamma\,(\tau-t)-r)}{4\pi\,\gamma\,r} + \frac{cI_1\left(\frac{c\sqrt{\gamma^2(\tau-t)^2-r^2}}{\gamma}\right)H(\gamma\,(\tau-t)-r)}{4\pi\gamma^2\sqrt{\gamma^2(\tau-t)^2-r^2}}.$

7.4.14. $K(x,t;\xi,\tau) = \sqrt{\frac{m}{2\pi\,\hbar^3(\tau-t)}}\;\exp\left[\frac{im(x-\xi)^2}{2\hbar(\tau-t)} - \frac{3i\pi}{4}\right] H(\tau - t)\,.$

7.4.17. (b): $K(x;\xi) = \begin{cases} \frac{e^{-k\xi}\cosh(kx)}{k}, & 0 < x < \xi, \\ \frac{e^{-kx}\cosh(k\xi)}{k}, & x > \xi. \end{cases}$

7.4.22.

$$K(x,z;\zeta) = \sum_{n=0}^{\infty} \frac{2i\sin\left[\frac{(2n+1)\pi\zeta}{2h}\right]\exp\left(\frac{i\sqrt{4k^2h^2-\pi^2(1+2n)^2}\,|x|}{2h}\right)\sin\left[\frac{(2n+1)\pi z}{2h}\right]}{\sqrt{4k^2h^2-\pi^2(1+2n)^2}}.$$

Section 7.5

7.5.1. (a): $u(\xi,\eta,\zeta) = \frac{1}{2\pi}\int_{\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x,y)\zeta}{\left((x-\xi)^2+(y-\eta)^2+\zeta^2\right)^{3/2}}\,dx\,dy.$

7.5.7. (a): $K(x,y,z;\xi,\eta,\zeta) = \frac{e^{ik\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}} - \frac{e^{ik\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}}}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}}.$

7.5.12. $u(\xi,\tau) = \begin{cases} 0, & \tau < \xi/\gamma, \\ -\gamma\int_0^{\tau-\xi/\gamma} h(t)\,J_0\left(\frac{c\sqrt{\gamma^2(t-\tau)^2-\xi^2}}{\gamma}\right)dt, & \xi/\gamma < \tau. \end{cases}$

7.5.20. $u(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\,\pi}\frac{\left(a^2-{\rho_0}^2\right)B}{{r_0}^2}\,d\theta.$

7.5.21. (b): $K(x,y;\xi,\eta) = -\frac{\pi}{2}\,\ln\left(\frac{\sqrt{(x-\xi)^2+(y-\eta)^2)}\sqrt{(x-\xi)^2+(y+\eta)^2)}}{\sqrt{(x+\xi)^2+(y-\eta)^2)}\sqrt{(x+\xi)^2+(y+\eta)^2)}}\right).$

Chapter 8

Section 8.1

8.1.3. $E\left(u\left(x\right)\right)=\left(u\left(l\right)\right)^2p\left(l\right)h_2+\left(u\left(0\right)\right)^2p\left(0\right)h_1+\int_0^l\left[\left(u'(x)\right)^2p\left(x\right)+q\left(x\right)\left(u\left(x\right)\right)^2\right]dx.$

8.1.5. $c=\frac{12}{\pi\left(2\,\pi^2-3\right)}$; $\quad E\left(u\left(x,y\right)\right)=\frac{2(2\,\pi^2+3)}{2\,\pi^2-3}.$

8.1.10. $p_M=2, p_m=1, \rho_M=1, \rho_m=1;\quad \lambda_n{}^{(M)}=4\,\pi^2n^2,\ \lambda_n{}^{(m)}=\pi^2n^2.$

8.1.15. Eigenvalues: $\lambda_{k,n}=\frac{1}{a^2}$ BesselJZeros $\left(\frac{\pi\,n}{\phi},k\right)^2,\ k,n=1,2,\ldots.$
Unnormalized Eigenfunctions: $M_{k,n}\left(r,\theta\right)=$ BesselJ $\left[\frac{\pi\,n}{\phi},\right.$ BesselJZeros $\left.\left(\frac{\pi n}{\phi},k\right)\frac{r}{a}\right]\sin\left[\frac{\pi n\theta}{\phi}\right].$

Section 8.2

8.2.2. $\hat{\phi_1}=1,\ \hat{\phi_2}=2\sqrt{3}\left(x-\frac{1}{2}\right),\ \hat{\phi_3}=6\sqrt{5}\left(x^2-x+\frac{1}{6}\right),\ \hat{\phi_4}=20\sqrt{7}\left(x^3-\frac{3x^2}{2}+\frac{3x}{5}-\frac{1}{20}\right).$

