GENERAL PRINCIPLES: FUNDAMENTAL LAWS AND EQUATIONS

A NUMBER OF THE FUNDAMENTAL "LAWS" OF CONTINUUM MECHANICS CAN BE EXPRESSED IN TERMS OF CONSERVATION OR BALANCE EQUATIONS.

1) CONSERVATION OF MASS
2) CONSERVATION OF LINEAR MOMENTUM (NEWTON'S 2ND LAW)
3) CONSERVATION OF ANGULAR MOMENTUM
4) CONSERVATION OF ENERGY (1ST LAW OF THERMODYNAMICS)

OF COURSE, THE CLAUSIS-DUHEM INEQUALITY (2ND LAW OF THERMODYNAMICS) PLACES RESTRICTIONS ON THE DIRECTION OF THE FLOW OF HEAT AND ENERGY.

IN ORDER TO PREFACE OUR DISCUSSION OF THE GENERAL PRINCIPLES OF CONTINUUM MECHANICS, IT IS USEFUL TO INTRODUCE THE NOTE OF THE MATERIAL TIME DERIVATIVE OF A VOLUME INTEGRAL

MATERIAL TIME DERIVATIVE OF A VOLUME INTEGRAL

⇨ IMPORTANT FOR AN EULERIAN DESCRIPTION EMPLOYING A CONTROL VOLUME

CONSIDER A VOLUME, V, IN THE CURRENT (DEFORMED) CONFIGURATION THAT CONTAINS A COLLECTION OF PARTICLES. CONSIDER AN ARBITRARY TENSOR, $\underset{\sim}{M}$, THAT IS ASSOCIATED WITH A PHYSICAL PROPERTY OF INTEREST. THEN

$$\frac{d}{dt}\int_V \underset{\sim}{M}\, dV = \int_V \frac{d}{dt}\left(\underset{\sim}{M}\, dV\right) = \int_V \left[\frac{d\underset{\sim}{M}}{dt}\cdot dV + \underset{\sim}{M}\,\frac{d}{dt}(dV)\right]$$

BUT $\frac{d}{dt}(dV) = d\dot{V} = \nabla \cdot \underset{\sim}{v}\, dV = v_{i,i}\, dV = D_{kk}\, dV$ (FROM BEFORE)

V = EULERIAN VOLUME

THEN $\underset{\sim}{v}$ = EULERIAN VELOCITY

$$\frac{d}{dt}\int_V \underset{\sim}{M}\, dV = \int_V \left[\frac{d}{dt}\underset{\sim}{M} + \underset{\sim}{M}\, \nabla \cdot \underset{\sim}{v}\right] dV = \int_V \left[\dot{\underset{\sim}{M}} + \underset{\sim}{M}\, \nabla \cdot \underset{\sim}{v}\right] dV$$

BUT $\frac{d}{dt}(\;) = \dot{(\;)} = \frac{\partial}{\partial t}(\;) + \underset{\sim}{v} \cdot \nabla (\;)$ EULERIAN

AND

$$\frac{d}{dt}\int_V \underset{\sim}{M}\, dV = \int_V \Big[\overbrace{\frac{\partial}{\partial t}\underset{\sim}{M} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{M}}^{\dot{\underset{\sim}{M}}} + \underset{\sim}{M}\, \nabla \cdot \underset{\sim}{v}\Big] dV = \int_V \left[\frac{\partial \underset{\sim}{M}}{\partial t} + \nabla \cdot (\underset{\sim}{v}\, \underset{\sim}{M})\right] dV$$

USING INDEX NOTATION

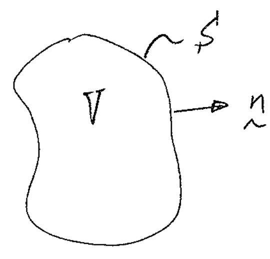
$$\frac{d}{dt}\int_V M_{ij\ldots}\, dV = \int_V \left[\frac{\partial}{\partial t} M_{ij\ldots} + v_p \frac{\partial}{\partial x_p} M_{ij\ldots} + v_{p,p} M_{ij\ldots}\right] dV$$

$$= \int_V \left[\frac{\partial}{\partial t} M_{ij\ldots} + \frac{\partial}{\partial x_p}\left(v_p M_{ij\ldots}\right)\right] dV$$

APPLY THE DIVERGENCE THEOREM TO THE 2ND TERM IN THE INTEGRAND.

RECALL

$$\underbrace{\int_V \nabla \cdot \underset{\sim}{A}\, dV}_{\text{VOLUME INTEGRAL IN } V} = \underbrace{\int_S \underset{\sim}{n} \cdot \underset{\sim}{A}\, dS}_{\text{SURFACE INTEGRAL ON } S}$$



OR

$$\int_V \frac{\partial}{\partial x_i} A_{ijkl\ldots}\, dV = \int_S n_i A_{ijkl\ldots}\, dS$$

$$\underbrace{\text{THEN}}_{\text{RATE OF INCREASE OF THE TOTAL AMOUNT OF } \underset{\sim}{M} \text{ INSIDE } \underset{\sim}{V}}$$

$$\boxed{\frac{d}{dt}\int_V M_{ij\ldots}\, dV = \int_V \frac{\partial}{\partial t} M_{ij\ldots}\, dV + \int_S n_i v_i M_{ij\ldots}\, dS}$$

OR

$$\boxed{\frac{d}{dt}\int_V \underset{\sim}{M}\, dV = \underbrace{\int_V \frac{\partial}{\partial t}\underset{\sim}{M}\, dV}_{} + \underbrace{\int_S (\underset{\sim}{n}\cdot\underset{\sim}{v})\underset{\sim}{M}\, dS}_{}}$$

GENERAL STATEMENT OF REYNOLDS TRANSPORT THEOREM

RATE OF INCREASE OF $\underset{\sim}{M}$ POSSESSED BY THE COLLECTION OF PARTICLES INSTANTANEOUSLY INSIDE V.
LOCAL CREATION / GENERATION OF $\underset{\sim}{M}$ IN V.

FLUX OF $\underset{\sim}{M}$ THROUGH THE SURFACE OF V.
CONVECTIVE FLUX TERM

CONSERVATION OF MASS (CONTINUITY EQUATION)

RECALL THE DEFINITION OF THE MASS DENSITY

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} = \frac{dm}{dV}$$

MASS DENSITY

m = MASS , V = VOLUME

THEN THE TOTAL MASS IN THE CONTINUUM IS GIVEN BY

$$m = \int_{V_0} \rho_0(\underset{\sim}{X}, t)\, dV_0 \qquad \text{(LAGRANGIAN)}$$

AND

$$m = \int_V \rho(\underset{\sim}{x}, t)\, dV \qquad \text{(EULERIAN)}$$

<u>CONSERVATION OF MASS</u>: THE MASS OF A BODY, OR ANY PORTION OF THE BODY, REMAINS CONSTANT IN EVERY CONFIGURATION (i.e., MASS IS INVARIANT UNDER MOTION).

USING AN EULERIAN DESCRIPTION, THIS SUGGESTS THAT $\frac{d}{dt} m = 0$

$$\frac{d}{dt} m = \frac{d}{dt}\int_V \rho \, dV = \int_V \Big[\frac{d\rho}{dt} dV + \rho \underbrace{\frac{d}{dt}(dV)}_{= v_{i,i}\, dV} \Big] = \int_V \Big[\frac{d}{dt}\rho + \rho \underset{\sim}{\nabla} \cdot \underset{\sim}{v} \Big] dV = 0$$

FOR ARBITRARY VOLUME V

$$\boxed{\begin{aligned} &\frac{d}{dt}\rho + \rho \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 \\ &\frac{d\rho}{dt} + \rho v_{i,i} = 0 \end{aligned}}$$

<u>CONTINUITY EQUATION</u>

(EULERIAN STATEMENT OF CONSERVATION OF MASS)

OR $\underbrace{\frac{\partial}{\partial t}\rho + \underset{\sim}{v} \cdot \underset{\sim}{\nabla}\rho}_{\frac{d\rho}{dt}} + \rho \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$

IF THE <u>DENSITY</u> OF INDIVIDUAL PARTICLES IS <u>CONSTANT</u>, THEN THE FLOW / MOTION IS <u>INCOMPRESSIBLE</u> AND $\overset{\circ}{\rho} = 0$

$$\cancelto{0}{\frac{d}{dt}\rho} + \rho \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0$$

$$\Rightarrow \boxed{\begin{aligned} &\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 \\ &v_{i,i} = 0 \end{aligned}}$$

<u>EULERIAN STATEMENT OF INCOMPRESSIBLE MOTION</u>

$\Rightarrow$ ISOCHORIC (VOLUME PRESERVING) FLOW

$\Rightarrow$ VELOCITY IS DIVERGENCE FREE

BASED UPON THE CONSERVATION OF MASS

$$m = \underbrace{\int_V \rho(\underset{\sim}{x}, t)\, dV}_{\text{EULERIAN}} = \underbrace{\int_{V_0} \rho_0(\underset{\sim}{X}, t)\, dV_0}_{\text{LAGRANGIAN}}$$

TOTAL MASS

ρ_0 = DENSITY IN β_0

ρ = DENSITY IN β

BUT $dV = J\, dV_0$

$$m = \int_{V_0} \rho(\underset{\sim}{X}, t)\, J\, dV_0 = \int_{V_0} \rho_0(\underset{\sim}{X}, t)\, dV_0$$

$$\Rightarrow \quad \int_{V_0} (\rho J - \rho_0)\, dV_0 = 0 \qquad \text{CONSERVATION OF MASS}$$

$$\boxed{\rho J = \rho_0}$$

LAGRANGIAN STATEMENT OF CONSERVATION OF MASS

AND

$$\boxed{J = 1}$$

LAGRANGIAN STATEMENT OF INCOMPRESSIBLE MOTION (ρ = CONSTANT)

$\Rightarrow$ ISOCHORIC (VOLUME PRESERVING) MOTION

REPEATING THE PREVIOUS RESULTS

$$\boxed{\begin{aligned} \dot{\rho} + \rho\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v} &= 0 \\ \dot{\rho} + \rho\, v_{i,i} &= 0 \end{aligned}}$$

EULERIAN STATEMENT OF CONSERVATION OF MASS

$$\boxed{\begin{aligned} \underset{\sim}{\nabla} \cdot \underset{\sim}{v} &= 0 \\ v_{i,i} &= 0 \end{aligned}}$$

EULERIAN STATEMENT OF INCOMPRESSIBLE MOTION $\left(\dot{\rho} = \frac{d}{dt}\rho = 0\right)$

ASIDE: CONSIDER THE INTEGRAL INVOLVING ARBITRARY TENSOR, $\underset{\sim}{M}$

SINCE $J\rho = \rho_0$

$$\frac{d}{dt}\int_V \rho \underset{\sim}{M}\, dV = \underbrace{\frac{d}{dt}\int_V \rho(\underset{\sim}{x},t)\, \underset{\sim}{M}(\underset{\sim}{x},t)\, dV}_{\text{EULERIAN}} = \underbrace{\frac{d}{dt}\int_{V_0} \rho(\underset{\sim}{X},t)\, \underset{\sim}{M}(\underset{\sim}{X},t)\, J\, dV_0}_{\text{LAGRANGIAN}}$$

$$\frac{d}{dt}() = \frac{\partial}{\partial t}() + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}() \qquad\qquad \frac{d}{dt}() = \frac{\partial}{\partial t}()$$

EVALUATE MATERIAL TIME DERIVATIVE USING LAGRANGIAN INTEGRAL NOTING $(\dot{dV_0}) = 0$

$$\frac{d}{dt}\int_V \rho \underset{\sim}{M}\, dV = \frac{d}{dt}\int_{V_0} \rho J \underset{\sim}{M}\, dV_0 = \int_{V_0}\left[\frac{d}{dt}(\rho J)\, \underset{\sim}{M} + \frac{d}{dt}\underset{\sim}{M}\, \rho J\right] dV_0$$

BUT $\rho J = \rho_0$ LAGRANGIAN FORM OF CONTINUITY EQUATION

$\therefore \quad \frac{d}{dt}(\rho J) = \frac{d}{dt}(\rho_0) = 0\,!$

THEN

$$\frac{d}{dt}\int_V \rho \underset{\sim}{M}\, dV = \int_{V_0} \rho \frac{d\underset{\sim}{M}}{dt} \underbrace{J\, dV_0}_{\equiv\, dV!} = \underbrace{\int_V \rho \dot{\underset{\sim}{M}}\, dV}_{\text{EULERIAN!}}$$

HENCE

$$\boxed{\frac{d}{dt}\int_V \rho \underset{\sim}{M}\, dV = \int_V \rho \frac{d\underset{\sim}{M}}{dt}\, dV}$$

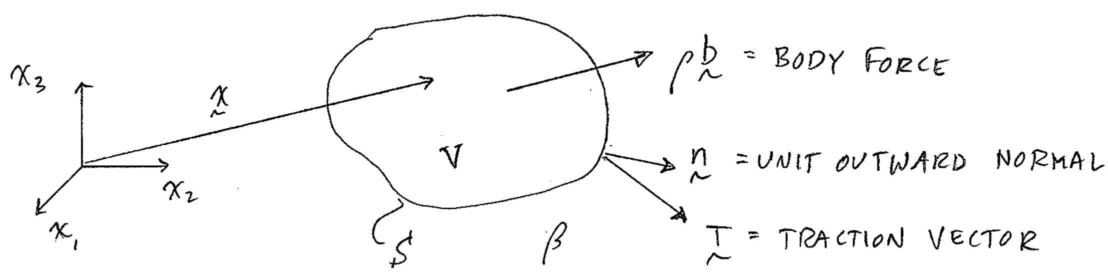
IDENTITY

$= 0$ FROM CONSERVATION OF MASS $(\dot{dm}) = 0$

OR $\quad \frac{d}{dt}\int_V \rho \underset{\sim}{M}\, dV = \int_V \left[\rho \dot{\underset{\sim}{M}}\, dV + \underset{\sim}{M}\, \cancel{(\dot{\rho\, dV})}\right] = \int_V \rho \dot{\underset{\sim}{M}}\, dV$

CONSERVATION OF LINEAR MOMENTUM (NEWTON'S 2^{ND} LAW): THE TIME RATE OF CHANGE OF LINEAR MOMENTUM FOR A BODY IS EQUAL TO THE RESULTANT FORCE ACTING ON THE BODY.

CONSIDER THE RESULTANT FORCES ACTING ON A FINITE VOLUME INSTANTANEOUSLY ENCLOSING A FIXED SET OF PARTICLES UNDER CONSIDERATION (EULERIAN)



$$\int_V \rho \underset{\sim}{v} \, dV = \text{LINEAR MOMENTUM VECTOR } (\sim m\underset{\sim}{v})$$

THEN BY CONSERVATION OF LINEAR MOMENTUM,

$$\frac{d}{dt} \int_V \rho \underset{\sim}{v} \, dV = \overbrace{\int_S \underset{\sim}{T} \, dS}^{\text{RESULTANT FORCE DUE TO SURFACE TRACTIONS ON } S} + \underbrace{\int_V \rho \underset{\sim}{b} \, dV}_{\text{RESULTANT FORCE DUE TO BODY FORCE IN V}}$$

BUT $\underset{\sim}{T} = \underset{\sim}{n} \cdot \underset{\sim}{\sigma}$ CAUCHY'S 1^{ST} LAW

WHERE $\underset{\sim}{\sigma}$ = CAUCHY STRESS TENSOR

$$\int_V \rho \dot{\underset{\sim}{v}} \, dV = \int_S \underset{\sim}{n} \cdot \underset{\sim}{\sigma} \, dS + \int_V \rho \underset{\sim}{b} \, dV \qquad \text{APPLY DIVERGENCE THEOREM}$$

$$\int_V \rho \dot{\underset{\sim}{v}} \, dV = \int_V \underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma} \, dV + \int_V \rho \underset{\sim}{b} \, dV$$

GLOBAL FORM OF BALANCE OF LINEAR MOMENTUM

SINCE $\underset{\sim}{v}$ IS ARBITRARY, THE INTEGRAND IN THE PRECEDING EQUATION MUST VANISH, i.e.,

$$\boxed{\underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma} + \rho \underset{\sim}{b} = \rho \mathring{\underset{\sim}{v}} = \rho \underset{\sim}{a}}$$

LOCAL FORM OF THE BALANCE OF LINEAR MOMENTUM (EQUATIONS OF MOTION)

OR

$$\boxed{\begin{aligned} &\sigma_{ji,j} + \rho b_i = \rho a_i = \rho \mathring{v}_i \\ &\frac{\partial \sigma_{ji}}{\partial x_j} + \rho b_i = \rho \frac{\partial v_i}{\partial t} + \rho v_j \frac{\partial v_i}{\partial x_j} \end{aligned}}$$

$\underset{\sim}{a}$ = EULERIAN ACCELERATION

$$\underset{\sim}{a} = \frac{d}{dt} \underset{\sim}{v} = \frac{\partial}{\partial t} \underset{\sim}{v} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} v$$

FOR THE STATIC CASE $\rho \mathring{v}_i = 0$

$$\sigma_{ji,j} + \rho b_i = 0$$

EQUILIBRIUM EQUATION (SAME AS BEFORE).

IMPORTANT: THE CAUCHY STRESS TENSOR, $\underset{\sim}{\sigma}$, DEFINES THE ACTUAL FORCE PER ACTUAL AREA IN THE DEFORMED CONFIGURATION, β.

⇨ $\underset{\sim}{\sigma}$ IS AN EULERIAN QUANTITY (TRUE STRESS)

⇨ IT IS POSSIBLE TO CONSTRUCT LAGRANGIAN MEASURES OF STRESS FOR MATHEMATICAL CONVENIENCE, ALTHOUGH SUCH MEASURES HAVE NO PHYSICAL INTERPRETATION

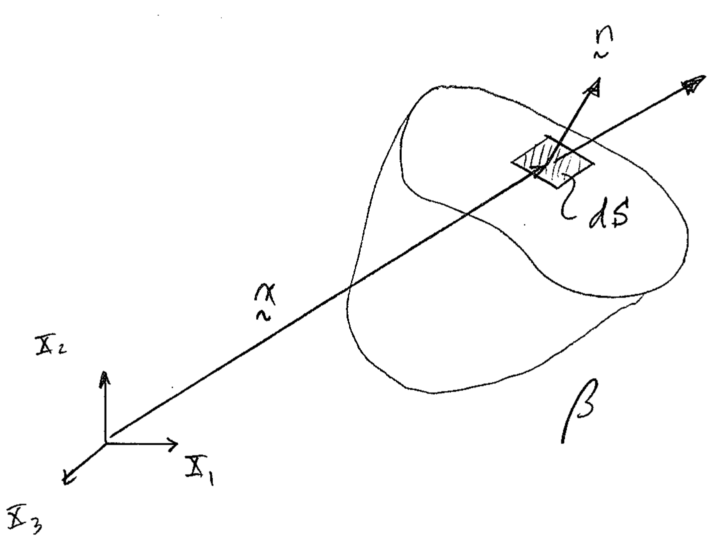
- EXAMPLES: 1^{ST} AND 2^{ND} PIOLA-KIRCHHOFF STRESS TENSORS

NOMINAL OR PIOLA KIRCHHOFF STRESSES (LAGRANGIAN)

THE CAUCHY STRESS TENSOR, $\underset{\sim}{\sigma}$, WAS DEFINED IN TERMS OF THE COMPONENTS OF TRACTION (ACTUAL FORCE PER UNIT ACTUAL AREA IN β) ACTING ON THREE MUTUALLY ORTHOGONAL PLANES IN THE DEFORMED CONFIGURATION, WHERE THE PLANES ARE PERPENDICULAR TO COORDINATE AXES.