8.2.8. $\hat{\phi_1}=\pi^2\left(\pi^2+4\right)/(4\pi^2-16).$

8.2.10. $\hat{\lambda_1}=5/2\pi^2.$

8.2.11. $\hat{\lambda_1}=5/6.$

8.2.12. $\hat{\lambda_1}=1+\epsilon\,\pi/2+O\left(\epsilon^2\right).$

8.2.14. $\hat{\lambda_1}=22.13493924.$

8.2.18. $w\left(x,y\right)=x^2+0.8006055180\,\cos\left(\pi\sqrt{x^2+y^2}/\,2\right).$

Section 8.3

8.3.4. $R\left(x,t;\xi,\tau\right)=\frac{1}{2\gamma}J_0\left(\frac{\sqrt{c}\sqrt{\gamma^2(t-\tau)^2-(x-\xi)^2}}{\gamma}\right).$

Section 8.5

8.5.5. If we assume that $l=1, c=1, f(x)=x(1-x)$ and $g(x)=0$, then the solution is

$$u\left(x,t\right)=\sum_{k=1}^{\infty}\frac{4\left[1-(-1)^k\right]\cos\left(\pi^2k^2t\right)\sin\left(k\pi\,x\right)}{\pi^3k^3}.$$

8.5.8. (b): $K\left(x;\xi\right)=\begin{cases}\frac{x\left(x^2l-x^2\xi-2\,l^2\xi-\xi^3+3\,l\xi^2\right)}{6l}, & 0<x<\xi,\\ -\frac{\xi\left(x^3-3\,x^2l+2\,xl^2+\xi^2x-l\xi^2\right)}{6l}, & \xi<x<l.\end{cases}$

8.5.15. $u\left(r,\theta\right)=\left(r^2-R^2\right)\left(\frac{3r\sin(\theta)}{8R^2}-\frac{r^3\sin(3\,\theta)}{8R^4}\right)+1.$

8.5.17. $u\left(r\right)=\frac{1}{64}F_0\left(R^4+r^4\right)-\frac{1}{32}F_0R^2r^2.$

8.5.32. (a): Cutoff Frequency: $c\pi\sqrt{\hat{l}^2+l^2}/\,l\hat{l}.$

Chapter 9

Section 9.2

9.2.1. $u(r) = \frac{r^2}{4} - \frac{1}{4} + \epsilon^2\left(-\frac{r^4}{64} + \frac{r^2}{16} - \frac{3}{64}\right) + O\left(\epsilon^4\right), \ r^2 = x^2 + y^2.$

9.2.3. $u(r) = \frac{r^2}{4} - \frac{1}{4} + \epsilon\left(-\frac{r^4}{32} + \frac{1}{32}\right) + O\left(\epsilon^2\right), \quad r^2 = x^2 + y^2.$

9.2.5.

$$u(x,y) = \sum_{k=1}^{\infty} \frac{2\sinh\left(k\left(y-\pi\right)\right)\left[-1+(-1)^k\right]\sin\left(kx\right)}{\sinh\left(k\pi\right)k\pi} +$$

$$\epsilon \sum_{k=1}^{\infty}\left[\frac{[-1+(-1)^k][\sinh(k(y-\pi))\,y - \cosh(k(y-\pi))\,ky^2]}{2k\pi\sinh(k\pi)} + \frac{\pi\left[-1+(-1)^k\right]\sinh(ky)}{2(\sinh(k\pi))^2}\right]\sin(kx) + O\left(\epsilon^2\right).$$

9.2.11.

Perturbation result: $u(x,t) = \left(\frac{1}{2} - \frac{\epsilon\,t}{4}\right)\left[f\left(x-ct\right) + f\left(x+ct\right)\right] + \frac{\epsilon}{4c}\int_{x-ct}^{x+ct} f(s)\ ds + O\left(\epsilon^2\right).$

Multiple scales result: $u(x,t) = \frac{1}{2}e^{-\epsilon\,t/2}\left[f\left(x-ct\right) + f\left(x+ct\right)\right] + \frac{\epsilon(1+\epsilon\,t/4)}{4c}\,e^{-\epsilon\,t/2}\int_{x-ct}^{x+ct} f(s)\,ds + O\left(\epsilon^2\right).$

9.2.17. $u(x,t) = \left\{A\cos\left[\omega\left(t - \frac{x}{c}\right)\right] - \frac{\epsilon\,A\omega}{c}\ \left(t - \frac{x}{c}\right)\sin\left[\omega\left(t-\frac{x}{c}\right)\right]\right\}H\left(t - \frac{x}{c}\right) + O\left(\epsilon^2\right).$

9.2.18. $u(x,y) = (1 + x^2 - y^2)/2 + \epsilon\left[a + b - (a-b)\left(x^2 - y^2\right)\right]/2 + O\left(\epsilon^2\right).$

9.2.21. $\lambda(\epsilon) = \pi^2 + \epsilon(-3 + 2\pi^2)/6\pi^2 + O\left(\epsilon^2\right).$

9.2.23. $\lambda_1(\epsilon) = 1/4 + 2\epsilon/\pi + O\left(\epsilon^2\right).$

Section 9.3

9.3.1.

Exact solution: $u(x,t) = \left[\epsilon\,e^{-\frac{t}{\epsilon}}\cos(x-t) + \epsilon^2 e^{-\frac{t}{\epsilon}}\sin(x-t) - \epsilon\,\cos(x) + \sin(x)\right]/\left(1+\epsilon^2\right).$

Regular perturbation result: $u(x,t) = \dfrac{\sin(x) - \epsilon\,\cos(x)}{1+\epsilon^2}.$

Boundary layer expansion: $u(x,t) = \left[\epsilon\,\cos(x) + (\epsilon\,t + \epsilon^2)\sin(x)\right]e^{-\frac{t}{\epsilon}} + \sin(x) - \epsilon\,\cos(x) - \epsilon^2\sin(x) + O\left(\epsilon^3\right).$

9.3.3. Both the exact and the boundary layer results break down when $t = \epsilon$.

Exact solution: $u(x,t) = \dfrac{\epsilon\,\sin(x-t)}{t\sin(x-t)+\epsilon}.$

Boundary layer result: $u(x,t) = \dfrac{\epsilon\,\sin(x)}{t\sin(x)+\epsilon} + O\left(\epsilon^2\right).$

9.3.9. (a): Composite result: $u(x,y) \approx e^{-\frac{x}{\epsilon}}\sin(\pi\,y/L) + x^2 e^{-\left(\frac{y}{\epsilon}-x\right)}.$

(c): Composite result: $u\,(x,y) \approx (x+y)^2\, e^{-(x+y)} - (x+L)^2\, e^{-(x+L)} e^{\frac{y-L}{\epsilon}} +$
$[\sin\,(\pi\, y/L) - y^2 e^{-y}] e^{-\frac{x}{\epsilon}}$.

9.3.11. Composite result: $u\,(x,y) \approx e^{-\frac{x}{\sqrt{\epsilon}}} + e^{\frac{x-\pi}{\sqrt{\epsilon}}} + e^{-\frac{y}{\sqrt{\epsilon}}} + e^{\frac{y-\pi}{\sqrt{\epsilon}}} - 1$.

9.3.13. Composite result: $u\,(x,y) \approx y + \operatorname{erfc}\left(\frac{y}{2\sqrt{\epsilon\,(x+1)}}\right)$.

9.3.19. (a): Parabolic boundary layers at $y = 0$ and $y = \pi$.
(b): A parabolic boundary layer at $y = 0$ and an ordinary boundary layer at $x = \pi$.
(c): An ordinary boundary layer at $x = \sqrt{1 - y^2}$.

Chapter 10

Section 10.1

10.1.1. $u_S e^{-i\omega\, t} = -\frac{\exp\left[i\left(kn\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}-\omega\, t\right)\right]}{4\pi\sqrt{(x-\xi)^2+(y-\eta)^2+(z+\zeta)^2}}.$

10.1.4.

$$K\left(x,y;\xi,\eta\right) = \frac{i}{4}\,\mathrm{HankelH1}\left(0,kn\sqrt{(x-\xi)^2+(y-\eta)^2}\right) - \frac{i}{4}\,\mathrm{HankelH1}\left(0,kn\sqrt{(x-\xi)^2+(y+\eta)^2}\right)$$
$$+\frac{i}{4}\,\mathrm{HankelH1}\left(0,kn\sqrt{(x+\xi)^2+(y-\eta)^2}\right) - \frac{i}{4}\,\mathrm{HankelH1}\left(0,kn\sqrt{(x+\xi)^2+(y+\eta)^2}\right).$$

10.1.6.