IN ELASTICITY THEORY, A LAGRANGIAN (OR MATERIAL) FORMULATION IS GENERALLY PREFERRED SINCE IT IS OFTEN ASSUMED THAT THERE EXISTS A NATURAL STATE (i.e., REFERENCE CONFIGURATION) TO WHICH THE BODY RETURNS WHEN UNLOADED. AS A CONSEQUENCE, IF THE STATE OF STRAIN IS DEFINED RELATIVE TO THE NATURAL (UNDEFORMED) STATE OF THE MATERIAL THEN THE STATE OF STRESS MUST ALSO BE DEFINED WITH RESPECT TO THE REFERENCE CONFIGURATION IN ORDER TO DEFINE CONSTITUTIVE EQUATIONS AND EQUATIONS OF MOTION IN β_0.

CONSIDER THE DEFINITION OF THE TRACTION VECTOR $\underset{\sim}{T}$ IN β.



$d\underset{\sim}{f}$ = DIFFERENTIAL FORCE IN β

$d\$$ = DIFFERENTIAL AREA IN β

$\underset{\sim}{T} = \dfrac{d\underset{\sim}{f}}{d\$}$ TRACTION VECTOR IN β

WE WILL DEFINE LAGRANGIAN MEASURES OF STRESS / TRACTION IN TWO WAYS.

1) BY ASSUMING THE DIFFERENTIAL FORCE VECTOR, $d\underset{\sim}{f}$, ACTS (WITHOUT CHANGE) IN THE UNDEFORMED CONFIGURATION

$\Rightarrow$ FORMS THE BASIS FOR THE 1ST PIOLA-KIRCHHOFF STRESS TENSOR, $\underset{\sim}{\sigma}^{PK(1)}$

$\Rightarrow$ $\underset{\sim}{\sigma}^{PK(1)}$ IS UNSYMMETRIC! THIS POSES PROBLEMS WHEN POSTULATING CONSTITUTIVE EQUATIONS IN TERMS OF THE SYMMETRIC GREEN'S STRAIN, $\underset{\sim}{E}$.

2) BY ASSUMING THE DIFFERENTIAL FORCE VECTOR, $d\underset{\sim}{f}$, IN β IS ALTERED (i.e., INCREASES / DECREASES IN MAGNITUDE AND ROTATES) DURING THE DEFORMATION IN A MANNER CONSISTENT WITH ELEMENTAL VECTORS, i.e., $d\underset{\sim}{x} = \underset{\sim}{F} \cdot d\underset{\sim}{X} \Rightarrow d\underset{\sim}{f} = \underset{\sim}{F} \cdot d\underset{\sim}{f}^{0}$

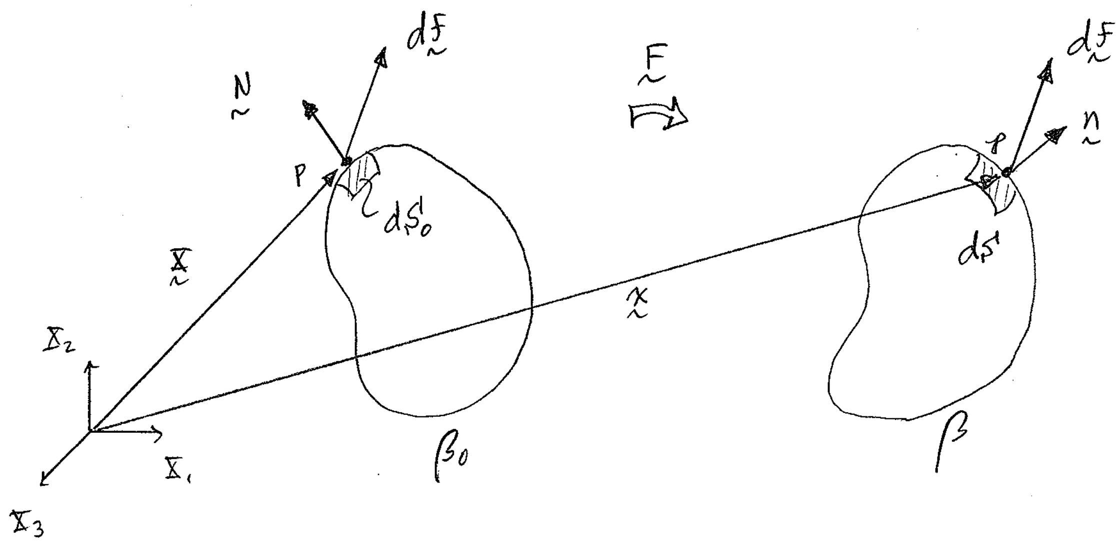
$\Rightarrow$ FORMS THE BASIS FOR THE 2ND PIOLA-KIRCHHOFF STRESS TENSOR, $\underset{\sim}{\sigma}^{PK(2)}$

$\Rightarrow$ $\underset{\sim}{\sigma}^{PK(2)}$ IS SYMMETRIC!

<u>IMPORTANT</u>: FOR FINITE DEFORMATION PROBLEMS, THE LAGRANGIAN STRESS MEASURES HAVE LITTLE PHYSICAL SIGNIFICANCE AND ARE INTRODUCED FOR MATHEMATICAL CONVENIENCE.

1^{st} PIOLA-KIRCHHOFF STRESS TENSOR

ASSUME THAT THE DIFFERENTIAL FORCE VECTOR, $d\underset{\sim}{F}$, DEFINED RELATIVE TO THE DEFORMED CONFIGURATION (β) ALSO ACTS IN THE UNDEFORMED (REFERENCE / MATERIAL) CONFIGURATION (β_0)



HERE: $\underset{\sim}{N}$ AND $\underset{\sim}{n}$ ARE THE UNIT NORMALS TO THE SURFACE AT POINT P IN β_0 AND β, RESPECTIVELY

dS'_0 AND dS' ARE THE DIFFERENTIAL ELEMENTS OF AREA IN β_0 AND β, RESPECTIVELY.

$\Rightarrow$ $d\underset{\sim}{F}$ IS THE DIFFERENTIAL FORCE ACTING IN BOTH β_0 AND β.

$d\underset{\sim}{F}$ REMAINS UNCHANGED DURING DEFORMATION

$$\overbrace{d\underset{\sim}{F}}^{\text{IN } \beta_0} = \overbrace{d\underset{\sim}{F}}^{\text{IN } \beta}$$

RECALL DEFINITION OF TRACTION $\quad \underset{\sim}{T} = \dfrac{d\underset{\sim}{F}}{dS'} \quad \Rightarrow \quad d\underset{\sim}{F} = \underset{\sim}{T}\, dS'$

THEN

$$\underbrace{d\tilde{F}}_{\substack{\text{DIFFERENTIAL} \\ \text{FORCE IN } \beta_0}} = \underbrace{\tilde{T}^{(1)}}_{\substack{\text{REFERENCE} \\ \text{TRACTION IN} \\ \beta_0}} \overbrace{dS_0}^{\substack{\text{ELEMENTAL} \\ \text{AREA IN } \beta_0}} = \underbrace{\tilde{T}}_{\substack{\text{TRACTION} \\ \text{VECTOR IN } \beta}} \overbrace{dS}^{\substack{\text{ELEMENTAL} \\ \text{AREA IN } \beta}} = \underbrace{d\tilde{f}}_{\substack{\text{DIFFERENTIAL} \\ \text{FORCE IN } \beta}}$$

<u>NOTING</u> <u>CAUCHY'S</u> <u>1ST</u> <u>LAW</u>: $\tilde{T} = \tilde{n} \cdot \tilde{\sigma}$

$$d\tilde{F} = \tilde{N} \cdot \tilde{\sigma}^{PK(1)} dS_0 = \tilde{n} \cdot \tilde{\sigma}\, dS$$

BUT $\tilde{n}\, dS = J \tilde{N} \cdot \tilde{F}^{-1} dS_0$!

$$\tilde{N} \cdot \tilde{\sigma}^{PK(1)} dS_0 = \tilde{N} \cdot (J \tilde{F}^{-1} \cdot \tilde{\sigma})\, dS_0$$

$$\boxed{\begin{aligned} \tilde{\sigma}^{PK(1)} &= J \tilde{F}^{-1} \cdot \tilde{\sigma} = |\tilde{F}|\, \tilde{F}^{-1} \cdot \tilde{\sigma} \\ \sigma_{ij}^{PK(1)} &= J \frac{\partial X_i}{\partial x_K} \sigma_{Kj} \end{aligned}}$$

<u>1ST</u> <u>PIOLA-KIRCHHOFF</u> <u>STRESS</u> (LAGRANGIAN)

LAGRANGIAN $\quad$ EULERIAN $\Rightarrow \tilde{\sigma}^{PK(1)}$ IS A TWO-POINT TENSOR

$\tilde{\sigma}^{PK(1)}$:

1) IS A TWO POINT TENSOR (DEPENDS ON BOTH LAGRANGIAN & EULERIAN COORDINATES)

2) IS A <u>LAGRANGIAN</u> MEASURE OF <u>ACTUAL</u> <u>FORCE</u> PER UNIT <u>UNDEFORMED</u> <u>AREA</u>

3) IS UNSYMMETRIC ($\tilde{\sigma}^{PK(1)}$ IS AWKWARD TO USE IN DEFINING CONSTITUTIVE EQUATIONS WITH A SYMMETRIC STRAIN TENSOR)

$\Rightarrow$ ILL SUITED FOR NUMERICAL SOLUTIONS TO INCREMENTAL PROBLEMS

$\Rightarrow$ REASONABLE FOR NUMERICAL SOLUTIONS TO NON-LINEAR ELASTIC PROBLEMS

4) HAS NO PHYSICAL SIGNIFICANCE

THE 1ST PIOLA-KIRCHHOFF STRESS MAY BE USED TO ENFORCE THE BALANCE OF LINEAR MOMENTUM IN THE REFERENCE CONFIGURATION, i.e.,

$$\frac{d}{dt}\int_{V_0} \rho_0 \underset{\sim}{v}(X,t)\, dV_0 = \frac{d}{dt}\int_{V_0} \rho_0 \dot{\underset{\sim}{u}}\, dV_0 = \int_{S_0} \underset{\sim}{N}\cdot\underset{\sim}{\sigma}^{PK(1)} dS_0 + \int_{V_0} \rho_0 \underset{\sim}{b}^0\, dV_0$$

WHERE

$$\underset{\sim}{T} = \underset{\sim}{N}\cdot\underset{\sim}{\sigma}^{PK(1)} = \text{LAGRANGIAN TRACTION}$$

$$\underset{\sim}{b}^0 = \text{BODY FORCE VECTOR IN } \beta_0$$

APPLYING THE DIVERGENCE THEOREM

$$\int_{V_0} \rho_0 \ddot{\underset{\sim}{u}}\, dV_0 = \int_{V_0} \underset{\sim}{\nabla}_X \cdot \underset{\sim}{\sigma}^{PK(1)} dV_0 + \int_{V_0} \rho_0 \underset{\sim}{b}^0\, dV_0$$

WHERE $\underset{\sim}{\nabla}_X = \frac{\partial}{\partial X_i}\underset{\sim}{e}_i$ LAGRANGIAN GRADIENT OPERATOR

FOR ARBITRARY REFERENCE VOLUME, V_0

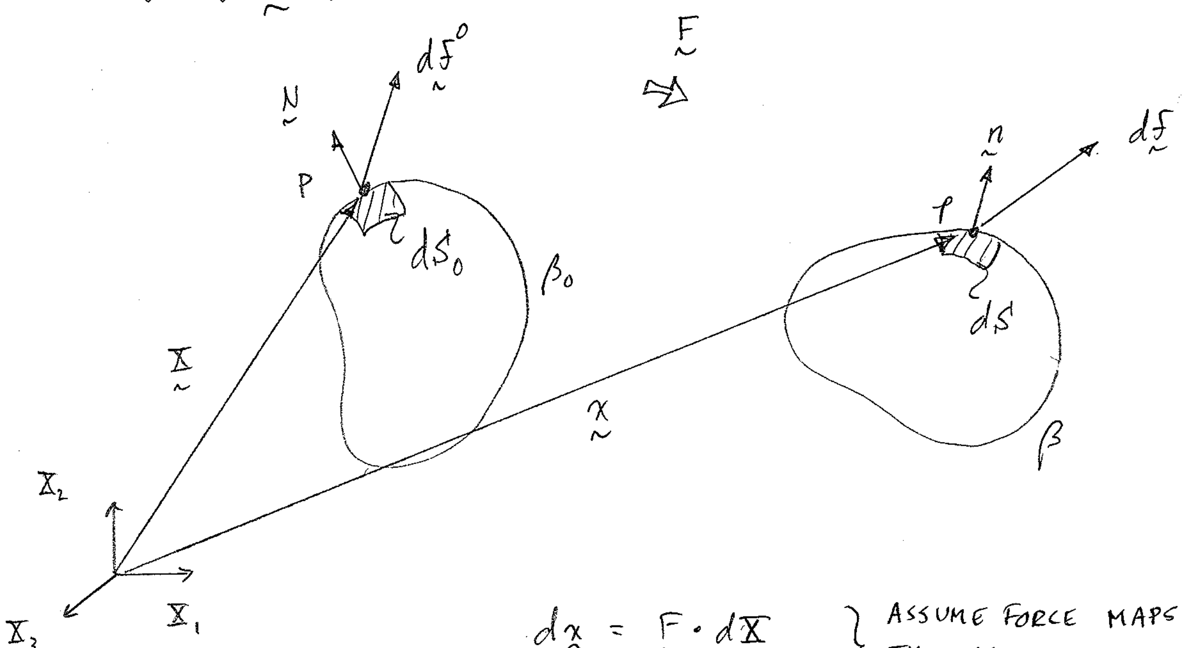
$$\boxed{\begin{aligned} &\underset{\sim}{\nabla}_X \cdot \underset{\sim}{\sigma}^{PK(1)} + \rho_0 \underset{\sim}{b}^0 = \rho_0 \ddot{\underset{\sim}{u}} = \rho_0 \underset{\sim}{a} \\ &\frac{\partial}{\partial X_j}\sigma_{ji}^{PK(1)} + \rho_0 b_i^0 = \rho_0 \ddot{u}_i = \rho_0 a_i \end{aligned}}$$

<u>LAGRANGIAN EQUATIONS OF MOTION</u>

THE LAGRANGIAN EXPRESSION FOR THE BALANCE OF LINEAR MOMENTUM IS OF THE SAME FORM AS IT'S EULERIAN COUNTERPART, BASED UPON $\underset{\sim}{\sigma}^{PK(1)}$. THIS WILL NOT HOLD TRUE WHEN CONSIDERING A BALANCE OF ANGULAR MOMENTUM SINCE $\sigma_{ij}^{PK(1)} \neq \sigma_{ji}^{PK(1)}$

2^{ND} PIOLA-KIRCHHOFF STRESS TENSOR

ASSUME THAT THE DIFFERENTIAL FORCE VECTOR, $d\underset{\sim}{f}$, DEFINED IN THE CURRENT CONFIGURATION (β) HAS CHANGED IN MAGNITUDE AND DIRECTION DURING THE DEFORMATION IN A MANNER CONSISTENT WITH THE MAPPING $d\underset{\sim}{x} = \underset{\sim}{F} \cdot d\underset{\sim}{X}$. HENCE, WE CAN DEFINE A DIFFERENTIAL "PSEUDO-FORCE" VECTOR, $d\underset{\sim}{f}^{o}$, BY MAPPING BACKWARDS FROM THE DEFORMED CONFIGURATION, β, TO THE REFERENCE STATE USING $\underset{\sim}{F}^{-1}$.



$$\left.\begin{aligned} d\underset{\sim}{x} &= \underset{\sim}{F} \cdot d\underset{\sim}{X} \\ d\underset{\sim}{f} &= \underset{\sim}{F} \cdot d\underset{\sim}{f}^{o} \end{aligned}\right\}$$ ASSUME FORCE MAPS INTO β IN THE SAME MANNER AS THE DEFORMATION OF MATERIAL LINE ELEMENTS

$\Rightarrow$ $d\underset{\sim}{f}^{o}$ IS "STRETCHED" AND ROTATED RELATIVE TO $d\underset{\sim}{f}$

$$d\underset{\sim}{f}^{o} = \underset{\sim}{F}^{-1} \cdot d\underset{\sim}{f} \qquad \text{INVERSE MAPPING}$$

BUT BASED UPON THE DEFINITION OF TRACTION IN THE REFERENCE (β_0) AND SPATIAL (β) CONFIGURATIONS

$$d\underset{\sim}{f}^{o} = \underset{\sim}{T}^{(2)} dS_0 = \underset{\sim}{F}^{-1} \cdot \underset{\sim}{T}\, dS = \underset{\sim}{F}^{-1} \cdot d\underset{\sim}{f}$$

HERE $\underset{\sim}{T}^{(2)}$ = REFERENCE TRACTION IN β_0

THEN APPLYING CAUCHY'S FIRST LAW ($\underset{\sim}{T} = \underset{\sim}{n} \cdot \underset{\sim}{\sigma}$ $IN\beta$)

$$d\underset{\sim}{f}^{0} = \underset{\sim}{N} \cdot \underset{\sim}{\sigma}^{PK(2)} dS_0 = \underset{\sim}{F}^{-1} \cdot \underset{\sim}{n} \cdot \underset{\sim}{\sigma}\, dS$$

BUT $\underset{\sim}{n}\, dS' = J \underset{\sim}{N} \cdot \underset{\sim}{F}^{-1} dS_0'$

$$dS_0 \underset{\sim}{N} \cdot \underset{\sim}{\sigma}^{PK(2)} = \underset{\sim}{F}^{-1} \cdot \underbrace{\left[J\, dS_0' \underset{\sim}{N} \cdot \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \right]}_{\text{VECTOR}} = J\, dS_0 \underset{\sim}{N} \cdot \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{F}^{-T}$$

THEN

$$\boxed{\begin{aligned} \underset{\sim}{\sigma}^{PK(2)} &= J \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{F}^{-T} \\ \sigma_{ij}^{PK(2)} &= J \frac{\partial X_i}{\partial x_m} \sigma_{mn} \frac{\partial X_j}{\partial x_m} \end{aligned}}$$

2ND PIOLA KIRCHHOFF STRESS (LAGRANGIAN) - SYMMETRIC

LAGRANGIAN, EULERIAN $\Rightarrow$ $\underset{\sim}{\sigma}^{PK(2)}$ IS A TWO-POINT TENSOR!