Rays: $[x\left(\sigma\right),y\left(\sigma\right),z\left(\sigma\right)]_+ = [a\cos\left(\alpha\right)+\sigma\,\cos\left(\alpha\right),\, a\sin\left(\alpha\right)+\sigma\,\sin\left(\alpha\right),\, \beta],$

$[x\left(\sigma\right),y\left(\sigma\right),z\left(\sigma\right)]_- = [a\cos\left(\alpha\right)-\sigma\,\cos\left(\alpha\right),\, a\sin\left(\alpha\right)-\sigma\,\sin\left(\alpha\right),\, \beta].$

Phase functions: $\phi_+ = 1+r-a, \quad \phi_- = 1-r+a, \quad r^2 = x^2+y^2+z^2.$

10.1.10. $u_S = i\sqrt{\frac{a}{4k\pi(y^2+a^2)}}\; e^{ik(x+a)-i\pi/4}.$

10.1.17. The solution of the parabolic equation: $u\left(x,y\right) = e^{i(kx-\pi/4)}\int_{-\frac{(1+y)\sqrt{2kx}}{2x}}^{\frac{(1-y)\sqrt{2kx}}{2x}} \frac{e^{i\sigma^2}}{\sqrt{\pi}}\,d\sigma.$

Section 10.2

10.2.1. $u\left(x,t\right) = \left[1-\frac{c^2t(x+\gamma\, t)}{2\gamma}+\frac{(c^4t^2+2c^2)(x+\gamma\, t)^2}{16\gamma^2}\right]H(x+\gamma\, t) - \left[\frac{c^2(x-\gamma\, t)^2}{8\gamma^2}\right]H\left(x-\gamma\, t\right)+$
$\dots.$

10.2.3. $u\left(x,t\right) = -\frac{e^{-\lambda\, t}H(x-\gamma\, t)}{2\gamma}+\frac{e^{-\lambda\, t}\lambda^2 t(x-\gamma\, t)H(x-\gamma\, t)}{4\gamma^2}+\dots.$

10.2.10.

$$\begin{bmatrix} v\left(x,t\right) \\ w\left(x,t\right) \end{bmatrix} = \frac{e^{-\lambda\, t}}{\gamma}\begin{bmatrix} \gamma \\ -1 \end{bmatrix}\delta\left(x-\gamma\, t\right)+\frac{e^{-\lambda\, t}}{\gamma}\begin{bmatrix} \gamma \\ 1 \end{bmatrix}\delta\left(x+\gamma\, t\right)+\begin{bmatrix} \frac{\lambda\, e^{-\lambda\, t}}{2\gamma} \\ 0 \end{bmatrix}[H\left(x-\gamma\, t\right)-H\left(x+\gamma\, t\right)]+\dots$$

Section 10.3

10.3.1. The parabolic equations:

$$v_t - cv_x - \frac{\epsilon}{2}\left(a^2-c^2\right)v_{xx} = O\left(\epsilon^2\right), \quad w_t + cw_x - \frac{\epsilon}{2}\left(a^2-c^2\right)w_{xx} = O\left(\epsilon^2\right).$$

10.3.3. Burgers' equation:

$$\frac{\partial\delta\left(x,t\right)}{\partial t} - \frac{\partial\delta\left(x,t\right)}{\partial x} - \epsilon\left[\frac{(\gamma+1)}{2}\delta\left(x,t\right)\frac{\partial\delta\left(x,t\right)}{\partial x}+\left(\frac{2\mu}{3}+\frac{k(\gamma-1)^2}{2}\right)\frac{\partial^2\delta\left(x,t\right)}{\partial x^2}\right] = 0.$$

10.3.5. Korteweg-deVries equation:

$$\frac{\partial \alpha\,(x,t)}{\partial t} - \frac{\partial \alpha\,(x,t)}{\partial x} - \frac{3\epsilon}{2}\,\alpha\,(x,t)\,\frac{\partial}{\partial x}\alpha\,(x,t) - \frac{\epsilon}{6}\,\frac{\partial^3 \alpha\,(x,t)}{\partial x^3} = 0.$$

10.3.7.

$$\frac{\epsilon^2}{4}\left[\frac{\partial}{\partial x}\left(\alpha\,(x,t)\,\frac{\partial \alpha\,(x,t)}{\partial x}\right)\right] - \frac{\epsilon}{4}\,\frac{\partial^2 \alpha\,(x,t)}{\partial x^2} + \frac{\partial \alpha\,(x,t)}{\partial t} = 0.$$

Section 10.4

10.4.3. Function: $1 + [\sin(x) + 1]H(x+1) + [1 - \sin(x)]H(x-1)$.

$$\text{Piecewise Function:} \quad \begin{cases} -1, & x \le -1, \\ \sin\,(x)\,, & -1 < x \le 1, \\ 1, & x > 1. \end{cases}$$

Chapter 11

Section 11.1

11.1.5.