$\underset{\sim}{\sigma}^{PK(2)}$:

1) IS A TWO POINT TENSOR

2) IS A LAGRANGIAN STRESS MEASURE BASED UPON THE NOTION OF THE PSEUDO-FORCE VECTOR, $d\underset{\sim}{F}^{0} = \underset{\sim}{F}^{-1} \cdot d\underset{\sim}{f}$

3) IS SYMMETRIC!

$\Rightarrow$ BETTER SUITED FOR NUMERICAL SOLUTIONS AND CONSTITUTIVE LAW DEVELOPMENT

4) HAS NO PHYSICAL SIGNIFICANCE.

THE 1^{ST} AND 2^{ND} PIOLA KIRCHHOFF STRESS TENSORS ARE RELATED, i.e.

$$\boxed{\begin{gathered} \underset{\sim}{\sigma}^{PK(2)} = J\, \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{F}^{-T} = \underset{\sim}{\sigma}^{PK(1)} \cdot \underset{\sim}{F}^{-T} \\ \sigma_{ij}^{PK(2)} = \sigma_{im}^{PK(1)} \frac{\partial X_j}{\partial x_m} \end{gathered}}$$

RELATIONSHIP BETWEEN $\underset{\sim}{\sigma}^{PK(1)}$ AND $\underset{\sim}{\sigma}^{PK(2)}$

★ ALSO $\quad \underset{\sim}{\sigma}^{PK(1)} = \underset{\sim}{\sigma}^{PK(2)} \cdot \underset{\sim}{F}^{T}$

THE LATTER EXPRESSION MAY BE SUBSTITUTED INTO THE PREVIOUS LAGRANGIAN EXPRESSION FOR THE BALANCE OF LINEAR MOMENTUM INVOLVING $\underset{\sim}{\sigma}^{PK(1)}$, i.e.;

$$\underset{\sim}{\nabla}_X \cdot \underset{\sim}{\sigma}^{PK(1)} + \rho_0 \underset{\sim}{b}^{0} = \rho_0 \ddot{\underset{\sim}{u}}$$

EQUATION OF MOTION

BUT $\underset{\sim}{\sigma}^{PK(2)} = \underset{\sim}{\sigma}^{PK(1)} \cdot \underset{\sim}{F}^{-T} \Rightarrow \underset{\sim}{\sigma}^{PK(2)} \cdot \underset{\sim}{F}^{T} = \underset{\sim}{\sigma}^{PK(1)}$

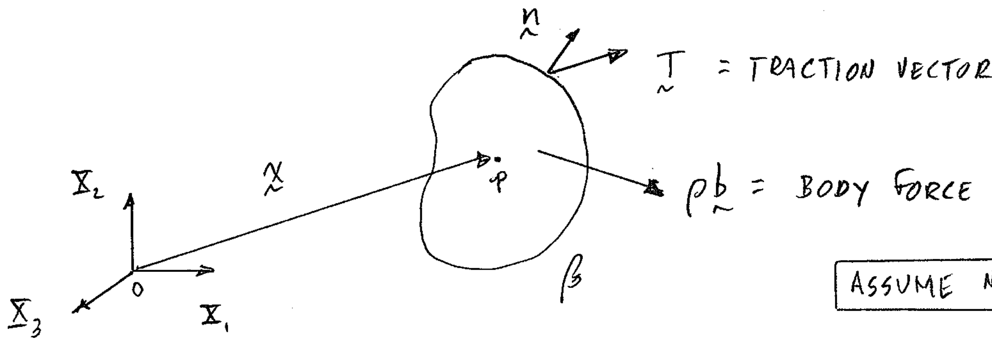
$$\boxed{\underset{\sim}{\nabla}_X \cdot \left[\underset{\sim}{\sigma}^{PK(2)} \cdot \underset{\sim}{F}^{T} \right] + \rho_0 \underset{\sim}{b}^{0} = \rho_0 \ddot{\underset{\sim}{u}}}$$

<u>BALANCE OF LINEAR MOMENTUM</u> IN TERMS OF $\underset{\sim}{\sigma}^{PK(2)}$

<u>NOTE</u>: SINCE $\underset{\sim}{\sigma}^{PK(2)}$ IS SYMMETRIC, IT IS MORE COMMONLY USED IN INCREMENTAL PROBLEMS THAN $\underset{\sim}{\sigma}^{PK(1)}$

BALANCE OF ANGULAR MOMENTUM (MOMENT OF MOMENTUM)

FROM CONSERVATION OF ANGULAR MOMENTUM, THE TIME RATE OF CHANGE OF THE MOMENT OF LINEAR MOMENTUM ABOUT SOME POINT IS EQUAL TO THE RESULTANT MOMENT ABOUT THAT POINT DUE TO SURFACE FORCES AND BODY FORCES ABOUT THE SAME POINT.



$\boxed{\text{ASSUME NO BODY MOMENT}}$

$$\underbrace{\frac{d}{dt}\underbrace{\int_V \underset{\sim}{x} \times \rho \underbrace{\underset{\sim}{v}\, dV}_{\sim\, \underset{\sim}{v}\, dm \,=\, \text{MOMENTUM}}}_{\text{MOMENT OF MOMENTUM}}}_{\text{TIME RATE OF CHANGE OF ANGULAR MOMENTUM}} = \underbrace{\int_V \underset{\sim}{x} \times \rho \underset{\sim}{b}\, dV}_{\text{MOMENT DUE TO BODY FORCE}} + \underbrace{\int_S \underset{\sim}{x} \times \underset{\sim}{T}\, dS}_{\text{MOMENT DUE TO SURFACE TRACTION}}$$

⇒ HERE MOMENT IS TAKEN WRT ORIGIN

USE CHAIN RULE IN EVALUATING $\frac{d}{dt}\int (\;)\, dV$

$$\int_V \underbrace{\frac{d\underset{\sim}{x}}{dt}}_{\underset{\sim}{v}} \cancel{\times \rho \underset{\sim}{v}}\, dV \;\; (\underset{\sim}{0} = \underset{\sim}{v} \times \underset{\sim}{v}) + \int_V \underset{\sim}{x} \times \underset{\sim}{v} \cancel{\frac{d}{dt}(\rho\, dV)} \;\; (0 \text{ CONSERVATION OF MASS}) + \int_V \underset{\sim}{x} \times \rho \frac{d\underset{\sim}{v}}{dt}\, dV =$$

$$\int_V \underset{\sim}{x} \times \rho \frac{d\underset{\sim}{v}}{dt}\, dV = \int_V \underset{\sim}{x} \times \rho \underset{\sim}{b}\, dV + \int_S \underset{\sim}{x} \times \overbrace{\underset{\sim}{n} \cdot \underset{\sim}{\sigma}}^{\underset{\sim}{T} = \underset{\sim}{n} \cdot \underset{\sim}{\sigma} \text{ CAUCHY FORMULA}}\, dS$$

<u>NOTE:</u> $\int_S \alpha \underset{\sim}{n} \cdot \underset{\sim}{A}\, dS = \int_V \underset{\sim}{n} \cdot (\alpha \underset{\sim}{A})\, dV$

USING INDICIAL NOTATION

$$\int_V \epsilon_{ijk} x_j \rho \dot{v}_k \, dV = \int_V \epsilon_{ijk} x_j \rho b_k \, dV + \int_S \epsilon_{ijk} x_j n_p \sigma_{pk} \, dS$$

$$'' \quad = \quad '' + \int_V \frac{\partial}{\partial x_p} \left(\epsilon_{ijk} x_j \sigma_{pk} \right) dV \qquad \text{USE CHAIN RULE}$$

$$\int_V \epsilon_{ijk} x_j \rho \dot{v}_k \, dV = \int_V \epsilon_{ijk} x_j \rho b_k \, dV + \int_V \epsilon_{ijk} x_j \sigma_{pk,p} \, dV + \int_V \epsilon_{ijk} \overbrace{x_{j,p}}^{\delta_{jp}} \sigma_{pk} \, dV$$

$$\therefore \int_V \epsilon_{ijk} x_j \underbrace{\left(\rho \dot{v}_k - \sigma_{pk,p} - \rho b_k \right)}_{\substack{=0 \text{ FROM BALANCE OF} \\ \text{LINEAR MOMENTUM}}} dV = \int_V \epsilon_{ijk} \sigma_{jk} \, dV = 0$$

FOR ARBITRARY VOLUME, V

$$\epsilon_{ijk} \sigma_{jk} = 0 \quad \Rightarrow \quad \boxed{\sigma_{jk} = \sigma_{kj}}$$

CAUCHY STRESS TENSOR IS SYMMETRIC (SAME RESULT AS BEFORE)

NOTE: THE MOMENT OF MOMENTUM EQUATIONS MAY ALSO BE EXPRESSED IN TERMS OF LAGRANGIAN INTEGRALS INVOLVING THE 1^{ST} PIOLA KIRCHHOFF STRESS TENSOR, $\underset{\sim}{\sigma}^{PK(1)}$, i.e.,

$$\frac{d}{dt} \int_{V_0} \underset{\sim}{x} \times \rho_0 \underset{\sim}{v} \, dV_0 = \int_{V_0} \underset{\sim}{x} \times \rho_0 \underset{\sim}{b}^0 \, dV_0 + \int_{S_0} \underset{\sim}{x} \times \underset{\sim}{T}^{(1)} \, dS_0$$

WHERE $\underset{\sim}{T}^{(1)} = \underset{\sim}{N} \cdot \underset{\sim}{\sigma}^{PK(1)}$

APPLICATION OF THE DIVERGENCE THEOREM $\int_{S_0} \underset{\sim}{N} \cdot \underset{\sim}{A} \, dV_0 = \int_{V_0} \underset{\sim}{\nabla}_X \cdot \underset{\sim}{A} \, dV_0$ AND USE OF INDICIAL NOTATION LEADS TO:

$$\int_{V_0} \epsilon_{ijk}\, x_j\, \rho_0 \dot{v}_k\, dV_0 = \int_{V_0} \epsilon_{ijk}\, x_j\, \rho_0\, b_k^{o}\, dV_0 + \int_{V_0} \frac{\partial}{\partial X_m}\left(\epsilon_{ijk}\, x_j\, \sigma_{mk}^{PK(1)}\right) dV_0$$

$$'' \quad = \quad '' + \int_{V_0} \epsilon_{ijk}\, x_j \frac{\partial}{\partial X_m} \sigma_{mk}^{PK(1)}\, dV_0 + \int_{V_0} \epsilon_{ijk} \frac{\partial x_j}{\partial X_m} \sigma_{mk}^{PK(1)}\, dV_0$$

OR

$$\int_{V_0} \epsilon_{ijk}\, x_j \underbrace{\left(\rho_0 \dot{v}_k - \frac{\partial}{\partial X_m} \sigma_{mk}^{PK(1)} - \rho_0 b_k^{o}\right)}_{=0 \text{ FROM CONSERVATION OF LINEAR MOMENTUM}} dV_0 = 0 = \int_{V_0} \epsilon_{ijk} \underbrace{\frac{\partial x_j}{\partial X_m}}_{F_{jm}} \sigma_{mk}^{PK(1)}\, dV_0$$

FOR ARBITRARY VOLUME V_0

$$\epsilon_{ijk} \frac{\partial x_j}{\partial X_m} \sigma_{mk}^{PK(1)} = 0$$

OR

$$\boxed{\frac{\partial x_j}{\partial X_m} \sigma_{mk}^{PK(1)} = \frac{\partial x_k}{\partial X_m} \sigma_{mj}^{PK(1)}}$$

<u>1^{ST} PIOLA KIRCHHOFF STRESS TENSOR IS NOT SYMMETRIC</u>

$$\underset{\sim}{F} \cdot \underset{\sim}{\sigma}^{PK(1)} = \left(\underset{\sim}{F} \cdot \underset{\sim}{\sigma}^{PK(1)}\right)^T$$

NOTE: $F_{ij} = \frac{\partial x_i}{\partial X_j}$ DEFORMATION GRADIENT TENSOR

BUT $\underset{\sim}{\sigma}^{PK(1)} = \underset{\sim}{\sigma}^{PK(2)} \cdot \underset{\sim}{F}^T$

THEN

$$\underset{\sim}{F} \cdot \left(\underset{\sim}{\sigma}^{PK(2)} \cdot \underset{\sim}{F}^T\right) = \left[\underset{\sim}{F} \cdot \left(\underset{\sim}{\sigma}^{PK(2)} \cdot \underset{\sim}{F}^T\right)\right]^T = \underset{\sim}{F} \cdot \underset{\sim}{\sigma}^{PK(2)^T} \cdot \underset{\sim}{F}^T$$

$$\boxed{\underset{\sim}{\sigma}^{PK(2)} = \left[\underset{\sim}{\sigma}^{PK(2)}\right]^T}$$

<u>2^{ND} PIOLA KIRCHHOFF STRESS TENSOR IS SYMMETRIC</u>

BALANCE OF LINEAR MOMENTUM

$$\boxed{\underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma} + \rho \underset{\sim}{b} = \rho \dot{\underset{\sim}{v}}}$$

EULERIAN

$$\underset{\sim}{\nabla}_{X} \cdot \underset{\sim}{\sigma}^{PK(1)} + \rho_0 \underset{\sim}{b}^{0} = \rho_0 \dot{\underset{\sim}{v}}$$

$$\underset{\sim}{\nabla}_{X} \cdot \left[\underset{\sim}{\sigma}^{PK(2)} \cdot \underset{\sim}{F}^{-T} \right] + \rho_0 \underset{\sim}{b}^{0} = \rho_0 \dot{\underset{\sim}{v}}$$

LAGRANGIAN

BALANCE OF ANGULAR MOMENTUM (ASSUMING NO BODY MOMENTS)

$$\boxed{\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^{T}}$$

EULERIAN

$$\underset{\sim}{F} \cdot \underset{\sim}{\sigma}^{PK(1)} = \left[\underset{\sim}{F} \cdot \underset{\sim}{\sigma}^{PK(1)} \right]^{T}$$

$$\underset{\sim}{\sigma}^{PK(2)} = \left[\underset{\sim}{\sigma}^{PK(2)} \right]^{T}$$

LAGRANGIAN

NOTES:

1) IN BOTH EULERIAN AND LAGRANGIAN COORDINATES, THE SAME PHYSICS IS DESCRIBED, NAMELY THE BEHAVIOR OF A FIXED SET OF PARTICLES (FIXED MASS) OCCUPYING THE VOLUME, V_0. IN LAGRANGIAN COORDINATES, IT IS ASSUMED THAT THE HISTORY OF MOTION OF THE PARTICLES IS KNOWN; IN EULERIAN COORDINATES THIS IS NOT KNOWN AND ONLY THE CHANGE WITH RESPECT TO A CONTROL VOLUME IS IMPORTANT.

2) UPDATED LAGRANGIAN APPROACHES (CURRENT CONFIGURATION = REFERENCE CONFIGURATION FOR THE NEXT TIME STEP) BASICALLY IMPLEMENT THE LAGRANGIAN APPROACH FOR THE MATERIAL TIME DERIVATIVE SINCE THE FLOW OF THE MATERIAL IS TRACKED THROUGHOUT THE HISTORY. THE CAUCHY STRESS IS USED BECAUSE THE GEOMETRY IS DEFORMED.

CONJUGATE STRESS AND STRAIN MEASURES (HILL)

FROM THE VIEWPOINT OF THERMODYNAMICS AND CONSISTENCY WITHIN THE ADOPTED FRAME OF REFERENCE (LAGRANGIAN / EULERIAN), THE STRESS AND STRAIN MEASURES USED MUST BE:

1) BASED ON THE SAME COORDINATES
2) A CONJUGATE PAIR (i.e., THE SCALAR PRODUCT OF STRESS AND STRAIN OR STRESS AND STRAIN RATE MUST BE THE SAME IN ALL CONFIGURATIONS)

DEFINE:

$$\boxed{\mathcal{S} = \int_V \underset{\sim}{\sigma} : \underset{\sim}{D}\, dV = \int_V \sigma_{ij} D_{ij}\, dV}$$

STRESS POWER (STRESS WORK)

$\underset{\sim}{\sigma}$ = CAUCHY STRESS TENSOR (EULERIAN)

$\underset{\sim}{D}$ = RATE OF DEFORMATION TENSOR (EULERIAN)

⇨ $\mathcal{S}$ DEFINES THE RATE OF WORK PERFORMED BY $\underset{\sim}{\sigma}$ IN V IN β.

⇨ $\underset{\sim}{\sigma}$ AND $\underset{\sim}{D}$ ARE A CONJUGATE PAIR IN THE CURRENT CONFIGURATION.

NOTE: $\underset{\sim}{\sigma} : \underset{\sim}{D} = \underset{\sim}{\sigma} : (\underset{\sim}{L} - \underset{\sim}{W}) = \underset{\sim}{\sigma} : \underset{\sim}{L}$ SINCE $\underbrace{\overbrace{\underset{\sim}{\sigma}}^{\text{SYMMETRIC}} : \underbrace{\underset{\sim}{W}}_{\text{ANTI-SYMMETRIC}} = 0}$

$\underset{\sim}{L} = \underset{\sim}{v}\overleftarrow{\underset{\sim}{\nabla}} = \underset{\sim}{D} + \underset{\sim}{W}$ = SPATIAL VELOCITY GRADIENT TENSOR

$\underset{\sim}{W}$ = VORTICITY TENSOR (ANTI-SYMMETRIC)

AND

$\underset{\sim}{L} = \underset{\sim}{v}\overleftarrow{\underset{\sim}{\nabla}} = \dot{\underset{\sim}{F}} \cdot \underset{\sim}{F}^{-1}$ WHERE $\underset{\sim}{F}$ = DEFORMATION GRADIENT TENSOR

THEN

$$\underset{\sim}{\sigma} : \underset{\sim}{D} = \underset{\sim}{\sigma} : \underset{\sim}{L} = \underbrace{tr}_{\text{"TRACE" MEANS CONTRACT ONCE MORE ON FREE INDICES}}\left(\underset{\sim}{\sigma} \cdot \underset{\sim}{L}\right) = tr\left(\underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{F}} \cdot \underset{\sim}{F}^{-1}\right) = \underbrace{tr\left(\underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{F}}\right)}_{\text{TENSOR MANIPULATION: CAN RE-ORDER INNER PRODUCTS WITHIN THE TRACE.}}$$

THEN

$$\underset{\sim}{\sigma} : \underset{\sim}{D} = \left(\underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma}\right) \cdot\cdot\, \dot{\underset{\sim}{F}}$$

<u>STRESS</u> <u>POWER</u> <u>PER</u> <u>UNIT</u> <u>VOLUME</u>

<u>RECALL</u>: $\underset{\sim}{A} \cdot\cdot \underset{\sim}{B} = A_{ij} B_{ji}$, $\underset{\sim}{A} : \underset{\sim}{B} = A_{ij} B_{ij}$ } SCALAR PRODUCTS

SUBSTITUTE THE PRECEDING EXPRESSION INTO THE INTEGRAL FOR THE STRESS POWER

$$S = \int_V \underset{\sim}{\sigma} : \underset{\sim}{D}\, dV = \underbrace{\int_V \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot\cdot\, \dot{\underset{\sim}{F}}\, dV}_{\text{EULERIAN}} = \underbrace{\int_{V_0} \overbrace{J \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma}}^{\underset{\sim}{\sigma}^{PK(1)}} \cdot\cdot\, \dot{\underset{\sim}{F}}\, dV_0}_{\text{LAGRANGIAN}} \qquad dV = J\, dV_0$$

$$S = \int_V \underset{\sim}{\sigma} : \underset{\sim}{D}\, dV = \int_{V_0} \underset{\sim}{\sigma}^{PK(1)} \cdot\cdot\, \dot{\underset{\sim}{F}}\, dV_0$$

$$S = \int_V \sigma_{ij} D_{ij}\, dV = \int_{V_0} \sigma_{ij}^{PK(1)}\, \dot{F}_{ji}\, dV_0$$

$\Rightarrow$ $\underset{\sim}{\sigma}^{PK(1)}$ AND $\dot{\underset{\sim}{F}}$ FORM A <u>WORK</u> <u>CONJUGATE</u> <u>PAIR</u> IN <u>LAGRANGIAN</u> <u>COORDINATES</u>

NOW CONSIDER THE 2^{ND} PIOLA KIRCHHOFF STRESS STRESS AND GREEN'S STRAIN RATE

$$\int_{V_0} \underset{\sim}{\sigma}^{PK(2)} : \dot{\underset{\sim}{E}}\, dV_0 = \int_{V_0} \underbrace{J\, \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{F}^{-T}}_{\underset{\sim}{\sigma}^{PK(2)}} : \underbrace{\underset{\sim}{F}^{T} \cdot \underset{\sim}{D} \cdot \underset{\sim}{F}}_{\dot{\underset{\sim}{E}}}\, dV_0$$

$$= \int_{V_0} tr\left(\underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot \underbrace{\underset{\sim}{F}^{-T} \cdot \underset{\sim}{F}^{T}}_{\underset{\sim}{I}} \cdot \underset{\sim}{D} \cdot \underset{\sim}{F}\right) \underbrace{J dV_0}_{dV!} \qquad \underline{\text{LAGRANGIAN}}$$

$$= \int_{V} tr\left(\underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{D} \cdot \underset{\sim}{F}\right) dV = \int_{V} tr\left(\underbrace{\underset{\sim}{F} \cdot \underset{\sim}{F}^{-1}}_{\underset{\sim}{I}} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{D}\right) dV \qquad \underline{\text{EULERIAN}}$$

$$\int_{V_0} \underset{\sim}{\sigma}^{PK(2)} : \dot{\underset{\sim}{E}}\, dV_0 = \int_{V} tr\left(\underset{\sim}{\sigma} \cdot \underset{\sim}{D}\right) dV = \int_{V} \underset{\sim}{\sigma} : D\, dV$$

$\Rightarrow$ $\underset{\sim}{\sigma}^{PK(2)}$ AND $\dot{\underset{\sim}{E}}$ FORM A <u>WORK</u> <u>CONJUGATE</u> <u>PAIR</u> IN <u>LAGRANGIAN</u> <u>COORDINATES</u>

REMINDER

$$\mathcal{S} = \int_{V} \underset{\sim}{\sigma} : \underset{\sim}{D}\, dV = \int_{V_0} \underset{\sim}{\sigma}^{PK(1)} \cdot\cdot\, \dot{\underset{\sim}{F}}\, dV_0 = \int_{V_0} \underset{\sim}{\sigma}^{PK(2)} : \dot{\underset{\sim}{E}}\, dV_0 \qquad \underline{\text{STRESS POWER}}$$

$$\mathcal{S} = \int_{V} \sigma_{ij} D_{ij}\, dV = \int_{V_0} \sigma_{ij}^{PK(1)} \dot{F}_{ji}\, dV_0 = \int_{V_0} \sigma_{ij}^{PK(2)} \dot{E}_{ij}\, dV_0$$

<u>CONJUGATE PAIRS</u>		<u>ALSO</u>
$\underset{\sim}{\sigma}, \underset{\sim}{D}$	<u>EULERIAN</u>	$\underset{\sim}{\sigma}, \underset{\sim}{e}$
$\underset{\sim}{\sigma}^{PK(1)}, \dot{\underset{\sim}{F}}$	<u>LAGRANGIAN</u>	$\underset{\sim}{\sigma}^{PK(1)}, ?$
$\underset{\sim}{\sigma}^{PK(2)}, \dot{\underset{\sim}{E}}$	<u>LAGRANGIAN</u>	$\underset{\sim}{\sigma}^{PK(2)}, \underset{\sim}{E}$

CONSERVATION OF ENERGY (1ST LAW OF THERMODYNAMICS)

⇨ ENERGY CAN BE NEITHER CREATED OR DESTROYED

FOR A CONTINUUM BODY OF VOLUME V, THE TOTAL ENERGY OF THE SYSTEM OF PARTICLES MAY BE EXPRESSED AS

$$E_{TOTAL} = K + U$$

WHERE

$$K = \frac{1}{2}\int_V \rho\, \underset{\sim}{v} \cdot \underset{\sim}{v}\, dV$$ KINETIC ENERGY OF A DEFORMABLE BODY

⇨ K IS THE MACROSCOPIC KINETIC ENERGY ASSOCIATED WITH THE MACROSCOPICALLY MEASURABLE VELOCITY

ρ = MASS DENSITY

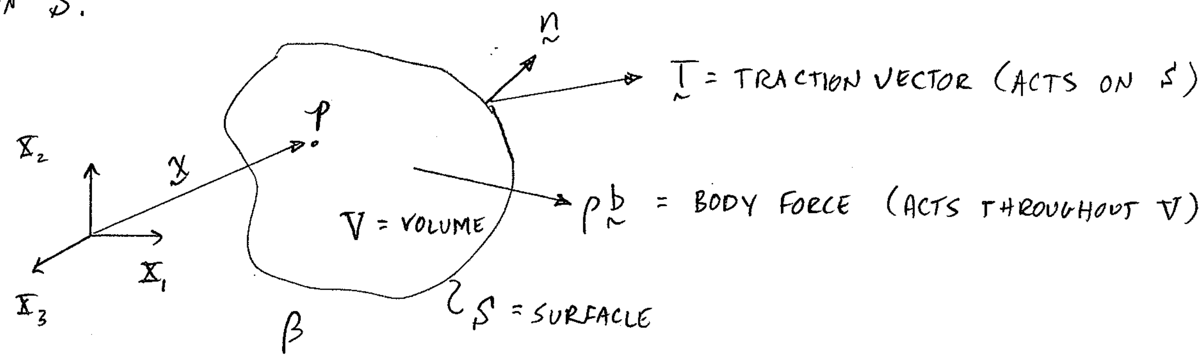
$\underset{\sim}{v}$ = EULERIAN VELOCITY

$$U = \int_V \rho\, u\, dV$$ INTERNAL ENERGY

⇨ U IS A MEASURE OF THE AVERAGE ENERGY STATE. THE INTERNAL ENERGY INCLUDES THE STORED ELASTIC ENERGY, UNRECOVERABLE ENERGY DUE TO DISSIPATION, AND POSSIBLY OTHER FORMS OF ENERGY NOT EXPLICITY SPECIFIED. THE KINETIC ENERGY OF RANDOM THERMAL MOTIONS OF MOLECULES, ASSOCIATED WITH TEMPERATURE CHANGES, ARE INCLUDED IN U.

u = INTERNAL ENERGY PER UNIT MASS OR "SPECIFIC" INTERNAL ENERGY.

CONSIDER THE RATE OF WORK OF THE BODY FORCES IN V AND SURFACE TRACTIONS ON S.



DEFINE MECHANICAL WORK RATE OR MECHANICAL POWER, P

$$P = \underbrace{\int_V \rho \underset{\sim}{b} \cdot \underset{\sim}{v}\, dV}_{\text{RATE OF WORK DUE TO BODY FORCE}} + \underbrace{\int_S \underset{\sim}{T} \cdot \underset{\sim}{v}\, dS}_{\text{RATE OF WORK DUE TO SURFACE TRACTIONS}}$$

MECHANICAL POWER

⇨ RATE OF WORK DONE ON THE BODY BY BODY FORCE AND SURFACE TRACTION

⇨ RATE OF MECHANICAL ENERGY SUPPLIED TO THE BODY

APPLY DIVERGENCE THEOREM

$$P = \int_V \rho b_i v_i\, dV + \int_S n_j \sigma_{ji} v_i\, dS = \int_V \left(\rho b_i v_i + \frac{\partial}{\partial x_j}(\sigma_{ji} v_i)\right) dV$$

$$P = \int_V \Big[v_i \underbrace{(\rho b_i + \sigma_{ji,j})}_{\equiv \rho \dot{v}_i \text{ FROM BALANCE OF LINEAR MOMENTUM}} + \sigma_{ji} v_{i,j}\Big] dV = \underbrace{\int_V \rho v_i \dot{v}_i\, dV}_{= \frac{1}{2}\frac{d}{dt}\int_V \rho \underset{\sim}{v}\cdot\underset{\sim}{v}\, dV} + \int_V \sigma_{ji} \underbrace{v_{i,j}}_{\underset{\sim}{L}} dV$$

$\frac{d}{dt}$ (KINETIC ENERGY)

$\underset{\sim}{L}$ = SPATIAL VELOCITY GRADIENT!

$$P = \overbrace{\frac{d}{dt}\int_V \rho v_i v_i\, dV}^{\dot{K}} + \overbrace{\int_V \sigma_{ji} D_{ji}\, dV}^{S = \underset{\sim}{\sigma} : \underset{\sim}{D} \text{ STRESS POWER}}$$

VORTICITY TENSOR

TRUE SINCE $\underset{\sim}{\sigma} \cdot\cdot \underset{\sim}{L} = \underset{\sim}{\sigma}^T : \underset{\sim}{L} = \underset{\sim}{\sigma} : \underset{\sim}{L} = \underset{\sim}{\sigma} : (\underset{\sim}{D} + \underset{\sim}{W}) = \underset{\sim}{\sigma} : \underset{\sim}{D}$

$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ RATE OF DEFORMATION TENSOR

THEN

$$P = \int_V \rho \underset{\sim}{b} \cdot \underset{\sim}{v} \, dV + \int_S \underset{\sim}{n} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{v} \, dS$$ MECHANICAL POWER

$$= \underbrace{\frac{1}{2} \frac{d}{dt} \int_V \rho \underset{\sim}{v} \cdot \underset{\sim}{v} \, dV}_{} + \underbrace{\int_V \underset{\sim}{\sigma} : \underset{\sim}{D} \, dV}_{} = \dot{K} + \mathcal{S}$$

↑ STRESS POWER!

A FRACTION OF THE POWER SUPPLIED BY SURFACE AND BODY FORCES RESULTS IN A CHANGE IN KINETIC ENERGY

A FRACTION OF THE MECHANICAL POWER CONTRIBUTES TO A CHANGE OF INTERNAL ENERGY!

NOW CONSIDER THE RATE OF THERMAL ENERGY ADDED TO THE BODY. FOR A THERMOMECHANICAL CONTINUUM, THIS CAN BE EXPRESSED IN TERMS OF THE HEAT INPUT RATE, Q.

$$Q = \int_V \rho r \, dV - \int_S \underset{\sim}{q} \cdot \underset{\sim}{n} \, dS$$

HEAT INPUT RATE

(HEAT FLOW INTO THE MASS CONTAINED INSTANTANEOUSLY IN V)

⇨ RATE OF THERMAL ENERGY TRANSMITTED TO V.

$$= \underbrace{\int_V \rho r \, dV}_{} - \underbrace{\int_S q_i n_i \, dS}_{}$$

$\left(\begin{array}{l}\text{RATE OF HEAT} \\ \text{GENERATION IN } V\end{array}\right) - \left(\begin{array}{l}\text{RATE OF HEAT OUTFLOW} \\ \text{THROUGH SURFACE } S \text{ OF } V\end{array}\right)$

POSSIBLY DUE TO RADIATION, MOLECULAR FRICTION

HEAT FLUX DUE TO CONDUCTION OUT OF V THROUGH SURFACE, S

r = SPECIFIC HEAT GENERATION = HEAT GENERATION PER UNIT MASS

⇨ DISTRIBUTED INTERNAL HEAT SOURCE

$\underset{\sim}{q}$ = HEAT FLUX VECTOR = MEASURE OF THE HEAT CONDUCTION PER UNIT AREA PER UNIT TIME THROUGH SURFACE dS WITH UNIT OUTWARD NORMAL, $\underset{\sim}{n}$

⇨ $\underset{\sim}{q}$ IS (+) OUT OF BODY; NOTE (−) SIGN IN EXPRESSION FOR Q.

OF COURSE, ONE MAY EXPRESS HEAT FLUX INTEGRAL IN TERMS OF AN EQUIVALENT VOLUME INTEGRAL USING THE DIVERGENCE THEOREM

$$Q = \int_V \rho r \, dV - \int_S \underset{\sim}{q} \cdot \underset{\sim}{n} \, dS \qquad \underline{\text{HEAT INPUT RATE}}$$

$$Q = \int_V (\rho r - \underset{\sim}{\nabla} \cdot q) \, dV$$

$$= \int_V (\rho r - q_{i,i}) \, dV$$

<u>ASIDE:</u> THE HEAT FLUX VECTOR IS OFTEN ASSUMED TO OBEY FOURIER'S LAW OF HEAT CONDUCTION, i.e.,

$$\boxed{\begin{aligned} \underset{\sim}{q} &= -\mathcal{K} \underset{\sim}{\nabla} \theta \\ q_i &= -\mathcal{K} \frac{\partial}{\partial x_i} \theta = -\mathcal{K} \theta_{,i} \end{aligned}}$$

<u>FOURIER'S LAW OF HEAT CONDUCTION</u> (HEAT FLUX THROUGH A SURFACE IS PROPORTIONAL TO THE TEMPERATURE GRADIENT)

HERE

$\mathcal{K}$ = THERMAL CONDUCTIVITY (ASSUMES ISOTROPIC CONTINUUM BEHAVIOR)

θ = TEMPERATURE

FOURIER'S "LAW" IS A <u>CONSTITUTIVE EQUATION</u> RELATING HEAT FLUX AND TEMPERATURE

⇨ NOT VALID FOR ALL MATERIALS

<u>1^{st} LAW OF THERMODYNAMICS</u> (<u>CONSERVATION OF ENERGY</u>): THE RATE OF CHANGE OF THE TOTAL ENERGY ASSOCIATED WITH THE MASS INSTANTANEOUSLY CONTAINED IN THE VOLUME V MUST BE EQUAL TO THE RATE OF MECHANICAL WORK PERFORMED BY THE BODY AND SURFACE FORCES PLUS THE RATE OF ENERGY TRANSFER INTO V FROM OTHER SOURCES (THERMAL, ELECTRICAL, MAGNETIC, ETC.)

NAMELY

$$\underbrace{\frac{d}{dt}\left(E_{TOTAL}\right)}_{\substack{\text{RATE OF CHANGE}\\\text{OF TOTAL ENERGY}\\\text{IN VOLUME, } V}} = \underbrace{P}_{\substack{\text{MECHANICAL POWER}\\\text{(WORK RATE) DUE}\\\text{TO } \rho \underset{\sim}{b} \text{ AND } \underset{\sim}{T}}} + \underbrace{\mathcal{Q}}_{\substack{\text{HEAT INPUT}\\\text{RATE DUE TO}\\\text{CONDUCTION AND}\\\text{HEAT GENERATION}}} + \underbrace{\dot{E}_{OTHER}}_{\substack{\text{RATE OF MAGNETIC,}\\\text{ELECTRICAL, CHEMICAL,}\\\text{OR OTHER ENERGY}}}$$

NEGLECTING $\dot{E}_{OTHER}$ AND NOTING $E_{TOTAL} = K + U$

$$\frac{d}{dt}\left(K + U\right) = P + \mathcal{Q}$$

<u>1^{ST} LAW OF THERMODYNAMICS</u>

(THERMOMECHANICAL CONTINUUM)

K = KINETIC ENERGY

U = INTERNAL ENERGY

OR

$$\overbrace{\frac{1}{2}\frac{d}{dt}\int_V \rho \underset{\sim}{v}\cdot\underset{\sim}{v}\, dV}^{\dot{K}} + \overbrace{\frac{d}{dt}\int_V \rho u \, dV}^{\dot{U}} =$$

$$= \underbrace{\int_V \rho \underset{\sim}{b}\cdot\underset{\sim}{v}\, dV + \int_S \underset{\sim}{n}\cdot\underset{\sim}{\sigma}\cdot\underset{\sim}{v}\, dS'}_{P} + \underbrace{\int_V \rho r \, dV - \int_S \underset{\sim}{q}\cdot\underset{\sim}{n}\, dS'}_{\mathcal{Q}}$$

BUT

$$P = \frac{1}{2}\frac{d}{dt}\int_V \rho \underset{\sim}{v}\cdot\underset{\sim}{v}\, dV + \int_V \underset{\sim}{\sigma} : \underset{\sim}{D}\, dV = \dot{K} + S$$

$$\mathcal{Q} = \int_V \left(\rho r - \underset{\sim}{\nabla}\cdot\underset{\sim}{q}\right) dV$$

HENCE,

$$\dot{K} + \dot{U} = P + Q = \underbrace{\dot{K} + \mathcal{S}}_{} + Q$$

P = MECHANICAL WORK RATE LEADS TO A CHANGE IN KINETIC ENERGY AND CHANGE IN INTERNAL ENERGY

OR

$$\boxed{\dot{U} = \mathcal{S} + Q}$$

<u>1ST LAW OF THERMODYNAMICS</u>

$$\frac{d}{dt}\int_V \rho u \, dV = \int_V \rho \frac{du}{dt} dV = \int_V \underset{\sim}{\sigma} : \underset{\sim}{D} \, d\underset{\sim}{V} + \int_V \left(\rho r - \underset{\sim}{\nabla} \cdot \underset{\sim}{q}\right) dV$$

SINCE $\frac{d}{dt}(\rho \, dV) = 0$
FROM CONSERVATION OF MASS

FOR ARBITRARY VOLUME, V, THE 1ST LAW MAY BE EXPRESSED IN LOCAL FORM

$$\boxed{\begin{aligned} \rho \frac{du}{dt} &= \underset{\sim}{\sigma} : \underset{\sim}{D} + \rho r - \underset{\sim}{\nabla} \cdot \underset{\sim}{q} \\ \rho \frac{du}{dt} &= \sigma_{ij} D_{ij} + \rho r - q_{i,i} \end{aligned}}$$

<u>ENERGY EQUATION</u>
(1ST LAW OF THERMODYNAMICS)
(LOCAL FORM) (EULERIAN)

$$\begin{pmatrix}\text{RATE OF CHANGE OF}\\ \text{INTERNAL ENERGY}\end{pmatrix} = \underbrace{\begin{pmatrix}\text{STRESS POWER: RATE OF}\\ \text{WORK OF INTERNAL STRESSES}\end{pmatrix}}_{\substack{\text{PART OF MECHANICAL POWER}\\ \text{NOT CONVERTED INTO KINETIC}\\ \text{ENERGY}}} + \underbrace{\begin{pmatrix}\text{RATE OF ADDITION}\\ \text{OF HEAT}\end{pmatrix}}_{\substack{\text{INTERNAL RATE OF HEAT}\\ \text{GENERATION MINUS HEAT}\\ \text{FLUX OUT OF BODY}}}$$

ENERGY EQUATION: $\boxed{\rho \dot{u} = \underset{\sim}{T} : \underset{\sim}{D} + \rho r - \underset{\sim}{\nabla} \cdot \underset{\sim}{q}}$

IMPORTANT NOTES:

1) THE 1^{ST} LAW OF THERMODYNAMICS CAN BE REGARDED AS AN EXPRESSION OF THE INTERCONVERTIBILITY OF HEAT AND WORK, MAINTAINING AN ENERGY BALANCE. IT PLACES NO RESTRICTIONS ON THE DIRECTION OF THE PROCESS.