$$\begin{bmatrix} \sin\left[\cos\left(xyz\right)\right] & \partial/\partial x & \partial/\partial y & \partial/\partial z & \partial^2/\partial x^2 & \partial^2/\partial y^2 & \partial^2/\partial z^2 & \partial^2/\partial y\partial x \\ \mathit{Exact} & 0.96126 & 0.48063 & 0.32042 & -22.122 & -5.5306 & -2.4581 & -10.580 \\ \mathit{ForwDiff} & 0.950225 & 0.477871 & 0.319194 & -22.0183 & -5.5179 & -2.4539 & -10.5579 \\ \mathit{BackwDiff} & 0.972347 & 0.483402 & 0.321652 & -22.2280 & -5.5437 & -2.4624 & -10.6035 \\ \mathit{CentDiff} & 0.961286 & 0.480636 & 0.320423 & -22.122 & -5.530 & -2.458 & -10.5805 \end{bmatrix}$$

Section 11.2

11.2.3. (a): NumHeatForw$(c^2, F(x,t), t = t_0..t_f, f(x), x = a..b, dirichlet, g(t), dirichlet, s(t), n, k)$.
(b): NumHeatForw$(c^2, F(x,t), t = t_0..t_f, f(x), x = a..b, 0, -g(t), 0, s(t), n, k)$.
(c): NumHeatForw$(c^2, F(x,t), t = t_0..t_f, f(x), x = a..b, \lambda_1, g(t), \lambda_2, s(t), n, k)$.
(d): NumHeatForw$(c^2, F(x,t), t = t_0..t_f, f(x), x = a..b, dirichlet, g(t), 0, s(t), n, k)$.

11.2.5.

$$\begin{bmatrix} x & u\left(x, 0.5\right) \\ 0.0 & 0.0 \\ 0.10 & 0.002044630895 \\ 0.20 & 0.003889119061 \\ 0.30 & 0.00005352913177 \\ 0.40 & 0.006292726828 \\ 0.50 & 0.006616564564 \\ 0.60 & 0.006292726828 \\ 0.70 & 0.005352913177 \\ 0.80 & 0.003889119061 \\ 0.90 & 0.002044630895 \\ 1.0 & 0.0 \end{bmatrix}$$

11.2.6. NumHeatForw $\left(1, 0, t = 0..0.5, \sin\left(\pi\, x\right), x = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, 0, 10, \frac{1}{600}\right)$.

11.2.14. NumHeatLines$\left(1, 2\,tx^2 - 2\,t^2, t = 0, 0, x = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, t^2, 4, t = 1\right)$.

11.2.15. NumHeatLines $\left(1, 2\,tx^2 - 2\,t^2, t = 0, 0, x = 0..1, 0, 0, 0, 2\,t^2, 4, t = 1\right)$.

Section 11.3

11.3.5. (a): Euler's method:

$$\text{NumWaveLines}\left(1, 2\,x^2 - 2\,t^2, t = 0, 0, 0, x = 0..1, 1, 0, 1, 3\,t^2, 4, \mathit{method} = \mathit{classical}[\mathit{foreuler}]\right).$$

$$[t = 1.0, u_0\,(t) = -0.0012184, \frac{du_0(t)}{dt} = -0.0027066, u_1\,(t) = 0.061012, \frac{du_1(t)}{dt} = 0.12313,$$
$$u_2(t) = 0.24859, \frac{du_2(t)}{dt} = 0.50169, u_3(t) = 0.56153, \frac{du_3(t)}{dt} = 1.1287, u_4(t) = 0.99909, \frac{du_4(t)}{dt} = 2.0056].$$

Section 11.4

11.4.6. $\quad \text{HilbertMatrix}\,(3) = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$

ConditionNumber(HilbertMatrix(3),1) = 748 in the L_1 norm.

11.4.7.(a):

$$\text{NumLaplace}\left(0, x = 0..1, \mathit{dirichlet}, 1, \mathit{dirichlet}, 0, y = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, 0, 4, 4, 200, 0, .1\ 10^{-7}, \mathit{Jacobi}\right).$$

$$\begin{bmatrix} & x & 0.0 & 0.250 & 0.500 & 0.750 & 1.0 \\ y & & --- & --- & --- & --- & --- \\ 0.0 & | & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.250 & | & 1.0 & 0.429 & 0.187 & 0.0714 & 0.0 \\ 0.500 & | & 1.0 & 0.527 & 0.250 & 0.0982 & 0.0 \\ 0.750 & | & 1.0 & 0.429 & 0.187 & 0.0714 & 0.0 \\ 1.0 & | & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

Section 11.6

11.6.3. $\quad \text{LaplaceMatrix}(0, x = 0..1, \mathit{dirichlet}, 1, \mathit{dirichlet}, 0, y = 0..1, \mathit{dirichlet}, 1, \mathit{dirichlet}, 0, 4, 4).$

$$\text{Coefficient Matrix} = \begin{bmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{bmatrix}$$

LaplaceConvergence($CoeffMatrix, Gauss - Seidel, r$) : $||H||_1 = 0.8671875$, $||H||_2 = 0.610696954$, $||H||_\infty = 0.8125$, Spectral Radius $= 0.5$, Convergence Rate $= 0.301029996$.

OptimalSOR($CoeffMatrix$, 1.2, .001) : SpectralRadius = .20, Convergence Rate = .6989700043, Relaxation Parameter = 1.2.

Section 11.7

11.7.4. NumHypSysExplicit([[4]], [[−1]], $[8\,xt - 8\,x + 4 + x^2t - 4\,t], t = 0..1, [-x^2 + e^{-x/4} + 8], x = 0..1, 5, 30$, *Forward − Backward*).

$$\begin{bmatrix} x & u(x,1) \\ 0.0 & 4.806196998 \\ 0.20 & 4.756014649 \\ 0.40 & 4.708271086 \\ 0.60 & 4.662847371 \\ 0.80 & 4.619630361 \\ 1.0 & 4.578512432 \end{bmatrix}$$

Section 11.8

11.8.4. NumHyperbolicLinesSL $\left(1, 0, 1, 0, 0, t = 0, x^2, \sin(\pi x), x = 0 \ldots 1, dirichlet, 1, dirichlet, 0, 5, t = 15\right)$.

$$\begin{bmatrix} x & u(x,15) \\ 0.0 & 1.0 \\ 0.20 & 0.7998 \\ 0.40 & 0.6000 \\ 0.60 & 0.4004 \\ 0.80 & 0.2004 \\ 1.0 & 0.0 \end{bmatrix}$$

NumParabolicLinesSL$\left(1, 0, 0, 0, t = 0, x^2, x = 0..1, dirichlet, 1, dirichlet, 0, 5, t = 15\right)$.

$$\begin{bmatrix} x & u(x,15) \\ 0.0 & 1.0 \\ 0.20 & 0.8000 \\ 0.40 & 0.6000 \\ 0.60 & 0.4000 \\ 0.80 & 0.2000 \\ 1.0 & 0.0 \end{bmatrix}$$

11.8.6. NumHypSystChar($[[u_1{}^3, 0], [5, u_2]], [[1, 0], [0, 1]], [[3\,u_1 + 2\,x^7 e^{4\,t} - 2\,x^2 e^t], [-u_2 + 10\,xe^t - 3\,x^2 t^2 + 3\,x^5 + 2\,t + t^2 - x^3]], [x^2, -x^3], [u_1, u_2], [x, t], [1, 0], [1.2, 0], 25$).

$$\text{Number of Iterations} = 8$$

$$\begin{bmatrix} x & t & \mathit{u1}\,(x,t) & \mathit{u2}\,(x,t) \\ 1.1027 & 0.064050 & 1.3023 & -1.3087 \end{bmatrix}$$

11.8.8. NumQuasiHypSystCharBack ($[[0, -1], [-x, 0]], [[1, 0], [0, 1]], [xu_1{}^2 - 4\,x^2 t^2, u_2{}^2 - 4\,x^4 t^2 + 2\,x^2 - 2\,xt^2], [u_1, u_2], t = 0.2, x = 0.5, t = 0, [0, 0]$).

$$\begin{bmatrix} x & t & u_1 & u_2 & t_Q & x_{Q1} & x_{Q2} \\ 0.5 & 0.2 & 0.03687 & 0.1002 & 0.0 & 0.3686 & 0.6514 \end{bmatrix}$$

11.8.9. NumEllipticSL($2 + \cos\left(y^2\right), 1 + x^2 + y^2, 1, 0, 0, \left(2 + \cos\left(y^2\right)\right)\left(6\,xe^{xy} + 6\,x^2 ye^{xy} + x^3 y^2 e^{xy}\right) + \left(1 + x^2 + y^2\right) x^5 e^{xy} + 3\,x^2 e^{xy} + x^3 ye^{xy}, u, x = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, e^y, y = 0..1, \mathit{dirichlet}, x^3, \mathit{dirichlet}, x^3 e^x, 10, 10, 1000, 0, 0.00001, 1$).

Part of the output is, $hx = 1/10, N = 10, hy = 1/10, M = 10$, Max Iterations $=$ 1000, Relaxation parameter $= 1$, Gauss-Seidel Method.