2) IN IDEAL ELASTICITY, HEAT TRANSFER IS CONSIDERED INSIGNIFICANT. ALL OF THE INPUT MECHANICAL WORK IS CONVERTED INTO INTERNAL ENERGY IN THE FORM OF RECOVERABLE STORED ELASTIC STRAIN ENERGY, WHICH CAN BE RECOVERED AS WORK WHEN THE BODY IS UNLOADED. IN GENERAL, HOWEVER, A MAJOR PART OF THE INPUT WORK INTO A DEFORMING MATERIAL IS NOT RECOVERABLY STORED, BUT DISSIPATED BY THE DEFORMATION PROCESS, CAUSING AN INCREASE IN THE BODY'S TEMPERATURE AND EVENTUALLY BEING CONDUCTED AWAY AS HEAT.

3) IN THE ABSENSE OF THERMAL PHENOMENON, KINETIC ENERGY AND POTENTIAL ENERGY MAY BE FULLY TRANSFORMED FROM ONE TO THE OTHER PROVIDED THERE IS NO FRICTION OR OTHER DISSIPATIVE MECHANISM (EXAMPLE: SWINGING PENDULUM). WHEN THERMAL PHENOMENA ARE INVOLVED, HOWEVER, MECHANICAL WORK MAY BE CONVERTED INTO HEAT, BUT NOT VICE-VERSA. FOR EXAMPLE, THE KINETIC ENERGY OF A FLYWHEEL CAN BE CONVERTED INTO INTERNAL ENERGY (NON-RECOVERABLE) BY MEANS OF A FRICTION BRAKE. IF THE ENTIRE SYSTEM IS INSULATED, THE INTERNAL ENERGY STAYS IN THE SYSTEM CAUSING IT'S TEMPERATURE TO RISE. ACCORDING TO THE 1^{ST} LAW OF THERMODYNAMICS, THIS PROCESS COULD BE REVERSED (THE FLYWHEEL COULD BE SET INTO MOTION ...

NOTES: (CONTINUED)

... BY CONVERTING INTERNAL ENERGY INTO KINETIC ENERGY, WHILE THE <u>TEMPERATURE</u> <u>DECREASED</u>.). OF COURSE, SUCH A REVERSAL <u>NEVER</u> OCCURS IN NATURE; THE FRICTIONAL DISSIPATION IS AN <u>IRREVERSIBLE</u> <u>PROCESS</u>

4) THE <u>2ND</u> <u>LAW</u> <u>OF</u> <u>THERMODYNAMICS</u> (CLAUSIUS-DUHEM INEQUALITY) PLACES RESTRICTIONS ON THE DIRECTION OF INTERCONVERTIBILITY BETWEEN WORK AND HEAT AND OTHER IRREVERSIBLE PROCESSES.

ENTROPY AND THE 2^{ND} LAW OF THERMODYNAMICS (CLAUSIUS-DUHEM INEQUALITY)

AS NOTED PREVIOUSLY, ENERGY CAN BE NEITHER CREATED NOR DESTROYED; IT CAN ONLY BE TRANSFORMED FROM ONE FORM TO ANOTHER. WHEN SUCH A TRANSFORMATION OCCURS, THE ENERGIES INVOLVED MUST OBEY THE 1^{ST} LAW OF THERMODYNAMICS, i.e.,

$$\rho \dot{u} = \frac{du}{dt} = \underset{\sim}{\sigma} : \underset{\sim}{D} + \rho r - q_{i,i}$$

1^{ST} LAW OF THERMODYNAMICS

u = INTERNAL ENERGY

WHILE SOME ENERGIES READILY TRANSFORM FROM ONE TYPE TO ANOTHER (KINETIC ENERGY $\xRightarrow{\text{FRICTION}}$ HEAT), THERE ARE OTHER TYPES OF TRANSFORMATIONS THAT ARE IMPOSSIBLE (COLD SYSTEM $\xRightarrow[\text{FLOW}]{\text{HEAT}}$ WARM SYSTEM).

THE 2^{ND} LAW OF THERMODYNAMICS LIMITS THE DIRECTION OF ENERGY TRANSFORMATION. THE 2^{ND} LAW POSTULATES THE EXISTENCE OF A STATE FUNCTION CALLED ENTROPY THAT EXPRESSES THE RATIO OF THE HEAT INPUT DURING A REVERSIBLE PROCESS TO THE TEMPERATURE

$$d\eta = \left(\frac{đh}{\theta}\right)_{REV}$$

DIFFERENTIAL EXPRESSION FOR SPECIFIC ENTROPY FOR A REVERSIBLE PROCESS

η = SPECIFIC ENTROPY = ENTROPY/MASS

$\Rightarrow$ A MEASURE OF DISORDER, DISSIPATION OR ABILITY TO DO WORK

h = HEAT INPUT PER UNIT MASS

θ = TEMPERATURE

HERE

$đ(\;)$ DENOTES AN IMPERFECT DIFFERENTIAL

$\Rightarrow$ PATH DEPENDENT

$d\eta = \left(\frac{đh}{\theta}\right)_{REV}$ = PERFECT DIFFERENTIAL $\Rightarrow$ NOT PATH DEPENDENT

THE 2^{ND} LAW OF THERMODYNAMICS POSTULATES THAT THE CHANGE IN ENTROPY MUST BE NON-NEGATIVE FOR IRREVERSIBLE PROCESSES, i.e.,

$$\Delta\eta = \eta_2 - \eta_1 = \int_1^2 \frac{dh}{\theta} \qquad \text{FOR A REVERSIBLE PROCESS}$$

⇒ FOR A REVERSIBLE PROCESS WITH NO HEAT INPUT $\Delta\eta = 0$!

AND

$$\Delta\eta = \eta_2 - \eta_1 > \int_1^2 \frac{dh}{\theta} \qquad \text{FOR AN IRREVERSIBLE PROCESS}$$

HERE 1, 2 DENOTE THE STARTING AND ENDING POINTS OF THE PROCESS

OR

$$\underbrace{\underbrace{\Delta\eta = \eta_2 - \eta_1}_{\substack{\text{TOTAL ENTROPY}\\ \text{PRODUCTION DURING}\\ \text{PROCESS 1-2}}} \geq \underbrace{\int_1^2 \frac{dh}{\theta}}_{\substack{\text{ENTROPY PRODUCTION DUE TO}\\ \text{HEAT TRANSER DURING}\\ \text{PROCESS 1-2 (ENTROPY INPUT)}}}}$$

$$\eta = \frac{\text{ENERGY}}{(\text{MASS})(\text{TEMPERATURE})}$$

SUGGESTS THAT ENTROPY PRODUCTION IS POSITIVE FOR ALL IRREVERSIBLE PROCESSES. ENERGY (PER UNIT MASS PER UNIT TEMPERATURE) IS DISSIPATED DURING AN IRREVERSIBLE PROCESS EVEN IF THERE IS NO HEAT TRANSFER ($dh = 0$); SUCH ENERGY MAY NOT BE RECOVERED.

DEFINE

$\int_V \rho\eta\, dV$ = TOTAL ENTROPY CONTAINED BY THE MASS (SET OF PARTICLES) INSTANTANEOUSLY CONTAINED IN V

THE 2ND LAW OF THERMODYNAMICS MAY BE EXPRESSED AS

$$\begin{pmatrix}\text{RATE OF ENTROPY}\\ \text{INCREASE}\end{pmatrix} \geq \begin{pmatrix}\text{ENTROPY INPUT}\\ \text{RATE DUE TO}\\ \text{HEAT TRANSFER}\end{pmatrix} = \frac{\text{HEAT INPUT RATE}}{\text{TEMPERATURE}}$$

INCLUDES ENERGY DISSIPATION DUE TO IRREVERSIBILITIES AND HEAT TRANSFER

OR

$$\underbrace{\frac{d}{dt}\int_V \rho\eta\, dV}_{\substack{\text{TOTAL ENTROPY}\\ \text{RATE (INCLUDES}\\ \text{HEAT TRANSFER}\\ \text{AND DISSIPATION)}}} \geq \underbrace{\underbrace{\int_V \frac{\rho r}{\theta}\, dV}_{\substack{\text{ENTROPY}\\ \text{INPUT RATE}\\ \text{DUE TO HEAT}\\ \text{SUPPLY}}} - \underbrace{\int_S \overbrace{\frac{\underset{\sim}{n}\cdot\underset{\sim}{q}}{\theta}\, dS}^{\text{APPLY DIVERGENCE THEOREM}}}_{\substack{\text{ENTROPY INPUT}\\ \text{RATE DUE TO}\\ \text{CONDUCTION}}}}_{\text{RATE OF ENTROPY INPUT DUE TO HEAT TRANSFER}}$$

<u>INTRINSIC</u> <u>ENTROPY</u> <u>RATE</u>

THIS TERM VANISHES FOR AN ISOLATED SYSTEM

$$\int_V \rho \frac{d\eta}{dt}\, dV \geq \int_V \left[\frac{\rho r}{\theta} - \underset{\sim}{\nabla}\cdot\left(\frac{\underset{\sim}{q}}{\theta}\right)\right] dV$$

FOR ARBITRARY VOLUME, V

$$\boxed{\begin{aligned} &\rho\dot{\eta} - \frac{\rho r}{\theta} + \underset{\sim}{\nabla}\cdot\left(\frac{\underset{\sim}{q}}{\theta}\right) \geq 0 \\ &\rho\dot{\eta} - \frac{\rho r}{\theta} + \left(\frac{q_i}{\theta}\right)_{,i} \geq 0 \end{aligned}}$$

<u>CLAUSIUS</u> - <u>DUHEM</u> <u>INEQUALITY</u>

2ND LAW OF THERMODYNAMICS

(LOCAL FORM)

OR

$$\rho\theta\dot{\eta} - \rho r + q_{i,i} - \frac{1}{\theta} q_i \theta_{,i} \geq 0$$

OR

$$\rho\theta\dot{\eta} - \rho r - \underset{\sim}{\nabla}\cdot\underset{\sim}{q} - \frac{1}{\theta}\underset{\sim}{q}\cdot\underset{\sim}{\nabla}\theta \geq 0$$

NOTE: THE CLAUSIUS-DUHEM INEQUALITY ASSERTS THAT ENTROPY PRODUCTION RATE IS NON-NEGATIVE, i.e.,

$$\underbrace{\rho\gamma}_{\substack{\text{INTERNAL ENTROPY}\\ \text{PRODUCTION RATE}}} = \underbrace{\rho\dot{\eta}}_{\substack{\text{TOTAL ENTROPY}\\ \text{PRODUCTION RATE}}} \underbrace{- \rho\frac{r}{\theta} + \left(\frac{q_i}{\theta}\right)_{,i}}_{\substack{\text{ENTROPY INPUT RATE}\\ \text{DUE TO HEAT TRANSFER}}} \geq 0$$

$\Downarrow$

DUE TO IRREVERSIBILITIES!
(FRICTION, METAL PLASTICITY, ETC.)

γ = SPECIFIC INTERNAL ENTROPY PRODUCTION RATE

THE C-D INEQUALITY MAY BE REWRITTEN TO DECOMPOSE THE ENTROPY PRODUCTION INTO A LOCAL INTRINSIC PRODUCTION TERM AND A HEAT CONDUCTION PRODUCTION TERM.

$$\rho\gamma = \rho\dot{\eta} - \rho\frac{r}{\theta} + \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right) \geq 0$$

EXPAND LAST TERM

$$\rho\gamma = \underbrace{\rho\dot{\eta} - \rho\frac{r}{\theta} + \frac{1}{\theta}\nabla\cdot\mathbf{q}}_{\rho\gamma_{LOCAL}} \underbrace{- \frac{1}{\theta^2}\mathbf{q}\cdot\nabla\theta}_{\rho\gamma_{CONDUCTION}} \geq 0$$

OR

$$\rho\gamma = \rho\gamma_{LOCAL} + \rho\gamma_{CONDUCTION} \geq 0$$

INTRINSIC ENTROPY PRODUCTION

WHERE $\rho\gamma_{LOCAL} = \rho\dot{\eta} - \rho\frac{r}{\theta} + \frac{1}{\theta}\nabla\cdot\mathbf{q}$ INTRINSIC ENTROPY PRODUCTION RATE OR LOCAL ENTROPY RATE

$\rho\gamma_{CONDUCTION} = -\frac{1}{\theta^2}\mathbf{q}\cdot\nabla\theta$ ENTROPY PRODUCTION ASSOCIATED WITH CONDUCTION

A STRONGER FORM OF THE C-D INEQUALITY MAY BE EXPRESSED AS

$$\rho\gamma = \rho\gamma_{LOCAL} + \rho\gamma_{CONDUCTION} \geq 0 \qquad \text{C-D INEQUALITY}$$

$$\boxed{\begin{aligned} \rho\gamma_{CONDUCTION} &= -\frac{1}{\theta^2}\,\underset{\sim}{q}\cdot\underset{\sim}{\nabla}\theta \geq 0 \\ \rho\gamma_{LOCAL} &= \rho\dot{\eta} - \frac{\rho r}{\theta} + \frac{1}{\theta}\underset{\sim}{\nabla}\cdot\underset{\sim}{q} \geq 0 \end{aligned}}$$

STRONG FORM OF C-D INEQUALITY

KELVIN DISSIPATION INEQUALITY

FOR A REVERSIBLE PROCESS THERE IS NO FRICTIONAL DISSIPATION, HENCE

$$\rho\gamma_{LOC} = 0$$

$$\rho\gamma_{CONDUCTION} = 0$$

NOW CONSIDER FOURIER'S LAW OF CONDUCTION FOR AN ANISOTROPIC MATERIAL

$$\underset{\sim}{q} = -\underset{\sim}{\kappa}\cdot\underset{\sim}{\nabla}\theta \qquad \text{WHERE } \underset{\sim}{\kappa} = \kappa_{ij}\,\underset{\sim}{e}_i\,\underset{\sim}{e}_j$$

2^{ND} RANK CONDUCTIVITY TENSOR

FROM THE STRONG FORM OF THE C-D INEQUALITY

$$\rho\gamma_{CONDUCTION} = -\frac{1}{\theta^2}\,\underset{\sim}{q}\cdot\underset{\sim}{\nabla}\theta = -\frac{1}{\theta^2}\left(-\underset{\sim}{\kappa}\cdot\underset{\sim}{\nabla}\theta\right)\cdot\underset{\sim}{\nabla}\theta \geq 0$$

THEN $\quad \kappa_{ij}\dfrac{\partial\theta}{\partial x_i}\dfrac{\partial\theta}{\partial x_j} = \kappa_{ij}\,\theta_{,i}\,\theta_{,j} \geq 0$

$\Rightarrow$ κ_{ij} IS POSITIVE SEMI-DEFINITE

FOR $\underset{\sim}{\kappa} = \kappa\underset{\sim}{I}$, THEN $\rho\gamma_{CONDUCTION} = -\dfrac{1}{\theta^2}\left(-\underset{\sim}{\kappa}\cdot\underset{\sim}{\nabla}\theta\right)\cdot\underset{\sim}{\nabla}\theta \geq 0$

$\therefore \quad \kappa\,\delta_{ij}\,\theta_{,i}\,\theta_{,j} = \kappa\,\theta_{,i}\,\theta_{,i} \geq 0 \quad \Rightarrow \quad \boxed{\kappa \geq 0}$ HEAT FLOWS FROM HOT TO COLD!

STATE FUNCTIONS, STATE VARIABLES, AND EQUATIONS OF STATE

THE ENERGY EQUATION (1ST LAW OF THERMODYNAMICS) INVOLVES THE RATE OF CHANGE OF THE SPECIFIC INTERNAL ENERGY, u

$$\rho \dot{u} = \underset{\sim}{T} : \underset{\sim}{D} + \rho r - \underset{\sim}{\nabla} \cdot \underset{\sim}{q} \qquad \text{ENERGY EQUATION}$$

HENCE, SPECIFICATION OF THE INTERNAL ENERGY DEFINES THE THERMODYNAMIC STATE AT A POINT IN A CONTINUUM.

$\Rightarrow$ u IS A STATE FUNCTION

e.g., THE SPECIFIC INTERNAL ENERGY CAN BE EXPRESSED AS A FUNCTION OF SPECIFIC ENTROPY (η) AS WELL AS A NUMBER OF SUBSTATE VARIABLES (ν_i), i.e.,

$$u = \hat{u}(\eta, \nu_1, \nu_2, \dots) = \hat{u}(\eta, \nu_i) \qquad \text{CALORIC EQUATION OF STATE}$$

$\Rightarrow$ SPECIFICATION OF η AND SUBSTATE VARIABLES (ν_i) UNIQUELY CHARACTERIZES THE THERMODYNAMIC STATE AT A POINT IN THE CONTINUUM

$\Rightarrow$ SPECIFICATION OF THE FUNCTION $\hat{u}(\eta, \nu_i)$ DEFINES AN EQUATION OF STATE

$\hat{(\;)}$ DENOTES FUNCTION OF ARGUEMENTS

HERE,

$$\nu_i = \text{SUBSTATE VARIABLES } (i = 1, \dots, n). \quad n = \text{TOTAL NUMBER}$$
$$= \text{THERMODYNAMIC "DISPLACEMENT" (GENERALIZED)}$$

$\Rightarrow$ SUBSTATE VARIABLES CAN BE SCALARS, VECTORS, OR HIGHER RANK TENSORS

$\Rightarrow$ IDEALLY SUBSTATE VARIABLES WILL BE OBSERVABLE

- EXAMPLES: KINEMATIC VARIABLES, $\underset{\sim}{\epsilon}$, $\underset{\sim}{E}$, $\underset{\sim}{e}$, $\underset{\sim}{F}$, $\underset{\sim}{D}$, etc.
- NOT ALWAYS POSSIBLE $\Rightarrow$ "INTERNAL STATE VARIABLES"

IMPORTANT: AS THE SPECIFIC ENTROPY INCREASES, THE CHANGE IN SPECIFIC INTERNAL ENERGY MANIFESTS ITSELF AS TEMPERATURE.