Section 11.9

11.9.3. A stable case: NumHeatForw2d ($1, 0, t = 0..5.0, xy^2, x = 0..1, 0, 0, 0, 0, y = 0..1, 0, 0, 0, 0, 10, 10, 0.001$).

An unstable case: NumHeatForw2d($1, 0, t = 0..5.0, xy^2, x = 0..1, 0, 0, 0, 0, y = 0..1, 0, 0, 0, 0, 10, 10, 0.01$).

11.9.5. Implicit backward scheme: NumHeatBackw2d ($1, 0, t = 0..2.0, xy^2, x = 0..1, 0, 0, 0, 0, y = 0..1, 0, 0, 0, 0, 10, 10, 0.01, 1$).

Crank-Nicolson scheme: NumHeatBackw2d ($1, 0, t = 0..2.0, xy^2, x = 0..1, 0, 0, 0, 0, y = 0..1, 0, 0, 0, 0, 10, 10, 0.01, 1/2$).

11.9.6. Peaceman-Rachford scheme: NumHeatPRADI2d ($1, \cos(\pi\,x)\left(-y^2 + \pi^2 y^2 - 2\right) e^{-t}, t = 0..1.0, \cos(\pi\,x)\,y^2, x = 0..1, 0, 0, \mathit{dirichlet}, -y^2 e^{-t}, y = 0..1, 1, 0, \mathit{dirichlet}, \cos(\pi\,x)\,e^{-t}, 4, 4, 0.01$).

Douglas-Rachford scheme: NumHeatDRADI2d($1, \cos(\pi\,x)\left(-y^2 + \pi^2 y^2 - 2\right) e^{-t}, t = 0..1.0, \cos(\pi\,x)\,y^2, x = 0..1, 0, 0, \mathit{dirichlet}, -y^2 e^{-t}, y = 0..1, 1, 0, \mathit{dirichlet}, \cos(\pi\,x)\,e^{-t}, 4, 4, 0.01$).

11.9.9. Unstable case: NumWaveForw2d($1, 2x^2 y^2 - 2\,y^2 t^2 - 2\,x^2 t^2, t = 0..1, 0, 0, x = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, y^2 t^2, y = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, x^2 t^2, 4, 4, 0.25$).

Stable case: NumWaveForw2d ($1, 2\,x^2 y^2 - 2\,y^2 t^2 - 2\,x^2 t^2, t = 0..1, 0, 0, x = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, y^2 t^2, y = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, x^2 t^2, 4, 4, 0.005$).

11.9.12. NumHeatLines3d ($1, -\left(x^2 + 2\right) yze^{-t}, t = 0, x^2 yz, x = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, yze^{-t}, y = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, x^2 ze^{-t}, z = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, x^2 ye^{-t}, 4, 4, 4, t = 1$).

11.9.14. Gauss-Seidel: NumLaplace3d ($0, x = 0, , 1, \mathit{dirichlet}, 1, \mathit{dirichlet}, 0, y = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, 0, z = 0..1, \mathit{dirichlet}, 0, \mathit{dirichlet}, 0, 4, 4, 4, 100, 4, 0.0000002, 1$).

Chapter 12

Section 12.4

12.4.3. BVList = [[0,0],[1,0],[0,1]].
VList = PolyygonTriang($BVList$), VList = [[0, 0], [1, 0], [1/3, 1/3], [1, 0], [0, 1], [1/3, 1/3], [0, 1], [0, 0], [1/3, 1/3]].
VVList = Vertexlist (PolygonTriang($BVList$),$BVList$) , VVList = [[1/3, 1/3], [0, 0], [1, 0], [0, 1]].

12.4.4. NumEllipticFEMCM($PolygonTriang((BVList), Vertexlist(PolygonTriang(BVList), BVList)$, $1, xy, [0,0], xy(x{+}y), [x,y], x{+}y, BVList[1..2], \sqrt{2}, BVList[2..3], y{-}1, 1, [BVList[3], BVList[1]]$).

$$[[1/3, 1/3, 0.6328203782], [0, 0, 0], [1, 0, 1], [0, 1, 0.8989925186]].$$

12.4.8. BPList = [[0, 0], [1/4, 0], [1/2, 0], [3/4, 0], [1, 0], [1, 1/4], [1, 1/2], [1, 3/4], [1, 1], [3/4, 1], [1/2, 1], [1/4, 1], [0, 1], [0, 3/4], [0, 1/2], [0, 1/4]].
VertL = PolygonTriang(*BPList*) , VList = Vertexlist(*VertL*,*BPList*,*p*) ,
VSList2 = RefineTriang(*VertL*,*BPList*, 2), VList2 = Vertexlist(*MListMod*,*BPListMod*,*p*).