DEFINE:

$$\boxed{\theta \equiv \left.\frac{\partial u}{\partial \eta}\right|_{\nu_i}}$$ THERMODYNAMIC TEMPERATURE

⇨ θ, η ARE THERMODYNAMIC CONJUGATE VARIABLES

- PRODUCT $\theta\eta$ HAS UNITS OF ENERGY (PER UNIT MASS)

ANALOGOUSLY, ONE MAY DEFINE A SET OF THERMODYNAMIC VARIABLES CONJUGATE TO THE SUBSTATE VARIABLES, ν_i, i.e.,

$$\boxed{\Upsilon_i \equiv \left.\frac{\partial u}{\partial \nu_i}\right|_{\eta}}$$ THERMODYNAMIC "TENSIONS" ($i = 1, \dots, n$)
(THERMODYNAMIC "FORCES" CONJUGATE TO THE GENERALIZED DISPLACEMENTS, ν_i)

⇨ Υ_i, ν_i ARE THERMODYNAMIC CONJUGATE (WORK CONJUGATE) VARIABLES

- SCALAR PRODUCT $\Upsilon_i \nu_i$ (OR $\underset{\sim}{\Upsilon} : \underset{\sim}{\nu}$) HAS UNITS OF WORK/ENERGY (PER UNIT MASS); Υ_i, ν_j ARE OF SAME RANK.

SINCE $u = \hat{u}(\eta, \nu_i)$, THEN $\theta = \hat{\theta}(\eta, \nu_i)$ AND $\Upsilon_i = \hat{\Upsilon}_i(\eta, \nu_j)$
ONE MAY EXPRESS THE DIFFERENTIAL SPECIFIC INTERNAL ENERGY AS

$$du = \left.\frac{\partial u}{\partial \eta}\right|_{\nu_i} d\eta + \sum_{i=1}^{n} \frac{\partial u}{\partial \nu_i} d\nu_i \qquad \text{(SCALAR)}$$

$$= \theta\, d\eta + \sum_{i=1}^{n} \Upsilon_i\, d\nu_i = \theta\, d\eta + \underbrace{\Upsilon_i\, d\nu_i}_{\text{SUMMATION CONVENTION}}$$

IN GENERAL, THE RELATIONSHIP FOR THERMODYNAMIC TEMPERATURE $\theta = \tilde{\theta}(\eta, \nu_i)$ MAY BE INVERTED TO DEFINE THE ENTROPY, i.e.,

$$\boxed{\eta = \hat{\eta}(\theta, \nu_i)}$$ <u>EQUATION OF STATE</u>

THEN THE <u>CALORIC EQUATION OF STATE</u> MAY BE EXPRESSED AS

$$\boxed{u = \hat{u}(\theta, \nu_i)}$$ <u>CALORIC EQUATION OF STATE</u> (ALTERNATE FORM)

GENERALIZED "DISPLACEMENTS" – PREFERABLY OBSERVABLE

θ = TEMPERATURE

ν_i = STATE VARIABLE

SIMILARLY, THE <u>THERMODYNAMIC TENSIONS</u> MAY BE EXPRESSED IN TERMS GENERALIZED DISPLACEMENTS θ AND ν_i

$$\gamma_i = \hat{\gamma}_i(\theta, \nu_j)$$ <u>THERMAL EQUATIONS OF STATE</u>

INVERTING

$$\nu_i = \hat{\nu}_i(\theta, \gamma_j)$$

⇨ THERMAL EQUATIONS OF STATE RESEMBLE STRESS-STRAIN RELATIONS, BUT SOME CAUTION IS REQUIRED IN INTERPRETING THE THERMODYNAMIC TENSIONS (γ_i) AS STRESSES AND SUBSTATE PARAMETERS (ν_i) AS STRAINS.

⇨ EVEN IF THE ν_j CORRESPOND TO ELASTIC STRAINS, THE γ_i MAY DIFFER FROM THE USUAL COMPONENTS OF THE STRESS TENSOR.

IN ADDITION TO THE SPECIFIC INTERNAL ENERGY, OTHER POTENTIAL FUNCTIONS MAY BE USED TO EXPRESS THE PRINCIPLES OF THERMODYNAMICS. THESE INCLUDE:

1) SPECIFIC HELMHOLTZ FREE ENERGY, ψ

$$\boxed{\psi = \underbrace{u}_{\text{INTERNAL ENERGY}} - \underbrace{\eta\theta}_{\text{NON-RECOVERABLE ENERGY}}}$$

SPECIFIC HELMHOLTZ FREE ENERGY

⇨ THE HELMHOLTZ FREE ENERGY CORRESPONDS TO THAT PART OF THE INTERNAL ENERGY AVAILABLE TO DO WORK AT CONSTANT TEMPERATURE

⇨ ψ IS GOOD FOR THE ANALYSIS OF SOLIDS

NOTE: $\eta = -\dfrac{\partial \psi}{\partial \theta}$ SPECIFIC ENTROPY

2) SPECIFIC ENTHALPY, χ

$$\boxed{\chi = u - \underbrace{\sum_{i=1}^{n} \gamma_i v_i}_{\text{SPECIFIC ENERGY ASSOCIATED WITH THERMODYNAMIC TENSIONS}}}$$

SPECIFIC ENTHALPY

⇨ THE SPECIFIC ENTHALPY CORRESPONDS TO THAT PORTION OF THE INTERNAL ENERGY THAT CAN BE RELEASED AS HEAT WHEN THE THERMODYNAMIC TENSIONS ARE HELD FIXED.

REPLACE PREVIOUS PAGES 246-255

3) GIBBS FREE ENERGY (FREE ENTHALPY)

$$\boxed{\varsigma = \chi - \eta\theta = u - \eta\theta - \sum_{i=1}^{n} \gamma_i v_j}$$

SPECIFIC GIBBS FREE ENTHALPY

NOTE: $u - \psi + \varsigma - \chi = 0$ RELATIONSHIP BETWEEN POTENTIALS

⇨ ς IS GOOD FOR THE ANALYSIS OF FLUIDS OR OTHER PROCESSES WHERE THE TEMPERATURE IS CONTROLLED AND THE HYDROSTATIC PRESSURE AND/OR DILATATION IS KNOWN

IN GENERAL, THE FOUR POTENTIALS MAY BE EXPRESSED IN TERMS OF ANY COMBINATION OF (TEMPERATURE OR ENTROPY) AND (SUBSTATE VARIABLES OR THERMODYNAMIC TENSIONS), i.e.,

COMMON FORMULATION

$$u = \hat{u}(\eta, \gamma_i) = \hat{u}(\theta, \gamma_i) = \hat{u}(\theta, v_i) = \underbrace{\hat{u}(\eta, v_i)}$$
$$\psi = \hat{\psi}(\eta, \gamma_i) = \hat{\psi}(\theta, \gamma_i) = \hat{\psi}(\eta, v_i) = \hat{\psi}(\theta, v_i)$$
$$\chi = \hat{\chi}(\theta, \gamma_i) = \hat{\chi}(\eta, v_i) = \chi(\theta, v_i) = \chi(\eta, \gamma_i)$$
$$\varsigma = \hat{\varsigma}(\eta, v_i) = \hat{\varsigma}(\theta, v_i) = \hat{\varsigma}(\eta, \gamma_i) = \hat{\varsigma}(\theta, \gamma_i)$$

ALTHOUGH THE LATTER FORMULATIONS LEAD TO PARTICULARLY CONVENIENT RESULTS. THE RELATIONSHIPS BETWEEN THERMODYNAMIC POTENTIALS ARE SUMMARIZED BELOW

POTENTIAL	RELATION TO u	INDEPENDENT VARIABLES	
INTERNAL ENERGY	u	η, v_i	
HELMHOLTZ FREE ENERGY	$\psi = u - \eta\theta$	θ, v_i	(SOLIDS)
ENTHALPY	$\chi = u - \gamma_i v_i$	η, γ_i	
FREE ENTHALPY, OR GIBBS FUNCTION	$\varsigma = u - \eta\theta - \gamma_i v_i$ $= \chi - \eta\theta$	θ, γ_i	(FLUIDS)

THE PRECEDING FOUR POTENTIALS CORRESPOND TO EQUATIONS OF STATE. THE DIFFERENTIAL INCREMENTS IN THE STATE FUNCTIONS $(u, \psi, \chi, \varsigma)$ MAY BE EXPRESSED AS

$$u = \hat{u}(\eta, v_j) \quad \Rightarrow \quad du = \theta\, d\eta + \tau_j\, dv_j$$

$$\psi = \hat{\psi}(\theta, v_j) = u - \eta\theta \quad \Rightarrow \quad d\psi = -\eta\, d\theta + \tau_j\, dv_j$$

$$\chi = \hat{\chi}(\eta, \tau_j) = u - \tau_j v_j \quad \Rightarrow \quad d\chi = \theta\, d\eta - v_j\, d\tau_j$$

$$\varsigma = \hat{\varsigma}(\theta, \tau_j) = \chi - \eta\theta \quad \Rightarrow \quad d\varsigma = -\eta\, d\theta - v_j\, d\tau_j$$
$$= u - \eta\theta - \tau_j v_j$$

CLEARLY THE RELATIONSHIP BETWEEN THERMODYNAMIC "DISPLACEMENTS" (θ, v_j) AND THERMODYNAMIC "FORCES" (η, τ_j) MAY BE EXPRESSED AS

$$\theta = \left.\frac{\partial u}{\partial \eta}\right|_{v_j} \qquad \tau_j = \left.\frac{\partial u}{\partial v_j}\right|_{\eta} \qquad \text{FROM } du = \theta\, d\eta + \tau_j v_j$$

$$\eta = -\left.\frac{\partial \psi}{\partial \theta}\right|_{v_j} \qquad \tau_j = \left.\frac{\partial \psi}{\partial v_j}\right|_{\theta} \qquad \text{FROM } d\psi = -\eta\, d\theta + \tau_j\, dv_j$$

$$\theta = \left.\frac{\partial \chi}{\partial \eta}\right|_{\tau_j} \qquad v_j = -\left.\frac{\partial \chi}{\partial \tau_j}\right|_{\eta} \qquad \text{FROM } d\chi = \theta\, d\eta - v_j\, d\tau_j$$

$$\eta = -\left.\frac{\partial \varsigma}{\partial \theta}\right|_{\tau_j} \qquad v_j = -\left.\frac{\partial \varsigma}{\partial \tau_j}\right|_{\theta} \qquad \text{FROM } d\varsigma = -\eta\, d\theta - v_j\, d\tau_j$$

NOTE: THE SPECIFIC INTERNAL ENERGY, u, IS REQUIRED FOR THE ENERGY EQUATION. FOR SOLIDS SPECIFICATION OF THE HELMHOLTZ FREE ENERGY, ψ, AND SPECIFIC ENTROPY, η, LEADS TO THE SPECIFIC INTERNAL ENERGY $u = \psi + \eta\theta$.

$\Rightarrow$ WE MAY INTRODUCE ψ AS A STATE FUNCTION PROVIDED THAT THE CONSTITUTIVE LAW FOR η SATISFIES THE 2ND LAW OF THERMODYNAMICS

ONE MAY EXPRESS BOTH THE 1ST LAW OF THERMODYNAMICS (ENERGY EQUATION) AND 2ND LAW OF THERMODYNAMICS (C-D INEQUALITY) IN TERMS OF THE HELMHOLTZ FREE ENERGY.

RECALL

$$\rho \dot{u} = \underset{\sim}{\sigma} : \underset{\sim}{D} + \rho r - \underset{\sim}{\nabla} \cdot \underset{\sim}{q} \qquad \text{ENERGY EQUATION}$$

OR

$$\underbrace{\rho r - \underset{\sim}{\nabla} \cdot \underset{\sim}{q}}_{\text{COMPARE}} = \rho \dot{u} - \underset{\sim}{\sigma} : \underset{\sim}{D}$$

AND

$$\rho \theta \dot{\eta} \overbrace{- \rho r + \underset{\sim}{\nabla} \cdot \underset{\sim}{q}} - \frac{1}{\theta} \underset{\sim}{q} \cdot \underset{\sim}{\nabla} \theta \geq 0 \qquad \text{C-D INEQUALITY}$$

SUBSTITUTE THE ENERGY EQUATION INTO THE C-D INEQUALITY

$$\rho \theta \dot{\eta} - \rho \dot{u} + \underset{\sim}{\sigma} : \underset{\sim}{D} - \frac{1}{\theta} \underset{\sim}{q} \cdot \underset{\sim}{\nabla} \theta \geq 0$$

BUT $\dot{\psi} = \dot{u} - \theta \dot{\eta} - \dot{\theta} \eta \Rightarrow \theta \dot{\eta} - \dot{u} = -\dot{\psi} - \dot{\theta} \eta$

$$\therefore \boxed{\underbrace{-\rho \dot{\psi} - \rho \eta \dot{\theta} + \underset{\sim}{\sigma} : \underset{\sim}{D}}_{\text{INTERNAL DISSIPATION}} \underbrace{- \frac{1}{\theta} \underset{\sim}{q} \cdot \underset{\sim}{\nabla} \theta}_{\text{DISSIPATION DUE TO HEAT CONDUCTION}} \geq 0}$$

LOCAL DISSIPATION INEQUALITY (C-D INEQUALITY CAST IN TERMS OF ψ)

☆ ERROR IN EQUATION 5.8-20 IN MASE

ANALOGOUSLY, FOR FLUIDS SPECIFICATION OF THE SPECIFIC ENTHALPY, $\chi = u - \gamma_i v_i$, OR SPECIFIC FREE ENTHALPY, $\zeta = \chi - \eta\theta$, LEADS TO THE SPECIFIC INTERNAL ENERGY, u.

ALTERNATIVELY, FOR MANY FLUIDS THERE IS A DIRECT RELATIONSHIP BETWEEN THE SPECIFIC INTERNAL ENERGY (u) AND ENTHALPY (χ) INVOLVING THERMODYNAMIC PRESSURE (p), DENSITY (ρ), AND TEMPERATURE (θ). IT IS THEN POSSIBLE TO EXPRESS THE CALORIC EQUATION OF STATE IN TERMS OF TEMPERATURE AND DENSITY, i.e.,

$$u = \hat{u}(\theta, v_i) = \hat{u}(\theta, \rho)$$ <u>CALORIC EQUATION OF STATE</u>

IT IS THEN POSSIBLE TO DEFINE A THERMAL EQUATION OF STATE

$$\gamma_i = \hat{\gamma}_i(\theta, v_i) = \hat{\gamma}_i(\theta, \rho)$$ <u>THERMAL EQUATION OF STATE</u>

WHERE THE THERMODYNAMIC TENSION (γ_i) CORRESPONDS TO THE THERMODYNAMIC PRESSURE (p), i.e., $\gamma_i = p$. THIS LEADS TO AN EQUATION OF STATE OF THE FORM

$$f(p, \theta, \rho) = 0$$ <u>KINETIC EQUATION OF STATE</u>

FOR AN IDEAL (PERFECT) GAS

$$\underbrace{p}_{\text{THERMODYNAMIC PRESSURE}} = \rho R \theta$$

<u>KINETIC EQUATION OF STATE</u>
(IDEAL GAS LAW)

R = GAS CONSTANT FOR A PARTICULAR GAS

<u>METHOD OF LOCAL STATE</u>: THE THERMODYNAMIC STATE OF A MATERIAL MEDIUM AT A GIVEN POINT AND INSTANT IS COMPLETELY DEFINED BY THE VALUES OF A CERTAIN NUMBER OF STATE VARIABLES AT THAT INSTANT, WHICH DEPEND ONLY ON THE POINT CONSIDERED.

EXAMPLE: $$u = \hat{u}(\underbrace{\theta, \nu_i}_{\text{STATE VARIABLES}}) = \hat{u}(\underbrace{\theta, \underset{\sim}{F}}_{\text{OBSERVABLE}}, \overbrace{\underset{\sim}{\xi_1}, \underset{\sim}{\xi_2}}^{\text{INTERNAL STATE VARIABLES}})$$

IN GENERAL, SINCE TIME DERIVATIVES OF STATE VARIABLES $(\dot{\theta}, \dot{\nu}_i)$ ARE NOT INVOLVED IN THE DEFINITION OF THE STATE, THIS HYPOTHESIS IMPLIES THAT ANY EVOLUTION CAN BE CONSIDERED AS A SUCCESSION OF EQUILIBRIUM STATES.

$\Rightarrow$ THE PROCESS MUST SATISFY THE C-D INEQUALITY AT EVERY INSTANT OF EVOLUTION

THE STATE VARIABLES CAN BE DIVIDED INTO TWO CLASSES:

1) <u>OBSERVABLE STATE VARIABLES</u> (STRAIN, TEMPERATURE, DENSITY, ETC.)
   - MEASURABLE QUANTITIES
   - FOR REVERSIBLE PROCESSES, THE THERMODYNAMIC STATE UNIQUELY DEPENDS ON THE OBSERVABLE VARIABLES ALONE

2) <u>INTERNAL STATE VARIABLES</u> (ISV's)
   - FOR DISSIPATIVE PHENOMENA, THE CURRENT STATE DEPENDS ON THE PAST HISTORY WHICH CAN BE DETERMINED BY A SET OF "HIDDEN" OR INTERNAL VARIABLES THAT REPRESENT THE INTERNAL STATE OF MATTER (DENSITY OF DISLOCATIONS, DENSITY AND DISTRIBUTION OF VOIDS AND MICROCRACKS) NOT AMENABLE TO DIRECT MEASUREMENT
   - ISV's DO NOT EXPLICITLY APPEAR IN THE CONSERVATION LAWS OR C-D INEQUALITY BUT MUST BE ACCOUNTED FOR WHEN DEVELOPING CONSTITUTIVE EQUATIONS.

CONSIDER THE GOVERNING EQUATIONS OF THERMOMECHANICS (EULERIAN)

1) $\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0$ CONSERVATION OF MASS (CONTINUITY)

2) $\sigma_{ji,j} + \rho b_i = \rho \dot{v}_i$ CONSERVATION OF LINEAR MOMENTUM

3) $\rho \dot{u} = \sigma_{ij} D_{ij} + \rho r - q_{i,i}$ CONSERVATION OF ENERGY (1ST LAW OF THERMODYNAMICS)

4) $\rho \dot{\eta} - \rho \frac{r}{\theta} + \left(\frac{q_i}{\theta}\right)_{,i} \geq 0$ CLAUSIUS-DUHEM INEQUALITY (2ND LAW OF THERMODYNAMICS)

PLUS

5) $\sigma_{ij} = \sigma_{ji}$ CONSERVATION OF ANGULAR MOMENTUM

6) $D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ RATE OF DEFORMATION TENSOR (OR STRAIN DISPLACEMENT RELATIONS)

IMPORTANT: ASSUMING THAT THE BODY FORCE (b_i) AND SPECIFIC HEAT SUPPLY (r) ARE KNOWN, THE FIRST THREE CONSERVATION LAWS REPRESENT FIVE EQUATIONS IN 14 UNKNOWNS ($\overset{1}{\rho}, \overset{3}{v_i}, \overset{6}{\sigma_{ij}}, \overset{1}{u}, \overset{3}{q_i}$). CONSIDERATION OF THE C-D INEQUALITY INTRODUCES TWO MORE UNKNOWNS (η, θ), BUT THE 2ND LAW OF THERMODYNAMICS MAY ONLY BE USED TO SOLVE FOR UNKNOWNS FOR REVERSIBLE (NON-DISSIPATING) PROCESSES (i.e., WHEN THE EQUALITY HOLDS)

HENCE, THERE ARE A TOTAL OF 16 UNKNOWNS BUT ONLY FIVE EQUATIONS (1) - (3)

⇒ NEED 11 MORE EQUATIONS

⇒ THESE ARE THE CONSTITUTIVE EQUATIONS.

CONSTITUTIVE EQUATIONS: RELATIONSHIPS CHARACTERIZING A SPECIFIC MATERIAL (SOLID, LIQUID, GAS) AND ITS REACTION TO THERMOMECHANICAL LOADS.

$\Rightarrow$ e.g., MACROSCOPIC IDEALIZATION OF THE CONTINUUM BEHAVIOR

EXAMPLES OF REQUISITE CONSTITUTIVE EQUATIONS (11 NEEDED)

$\underset{\sim}{q} = -\underset{\sim}{k} \cdot \underset{\sim}{\nabla}\theta$ — FOURIER'S "LAW" OF HEAT CONDUCTION $\Rightarrow$ 3 EQUATIONS

+   +

$\underset{\sim}{\sigma} = \hat{\underset{\sim}{\sigma}}(\underset{\sim}{D})$ OR $\underset{\sim}{\sigma} = \hat{\underset{\sim}{\sigma}}(\underset{\sim}{e})$ — STRESS - STRAIN EQUATIONS $\Rightarrow$ 6 EQUATIONS

+   +

$\psi = \hat{\psi}(\underset{\sim}{E}, \theta)$ AND $\eta = \hat{\eta}(\underset{\sim}{E}, \theta)$ — EQUATIONS OF STATE FOR SOLIDS $\Rightarrow$ 2 EQUATIONS

ψ = HELMHOLTZ FREE ENERGY
η = SPECIFIC ENTROPY

OR   OR

$p = \hat{p}(\rho, \theta)$ AND $u = \hat{u}(\rho, \theta)$ — KINETIC EQUATION OF STATE AND CALORIC EQUATION OF STATE FOR COMPRESSIBLE FLUIDS $\Rightarrow$ 2 EQUATIONS

THIS IS EQUIVALENT TO SPECIFYING THE GIBBS FUNCTION PLUS KINETIC EQUATION OF STATE

u = INTERNAL ENERGY
p = THERMODYNAMIC PRESSURE

TOTAL: 11 EQUATIONS

ASIDE: FOR IDEAL GASES: $p = \rho R \theta$ KINETIC EQUATION OF STATE

FOR COMPRESSIBLE FLUIDS: $p \neq \frac{\sigma_{kk}}{3}$ IN GENERAL

FOR INCOMPRESSIBLE FLUIDS: $p = \frac{\sigma_{kk}}{3}$ HYDROSTATIC PRESSURE

<u>NOTE:</u> FOR A FULLY COUPLED THERMOMECHANICAL PROBLEM, THE FULL SET OF GOVERNING EQUATIONS (CONSERVATION LAWS, C-D INEQUALITY, STRAIN-DISPLACEMENT RELATIONS, PLUS CONSTITUTIVE EQUATIONS) ARE REQUIRED TO SOLVE THE PROBLEM.

FOR A <u>PURELY</u> <u>MECHANICAL</u> <u>PROCESS</u>, THE TEMPERATURE DOES NOT DEPEND ON THE DEFORMATION, i.e., $\theta \neq \hat{\theta}(\underset{\sim}{u})$ AND $\theta \neq \hat{\theta}(\underset{\sim}{v})$. AN ASSUMPTION OF PURELY MECHANICAL BEHAVIOR IS OFTEN INVOKED FOR SOLIDS UNDERGOING SMALL INELASTIC (NONLINEAR) STRAINS. FOR SUCH CASES, ONLY THE CONTINUITY, EQUILIBRIUM, AND STRESS-STRAIN RELATIONSHIPS ARE USED.

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + (\rho v_i)_{,i} = 0 \\ &\sigma_{ji,j} + \rho b_i = 0 \\ &\sigma_{ij} = C_{ijkl}\,\epsilon_{kl} \\ &\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \end{aligned}\right\} \quad \begin{array}{l} \text{10 EQUATIONS IN 10 UNKNOWNS} \\ \text{FOR } \rho,\ \underset{\sim}{\sigma},\ \underset{\sim}{u} \end{array}$$

THE TEMPERATURE IS FOUND FROM A PARALLEL INDEPENDENT SOLUTION OF THE ENERGY EQUATION, NEGLECTING THE STRESS-POWER $(\underset{\sim}{\sigma} : \underset{\sim}{D})$. THIS LEADS TO AN ENERGY EQUATION OF THE FORM

$$\underbrace{\rho \dot{u} = \rho C_v \frac{\partial \theta}{\partial t}}_{\substack{\text{RATE OF ENERGY} \\ \text{STORAGE IN } V}} = \underbrace{\rho r}_{\substack{\text{RATE OF HEAT} \\ \text{GENERATION IN } V}} - \underbrace{q_{i,i}}_{\substack{\text{RATE OF HEAT} \\ \text{CONDUCTION OUT OF } V}}$$

HEAT EQUATION (HEAT DIFFUSION EQUATION)

C_v = SPECIFIC HEAT AT CONSTANT VOLUME (DEFORMATION)

IF $\underset{\sim}{q} = k \underset{\sim}{\nabla} \theta$ FOURIER'S LAW, THEN

$$\rho C_v \frac{\partial \theta}{\partial t} = \rho r - \frac{\partial}{\partial x_i}\left(k \frac{\partial \theta}{\partial x_i}\right) \qquad \text{HEAT EQUATION}$$

⇒ THE HEAT EQUATION PROVIDES THE BASIC TOOL FOR HEAT CONDUCTION ANALYSIS NECESSARY TO FIND THE TEMPERATURE DISTRIBUTION $\theta(\underset{\sim}{x}, t)$

FUNDAMENTAL RESTRICTIONS ON THE FORMULATIONS OF CONSTITUTIVE LAWS

1) PHYSICAL ADMISSIBILITY: ALL CONSTITUTIVE EQUATIONS MUST BE CONSISTENT WITH:

   i) CONSERVATION OF MASS

   ii) CONSERVATION OF LINEAR AND ANGULAR MOMENTUM

   iii) CONSERVATION OF ENERGY

   iv) 2^{ND} LAW OF THERMODYNAMICS

   v) EXPERIMENTAL OBSERVATIONS ⇨ MOST IMPORTANT

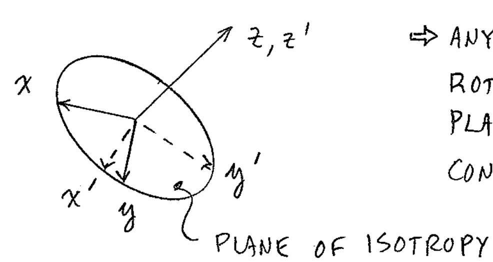
2) PRINCIPLE OF LOCAL ACTION: ONLY THE DEFORMATION WITHIN AN INFINITESIMAL NEIGHBORHOOD OF THE POINT $\underset{\sim}{x}$ AFFECTS THE MATERIAL BEHAVIOR AT $\underset{\sim}{x}$.

3) PRINCIPLE OF EQUIPRESENCE: A STATE VARIABLE ASSUMED TO BE PRESENT IN ONE CONSTITUTIVE EQUATION OF A MATERIAL SHOULD BE SO PRESENT IN ALL, UNLESS IT'S PRESENCE IS IN DIRECT CONTRADICTION WITH THE ASSUMED SYMMETRY OF THE MATERIAL, THE PRINCIPAL OF MATERIAL OBJECTIVITY, OR THE LAWS OF THERMODYNAMICS.

   ⇨ WEAK RESTRICTION, OFTEN VIOLATED

4) MATERIAL SYMMETRY: THE CONSTITUTIVE EQUATIONS MUST BE FORM-INVARIANT WITH RESPECT TO COORDINATE TRANSFORMATIONS COMPRISING THE SYMMETRY GROUP OR CLASS OF THE MATERIAL (VERY IMPORTANT)

   EXAMPLE: TRANSVERSELY ISOTROPIC MATERIAL



⇨ ANY CHANGE OF COORDINATES THAT CONSISTS OF A ROTATION ABOUT THE AXIS PERPENDICULAR TO THE PLANE OF ISOTROPY, RESULTS IN NO CHANGE IN CONSTITUTIVE EQUATIONS.

RESTRICTIONS (CONT'D)

5) MATERIAL FRAME INDIFFERENCE (MATERIAL OBJECTIVITY): THE RESPONSE OF A MATERIAL MUST BE INDEPENDENT OF THE SPATIAL (EULERIAN) REFERENCE FRAME USED TO DESCRIBE IT.

⇨ ALL EQUATIONS ARE INVARIANT UNDER ARBITRARY OBSERVER TRANSFORMATIONS (TRANSLATIONS AND ROTATIONS) IN A GALILEAN REFERENCE FRAME.

⇨ AN EQUATION IS CONSIDERED FRAME INDIFFERENT IF ALL TERMS ARE FRAME INDIFFERENT

⇨ EQUATIONS EXPRESSED IN TERMS OF LAGRANGIAN COORDINATES ARE FRAME INDIFFERENT; USE OF LAGRANGIAN COORDINATES WOULD INVOKE A SPECIAL (MATERIAL) FRAME OF REFERENCE WHICH IS NOT SUBJECT TO THE NOTION OF DIFFERENCE BETWEEN SPATIAL FRAMES.

⇨ NOTE: THE ACCELERATION VECTOR IS NOT FRAME INDIFFERENT

NOTES ON OBJECTIVITY (MATERIAL FRAME INDIFFERENCE)

IF A CONSTITUTIVE EQUATION IS SATISFIED FOR A DYNAMICAL PROCESS WITH A MOTION $\underset{\sim}{x} = x(\underset{\sim}{X}, t)$ AND A SYMMETRIC STRESS TENSOR $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(\underset{\sim}{X}, t) = \underset{\sim}{\sigma}(\underset{\sim}{x}(\underset{\sim}{X}), t)$, THEN IT IS SATISFIED FOR AN EQUIVALENT PROCESS

$$\left.\begin{aligned} \underset{\sim}{x}^*(\underset{\sim}{X}, t^*) &= \underset{\sim}{c}(t) + \underset{\sim}{Q}(t)\cdot\underset{\sim}{x}(\underset{\sim}{X}, t) \\ \text{AND} \qquad \underset{\sim}{\sigma}^*(\underset{\sim}{X}, t^*) &= \underset{\sim}{Q}(t)\cdot\underset{\sim}{\sigma}\cdot\underset{\sim}{Q}^T(t) \end{aligned}\right\} \text{PROCESS AS OBSERVED BY SOMEONE IN A DIFFERENT FRAME OF REFERENCE}$$

WHERE

$\underset{\sim}{c}(t)$ = RELATIVE TRANSLATION BETWEEN OBSERVERS

$\underset{\sim}{Q}(t)$ = RELATIVE RIGID ROTATION BETWEEN OBSERVERS

$$\text{WHERE} \quad \left.\begin{aligned} \underset{\sim}{Q}(t)\cdot\underset{\sim}{Q}(t)^T &= \underset{\sim}{I} \\ |\underset{\sim}{Q}(t)| &= +1 \end{aligned}\right\} \text{PROPER ORTHOGONAL}$$

$t^* = t - a$ = TIME OFFSET

HERE $\underset{\sim}{c}(t)$, $\underset{\sim}{Q}(t)$, AND a SERVE TO IDENTIFY THE SAME PROCESS AS OBSERVED BY SOMEONE IN ANOTHER REFERENCE FRAME.

VECTORS AND 2^{ND} RANK TENSORS ARE FRAME INDIFFERENT IF THEY OBEY:

$$\boxed{\begin{aligned} \underset{\sim}{v}^*(\underset{\sim}{X}, t^*) &= \underset{\sim}{Q}(t)\cdot\underset{\sim}{v} \\ \underset{\sim}{A}^*(\underset{\sim}{X}, t^*) &= \underset{\sim}{Q}(t)\cdot\underset{\sim}{A}\cdot\underset{\sim}{Q}^T(t) \end{aligned}}$$

FRAME INDIFFERENT TRANSFORMATION LAWS FOR VECTORS AND 2^{ND} RANK TENSORS

$\underset{\sim}{v}^*$ = VECTOR

$\underset{\sim}{A}^*$ = 2^{ND} RANK TENSOR

NOTE: THE DEFORMATION GRADIENT TENSOR TRANSFORMS LIKE A VECTOR SINCE IT IS A TWO-POINT TENSOR ($\underset{\sim}{F}$ DEPENDS ON BOTH LAGRANGIAN AND EULERIAN COORDINATES)

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

DEFORMATION GRADIENT TENSOR (TWO POINT TENSOR)

i: REFERS TO $\underset{\sim}{x}$ — PART OF $\underset{\sim}{F}$ THAT REFERS TO THE SPATIAL FRAME

j: REFERS TO $\underset{\sim}{X}$

IN GENERAL

$$d\underset{\sim}{x}^* = \underbrace{\underset{\sim}{F}^*}_{\text{DEFORMATION GRADIENT AS SEEN BY OBSERVER B}} \cdot \underbrace{d\underset{\sim}{X}}_{\text{UNDEFORMED ELEMENT}}$$

DEFORMATION MAPPING AS SEEN BY OBSERVER B IN $\underset{\sim}{x}^*$

AND

$$\underbrace{d\underset{\sim}{x}^*}_{\text{DEFORMED ELEMENT AS SEEN BY OBSERVER B}} = \underset{\sim}{Q} \cdot \underbrace{d\underset{\sim}{x}}_{\text{DEFORMED ELEMENT AS SEEN BY OBSERVER A IN } \underset{\sim}{x}}$$

FRAME INDIFFERENT VECTOR TRANSFORMATION EQUATION

THEN

$$d\underset{\sim}{x}^* = \underset{\sim}{F}^* \cdot d\underset{\sim}{X} = \underset{\sim}{Q} \cdot d\underset{\sim}{x} = \underset{\sim}{Q} \cdot \underbrace{\underset{\sim}{F}}_{\text{DEFORMATION GRADIENT AS SEEN BY OBSERVER A}} \cdot d\underset{\sim}{X}$$

$$\therefore \quad \boxed{\underset{\sim}{F}^* = \underset{\sim}{Q} \cdot \underset{\sim}{F}}$$

DEFORMATION GRADIENT IS NOT FRAME INDIFFERENT $\underset{\sim}{F}^* \neq \underset{\sim}{Q} \cdot \underset{\sim}{F} \cdot \underset{\sim}{Q}^T$

$\underset{\sim}{A}^* = \underset{\sim}{Q} \cdot \underset{\sim}{A} \cdot \underset{\sim}{Q}^T$ FRAME INDIFFERENT LAW FOR 2^{ND} RANK TENSOR.

$\underset{\sim}{Q} = \underset{\sim}{Q}(t)$ = RIGID ROTATION TENSOR

EXAMPLE: OBJECTIVITY OF THE RATE OF DEFORMATION TENSOR, $\underset{\sim}{D}$ (EULERIAN)

CONSIDER THE DEFORMATION GRADIENT AS SEEN BY TWO OBSERVERS

$$\underset{\sim}{F}^* = \underset{\sim}{Q} \cdot \underset{\sim}{F}$$ TAKE MATERIAL TIME DERIVATIVE

$$\dot{\underset{\sim}{F}}^* = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{F}} + \dot{\underset{\sim}{Q}} \cdot \underset{\sim}{F} = \underset{\sim}{Q} \cdot \left(\dot{\underset{\sim}{F}} + \underset{\sim}{Q}^T \cdot \dot{\underset{\sim}{Q}} \cdot \underset{\sim}{F} \right)$$

LATTER TERM IS TRUE SINCE $\underset{\sim}{Q} \cdot \underset{\sim}{Q}^T = \underset{\sim}{I}$

RECALL DEFINITION OF SPATIAL VELOCITY GRADIENT TENSOR: $\underset{\sim}{L} = \dot{\underset{\sim}{F}} \cdot \underset{\sim}{F}^{-1}$. THEN

$$\underset{\sim}{L}^* = \dot{\underset{\sim}{F}}^* \cdot \underset{\sim}{F}^{*-1} = \underbrace{\underset{\sim}{Q} \cdot \left(\dot{\underset{\sim}{F}} + \underset{\sim}{Q}^T \cdot \dot{\underset{\sim}{Q}} \cdot \underset{\sim}{F} \right)}_{\dot{\underset{\sim}{F}}^*} \cdot \underbrace{\left(\underset{\sim}{Q} \cdot \underset{\sim}{F} \right)^{-1}}_{\underset{\sim}{F}^{*-1}}$$

$\underset{\sim}{Q} \cdot \underset{\sim}{Q}^T = \underset{\sim}{I}$

$\underset{\sim}{Q}$ IS PROPER ORTHOGONAL

$$\text{BUT } \left(\underset{\sim}{A} \cdot \underset{\sim}{B} \right)^{-1} = \underset{\sim}{B}^{-1} \cdot \underset{\sim}{A}^{-1} \Rightarrow \left(\underset{\sim}{Q} \cdot \underset{\sim}{F} \right)^{-1} = \underset{\sim}{F}^{-1} \cdot \underset{\sim}{Q}^{-1} = \underset{\sim}{F}^{-1} \cdot \underset{\sim}{Q}^T$$

THEN

$$\underset{\sim}{L}^* = \underset{\sim}{Q} \cdot \left(\dot{\underset{\sim}{F}} + \underset{\sim}{Q}^T \cdot \dot{\underset{\sim}{Q}} \cdot \underset{\sim}{F} \right) \cdot \left(\underset{\sim}{F}^{-1} \cdot \underset{\sim}{Q}^T \right) = \underset{\sim}{Q} \cdot \left(\underbrace{\dot{\underset{\sim}{F}} \cdot \underset{\sim}{F}^{-1}}_{\equiv \underset{\sim}{L}} + \underset{\sim}{Q}^T \cdot \dot{\underset{\sim}{Q}} \right) \cdot \underset{\sim}{Q}^T$$

$\underset{\sim}{F} \cdot \underset{\sim}{F}^{-1} = \underset{\sim}{I}$

$$\underset{\sim}{L}^* = \underset{\sim}{Q} \cdot \left(\underset{\sim}{L} + \underset{\sim}{Q}^T \cdot \dot{\underset{\sim}{Q}} \right) \cdot \underset{\sim}{Q}^T = \underset{\sim}{Q} \cdot \left(\overbrace{\underset{\sim}{D} + \underset{\sim}{W}}^{\equiv \underset{\sim}{L}} + \underset{\sim}{Q}^T \cdot \dot{\underset{\sim}{Q}} \right) \cdot \underset{\sim}{Q}^T$$

$$\text{BUT} \quad \underset{\sim}{L}^* = \underset{\sim}{D}^* + \underset{\sim}{W}^* = \underbrace{\underset{\sim}{Q} \cdot \underset{\sim}{D} \cdot \underset{\sim}{Q}^T}_{\underset{\sim}{D}^*} + \underbrace{\underset{\sim}{Q} \cdot \left(\underset{\sim}{W} + \underset{\sim}{Q}^T \cdot \dot{\underset{\sim}{Q}} \right) \cdot \underset{\sim}{Q}^T}_{\underset{\sim}{W}^*}$$

$$\Rightarrow \boxed{\underset{\sim}{D}^* = \underset{\sim}{Q} \cdot \underset{\sim}{D} \cdot \underset{\sim}{Q}^T}$$ RATE OF DEFORMATION TENSOR IS FRAME INDIFFERENT

$$\boxed{\underset{\sim}{W}^* = \underset{\sim}{Q} \cdot \left(\underset{\sim}{W} + \underbrace{\underset{\sim}{Q}^T \cdot \dot{\underset{\sim}{Q}}}_{\text{MUST BE ANTI-SYMMETRIC}} \right) \cdot \underset{\sim}{Q}^T}$$

VORTICITY TENSOR IS NOT FRAME INDIFFERENT
$\underset{\sim}{W}^* \neq \underset{\sim}{Q} \cdot \underset{\sim}{W} \cdot \underset{\sim}{Q}^T$

IN THE PRECEDING DEVELOPMENT

$$\underset{\sim}{W}^* = \underset{\sim}{Q} \cdot (\underset{\sim}{W} + \underset{\sim}{Q}^T \cdot \dot{\underset{\sim}{Q}}) \cdot \underset{\sim}{Q}^T = \underset{\sim}{Q} \cdot \underset{\sim}{W} \cdot \underset{\sim}{Q}^T + \underbrace{\dot{\underset{\sim}{Q}} \cdot \underset{\sim}{Q}^T}^{\text{MUST BE ANTI-SYMMETRIC}}$$

HERE $\dot{\underset{\sim}{Q}} \cdot \underset{\sim}{Q}^T$ = ANGULAR VELOCITY TENSOR OF THE STARRED (*) FRAME RELATIVE TO THE UNSTARRED FRAME

$= -\underset{\sim}{Q} \cdot \dot{\underset{\sim}{Q}}^T$ ANTI-SYMMETRIC

= ADDITIONAL COMPONENT OF VORTICITY SEEN BY OBSERVER IN $\underset{\sim}{x}^*$ DUE TO RELATIVE ROTATION BETWEEN OBSERVERS

<u>EXAMPLE</u>: IS THE GREEN'S STRAIN RATE MATERIAL FRAME INDIFFERENT?