NumEllipticFEMCM (*VSList2*, *VList2*, $e^x, 1{+}x^2{+}y^2, [\sinh(x-y), 1/\left(1+x^4\right)], e^x \sin(xy)\, y + e^x \cos(xy)\, y^2 + e^x \cos(xy)\, x^2 - \sinh(x-y)\sin(xy)\, y - \sin(xy)\, x/(1+x^4) + \left(1+x^2+y^2\right)\cos(xy)$
, $[x, y], \cos(xy)$, *BPListMod*, *NONE*, [], *NONE*, *NONE*, []).

Section 12.5

12.5.4. BSL = [[0, 0], [1, 0], [1, 1], [0, 1]], VSL= *PolygonTriang* (*BSL*), VVSL = *Vertexlist* (*VSL*, *BSL*)].

NumParabolicFEMCM (*VSL*, *VVSL*, $1, 1, 0, [0,0], -(x+y)\,e^{-t}, [x,y], t=0, x{+}y, (x+y)\,e^{-t}$, *BSL*, *NONE*, [], *NONE*, *NONE*, [])

$$[[1/2, 1/2, c_1(t)], [0, 0, 0], [1, 0, e^{-t}], [1, 1, 2\,e^{-t}], [0, 1, e^{-t}]].$$

$$\text{SParFEMCM}(1) = [t = 1.0, c_1(t) = 0.3679].$$

12.5.5. NumParabolicFEMCM(*VSL*, *VVSL*, $1, 1, 0, [0,0], -(x+y)\,e^{-t}, [x,y], t=0, x{+}y$, *NONE*, [], $[-e^{-t}, e^{-t}, e^{-t}, -e^{-t}]$, [*BSL*[1..2], *BSL*[2..3], *BSL*[3..4], [*BSL*[4], *BSL*[1]]], *NONE*, *NONE*, []).

$$[[1/2, 1/2, c_1(t)], [0, 0, c_2(t)], [1, 0, c_3(t)], [1, 1, c_4(t)], [0, 1, c_5(t)]]).$$

SParFEMCM(0.5) = $[t = 0.5, c_1(t) = 0.6065, c_2(t) = -0.01830, c_3(t) = 0.6065, c_4(t) = 1.231, c_5(t) = 0.6065]$.

Section 12.6

12.6.3. BSL = [[0, 0], [1, 0], [1, 1], [0, 1]], VSL=PolygonTriang(*BSL*) ,VVSL=Vertexlist(*VSL*,*BSL*).

NumHyperbolicFEMCM(*VSL*, *VVSL*, $1, 1, 0, [0,0], (x+y)\,e^{-t}, [x,y], t=0, x+y, -x-y, (x+y)\,e^{-t}$, *BSL*, *NONE*, [], *NONE*, *NONE*, []).

Section 12.7

12.7.6. 37 Finite Element eigenvalues:

[$\lambda_1 = 51.67019988$, $\lambda_2 = 113.9650108$, $\lambda_3 = 144.8992399$, $\lambda_4 = 210.2683833$, $\lambda_5 = 255.8830049$, $\lambda_6 = 295.8409441$, $\lambda_7 = 372.4582517$, $\lambda_8 = 431.1191756$, $\lambda_9 = 456.2012635$, $\lambda_{10} = 518.6398986$, $\lambda_{11} = 536.1086420$, $\lambda_{12} = 687.3672655$, $\lambda_{13} = 707.0618019$, $\lambda_{14} = 761.7843666$, $\lambda_{15} = 846.1326733$, $\lambda_{16} = 963.0363071$, $\lambda_{17} = 1001.755450$, $\lambda_{18} = 1045.729021$, $\lambda_{19} = 1046.754610$, $\lambda_{20} = 1116.979917$, $\lambda_{21} = 1141.923527$, $\lambda_{22} = 1273.949205$, $\lambda_{23} = 1399.375427$, $\lambda_{24} = 1464.620907$, $\lambda_{25} = 1561.983932$, $\lambda_{26} = 1566.093911$, $\lambda_{27} = 1629.287500$, $\lambda_{28} = 1691.570093$, $\lambda_{29} = 1713.167135$, $\lambda_{30} = 2117.208564$, $\lambda_{31} = 2227.000970$, $\lambda_{32} = 2458.046626$, $\lambda_{33} = 2790.561809$, $\lambda_{34} = 2806.605985$, $\lambda_{35} = 3282.964567$, $\lambda_{36} = 3342.954327$, $\lambda_{37} = 3542.819149$].