$\dot{\underset{\sim}{E}} = \underset{\sim}{F}^T \cdot \underset{\sim}{D} \cdot \underset{\sim}{F}$ GREEN'S STRAIN RATE

$\Rightarrow$ YES! $\dot{\underset{\sim}{E}}$ IS OBJECTIVE (MATERIAL FRAME INDIFFERENT) SINCE IT IS BASED ON <u>LAGRANGIAN</u> COORDINATES.

EXAMPLE: OBJECTIVITY OF THE TIME RATE OF CHANGE OF THE CAUCHY STRESS TENSOR AND THE JAUMANN RATE.

CONSIDER THE CAUCHY STRESS TENSOR AS SEEN BY AN OBSERVER IN $\underset{\sim}{x}^*$

$$\underset{\sim}{\sigma}^* = \underset{\sim}{Q} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{Q}^T$$

CAUCHY STRESS TENSOR IS OBJECTIVE (MATERIAL FRAME INDIFFERENT)

EVALUATE MATERIAL TIME DERIVATIVE

$$\dot{\underset{\sim}{\sigma}}^* = \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}} \cdot \underset{\sim}{Q}^T + \dot{\underset{\sim}{Q}} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{Q}^T + \underset{\sim}{Q} \cdot \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{Q}}^T$$

CAUCHY STRESS RATE IS NOT MATERIAL FRAME INDIFFERENT $\dot{\underset{\sim}{\sigma}}^* \neq \underset{\sim}{Q} \cdot \dot{\underset{\sim}{\sigma}} \cdot \underset{\sim}{Q}^T$

⇨ IN ORDER TO DEVELOP AN OBJECTIVE STRESS RATE MEASURE FOR USE IN CONSTITUTIVE LAW DEVELOPMENT, MUST AUGMENT THE PRECEDING EXPRESSION WITH ADDITIONAL ROTATIONAL TERMS. FROM THE PRECEDING EXAMPLE

$$\underset{\sim}{W}^* = \underset{\sim}{Q} \cdot \underset{\sim}{W} \cdot \underset{\sim}{Q}^T + \dot{\underset{\sim}{Q}} \cdot \underset{\sim}{Q}^T$$

VORTICITY TENSOR

OR

$$\dot{\underset{\sim}{Q}} \cdot \underset{\sim}{Q}^T = \underset{\sim}{W}^* - \underset{\sim}{Q} \cdot \underset{\sim}{W} \cdot \underset{\sim}{Q}^T$$

TAKE INNER PRODUCT WITH $\underset{\sim}{Q}$

$$\dot{\underset{\sim}{Q}} = \underset{\sim}{W}^* \cdot \underset{\sim}{Q} - \underset{\sim}{Q} \cdot \underset{\sim}{W}$$

NOTE: $(\underset{\sim}{A} \cdot \underset{\sim}{B})^T = \underset{\sim}{B}^T \cdot \underset{\sim}{A}^T$

$$\dot{\underset{\sim}{Q}}^T = \underset{\sim}{Q}^T \cdot \underbrace{\underset{\sim}{W}^{*T}}_{\equiv -\underset{\sim}{W}^*} - \underbrace{\underset{\sim}{W}^T}_{\equiv -\underset{\sim}{W}} \cdot \underset{\sim}{Q}^T = -\underset{\sim}{Q}^T \cdot \underset{\sim}{W}^* + \underset{\sim}{W} \cdot \underset{\sim}{Q}^T$$

★ ★ SINCE VORTICITY TENSOR IS ANTI-SYMMETRIC

SUBSTITUTE EXPRESSIONS FOR $\mathring{\boldsymbol{Q}}$ AND $\mathring{\boldsymbol{Q}}^T$ INTO CAUCHY STRESS RATE

$$\mathring{\boldsymbol{\sigma}}^* = \boldsymbol{Q}\cdot\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{Q}^T + \mathring{\boldsymbol{Q}}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^T + \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\mathring{\boldsymbol{Q}}^T \qquad \text{CAUCHY STRESS RATE}$$

$$\dot{\boldsymbol{\sigma}}^* = \boldsymbol{Q}\cdot\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{Q}^T + \underbrace{(\boldsymbol{W}^*\cdot\boldsymbol{Q} - \boldsymbol{Q}\cdot\boldsymbol{W})}_{\dot{\boldsymbol{Q}}}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^T + \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\underbrace{(-\boldsymbol{Q}^T\cdot\boldsymbol{W}^* + \boldsymbol{W}\cdot\boldsymbol{Q}^T)}_{\dot{\boldsymbol{Q}}^T}$$

$$= \boldsymbol{Q}\cdot\dot{\boldsymbol{\sigma}}\cdot\boldsymbol{Q}^T + \boldsymbol{W}^*\cdot\underbrace{\boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^T}_{\equiv\,\boldsymbol{\sigma}^*} - \boldsymbol{Q}\cdot\boldsymbol{W}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^T - \underbrace{\boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^T}_{\equiv\,\boldsymbol{\sigma}^*}\cdot\boldsymbol{W}^* + \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{W}\cdot\boldsymbol{Q}^T$$

COLLECT TERMS

$$\mathring{\boldsymbol{\sigma}}^* = \boldsymbol{Q}\cdot(\dot{\boldsymbol{\sigma}} - \boldsymbol{W}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\boldsymbol{W})\cdot\boldsymbol{Q}^T + \boldsymbol{W}^*\cdot\boldsymbol{\sigma}^* - \boldsymbol{\sigma}^*\cdot\boldsymbol{W}^*$$

REARANGING

$$\underbrace{\mathring{\boldsymbol{\sigma}}^* - \boldsymbol{W}^*\cdot\boldsymbol{\sigma}^* + \boldsymbol{\sigma}^*\cdot\boldsymbol{W}^*}_{\equiv\,\overset{\triangledown}{\boldsymbol{\sigma}}^*} = \boldsymbol{Q}\cdot\underbrace{(\dot{\boldsymbol{\sigma}} - \boldsymbol{W}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\boldsymbol{W})}_{\equiv\,\overset{\triangledown}{\boldsymbol{\sigma}}}\cdot\boldsymbol{Q}^T$$

$$\boxed{\overset{\triangledown}{\boldsymbol{\sigma}}^* = \boldsymbol{Q}\cdot\overset{\triangledown}{\boldsymbol{\sigma}}\cdot\boldsymbol{Q}^T}$$

<u>JAUMANN</u> <u>STRESS</u> <u>RATE</u> <u>IS</u> <u>FRAME</u> <u>INDIFFERENT</u>

$$\boxed{\overset{\triangledown}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{W}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\boldsymbol{W}}$$

<u>JAUMANN</u> <u>STRESS</u> <u>RATE</u>

(<u>CO-ROTATIONAL</u> <u>STRESS</u> <u>RATE</u>)

$\boldsymbol{W}$ = VORTICITY (SPIN) TENSOR

$\Rightarrow$ $\overset{\triangledown}{\boldsymbol{\sigma}}$ IS IMPORTANT IN DEVELOPING VISCOELASTIC AND VISCOPLASTIC CONSTITUTIVE EQUATIONS.

EXAMPLE: NAVIER POISSON LAW FOR A NEWTONIAN FLUID WITH NO BULK VISCOSITY

$$S_{ij} = \underbrace{\sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij}}_{\boldsymbol{S} \,=\, \text{DEVIATORIC STRESS}} = 2\mu\Big(\underbrace{D_{ij} - \frac{D_{kk}}{3}\delta_{ij}}_{\text{DEVIATORIC PART OF RATE OF DEFORMATION TENSOR}}\Big) = 2\mu\,\beta_{ij}$$

OR $\boldsymbol{S} = 2\mu\boldsymbol{\beta}$ NAVIER - POISSON LAW FOR NEWTONIAN FLUID

μ = SHEAR VISCOSITY COEFFICIENT (DYNAMIC VISCOSITY)

SINCE $\boldsymbol{\sigma}^* = \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^T \Rightarrow \boldsymbol{S}^* = \boldsymbol{Q}\cdot\boldsymbol{S}\cdot\boldsymbol{Q}^T$

AND

$\boldsymbol{D}^* = \boldsymbol{Q}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{Q}^T \Rightarrow \boldsymbol{\beta}^* = \boldsymbol{Q}\cdot\boldsymbol{\beta}\cdot\boldsymbol{Q}^T$

$\Rightarrow$ CONSTITUTIVE LAW $\boldsymbol{S} = \mu\boldsymbol{\beta}$ IS FRAME INDIFFERENT

EXAMPLE:

CONSIDER LAGRANGIAN CONSTITUTIVE EQUATION

$$\sigma_{ij}^{PK(2)} = C_{ijkl}\,E_{kl}$$

BY DEFINITION FRAME INDIFFERENT (LAGRANGIAN)

LOOKING AT A TIME-DEPENDENT RELATION USING AN EULERIAN (SPATIAL) DESCRIPTION

$$\underbrace{\overset{\circ}{\sigma}_{ij}}_{\text{NOT FRAME INDIFFERENT}} = C_{ijkl}\underbrace{D_{kl}}_{\text{FRAME INDIFFERENT}}$$

INVALID CONSTITUTIVE RELATIONSHIP

BETTER CHOICE: $\overset{\nabla}{\sigma}_{ij} = C_{ijkl}\,D_{kl}$ FRAME INDIFFERENT CONSTITUTIVE EQUATION

RESTRICTIONS ON CONSTITUTIVE LAWS BY THE 2^{ND} LAW OF THERMODYNAMICS

IN ORDER TO SPECIFY A THERMODYNAMIC PROCESS, A NUMBER OF THERMODYNAMIC FUNCTIONS, INVOLVING BOTH MECHANICAL AND THERMODYNAMIC QUANTITIES, MUST BE SPECIFIED. THESE INCLUDE

1) SPATIAL POSITION: $\underset{\sim}{x} = \hat{\underset{\sim}{x}}(\underset{\sim}{X}, t)$

   $\Rightarrow$ USED TO CHARACTERIZE DEFORMATION, MOTION, FLOW, ETC.

2) DENSITY: $\rho = \hat{\rho}(\underset{\sim}{X}, t)$
3) STRESS TENSOR: $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}(\underset{\sim}{X}, t)$
4) BODY FORCE PER UNIT MASS: $\underset{\sim}{b} = \hat{\underset{\sim}{b}}(\underset{\sim}{X}, t)$ } USUALLY PRESCRIBED
5) SPECIFIC HEAT SUPPLY: $r = \hat{r}(\underset{\sim}{X}, t)$ } USUALLY PRESCRIBED
6) SPECIFIC INTERNAL ENERGY: $u = u(\underset{\sim}{X}, t)$
7) HEAT FLUX VECTOR: $\underset{\sim}{q} = \hat{\underset{\sim}{q}}(\underset{\sim}{X}, t)$
8) SPECIFIC ENTROPY: $\eta = \hat{\eta}(\underset{\sim}{X}, t)$
9) THERMODYNAMIC TEMPERATURE: $\theta = \hat{\theta}(\underset{\sim}{X}, t)$ (ALWAYS POSITIVE)

OF COURSE, THESE FUNCTIONS MUST SATISFY THE BALANCE EQUATIONS AND C-D INEQUALITY

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{v}) = 0 \qquad \text{CONTINUITY EQUATION}$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{\sigma} + \rho \underset{\sim}{b} = \rho \dot{\underset{\sim}{v}} \quad ; \quad \underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T \qquad \text{BALANCE OF LINEAR AND ANGULAR MOMENTUM}$$

$$\rho \dot{u} = \underset{\sim}{\sigma} : \underset{\sim}{D} + \rho r - \underset{\sim}{\nabla} \cdot \underset{\sim}{q} \qquad \text{CONSERVATION OF ENERGY}$$

$$\rho \theta \dot{\eta} - \rho \dot{u} + \underset{\sim}{D} : \underset{\sim}{\sigma} - \frac{1}{\theta} \underset{\sim}{q} \cdot \underset{\sim}{\nabla} \theta \geq 0 \qquad \text{C-D INEQUALITY}$$

IN ADDITION, THE STANDARD KINEMATIC AND COMPATIBILITY RELATIONSHIPS APPLY,

$\underset{\sim}{L} = \underset{\sim}{v} \otimes \overleftarrow{\underset{\sim}{\nabla}}$, $\underset{\sim}{E} = \frac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I})$, etc...

ONE OF THE PRINCIPLE USES OF THE 2ND LAW OF THERMODYNAMICS IS TO INFER RETRICTIONS ON THE FUNCTIONAL FORM OF THE CONSTITUTIVE RESPONSES.

CONSIDER AN ELASTIC* SOLID* MATERIAL AND NOTE THAT THE HELMHOLTZ FREE ENERGY MAY BE EXPRESSED AS $\psi = u - \eta\theta$. THEN THE C-D INEQUALITY MAY BE RE-EXPRESSED AS

$$\underbrace{-\rho\dot{\psi} - \rho\eta\dot{\theta} + \underset{\sim}{\sigma}:\underset{\sim}{D}}_{\text{LOCAL INTERNAL DISSIPATION}} \underbrace{- \frac{1}{\theta}\underset{\sim}{q}\cdot\underset{\sim}{\nabla}\theta}_{\text{DISSIPATION DUE TO HEAT CONDUCTION}} \geq 0 \qquad \text{LOCAL DISSIPATION INEQUALITY}$$

ASIDE: THE STRESS POWER TERM COULD HAVE EASILY BEEN EXPRESSED AS $\underset{\sim}{\sigma}:\underset{\sim}{L} = \underset{\sim}{\sigma}:(\underset{\sim}{D} + \underset{\sim}{W}) = \underset{\sim}{\sigma}:\underset{\sim}{D} + \cancelto{0}{\underset{\sim}{\sigma}:\underset{\sim}{W}}$ SINCE $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^T$ AND THE VORTICITY TENSOR, $\underset{\sim}{W}$, IS ANTI-SYMMETRIC.

THE PRECEDING EXPRESSION SUGGESTS THAT THE RESPONSE FUNCTIONS $\psi, \eta, \underset{\sim}{\sigma}$, AND $\underset{\sim}{q}$ SHOULD BE EXPRESSED IN TERMS OF THE TEMPERATURE (θ), TEMPERATURE GRADIENT ($\underset{\sim}{\nabla}\theta$), AND AN APPROPRIATE KINEMATIC VARIABLE. INVOKING THE PRINCIPLE OF EQUIPRESENCE, THIS SUGGESTS

$$\eta = \hat{\eta}(\underset{\sim}{F}, \theta, \underset{\sim}{\nabla}\theta) \qquad \underset{\sim}{q} = \hat{\underset{\sim}{q}}(\underset{\sim}{F}, \theta, \underset{\sim}{\nabla}\theta)$$

$$\underset{\sim}{\sigma} = \hat{\underset{\sim}{\sigma}}(\underset{\sim}{F}, \theta, \underset{\sim}{\nabla}\theta) \qquad \psi = \hat{\psi}(\underset{\sim}{F}, \theta, \underset{\sim}{\nabla}\theta)$$

HERE THE DEFORMATION GRADIENT TENSOR ($\underset{\sim}{F}$) IS SELECTED AS THE KINEMATIC VARIABLE OF CHOICE. THEN THE RESPONSE FUNCTIONS $\eta, \underset{\sim}{q}, \underset{\sim}{\sigma}$, AND ψ MAY ALL BE EXPRESSED IN TERMS OF THE STATE VARIABLES $\underset{\sim}{F}, \theta$, AND $\underset{\sim}{\nabla}\theta$.

IMPORTANT: RECALL THAT THE C-D INEQUALITY MAY BE DECOMPOSED INTO A LOCAL INTRINSIC ENTROPY PRODUCTION TERM PLUS AN ENTROPY PRODUCTION TERM ASSOCIATED WITH HEAT CONDUCTION, i.e.,

$$\rho\gamma_{LOCAL} + \rho\gamma_{CONDUCTION} \geq 0$$

OR

$$\underbrace{\rho\dot{\eta} - \frac{\rho r}{\theta} + \frac{1}{\theta}\nabla\cdot q}_{\rho\gamma_{LOCAL}} \quad \underbrace{-\frac{1}{\theta^2} q\cdot\nabla\theta}_{\rho\gamma_{CONDUCTION}} \geq 0$$

GENERAL FORM FOR 2ND LAW OF THERMODYNAMICS

THEN

$$\boxed{\begin{array}{ll} \rho\gamma_{LOCAL} & \geq 0 \\ \rho\gamma_{CONDUCTION} & \geq 0 \end{array}}$$

STRONG FORM OF C-D INEQUALITY

$\rho\gamma_{LOCAL} = 0$ FOR AN ELASTIC (REVERSIBLE) PROCESS

COMPARE THE GENERAL EXPRESSION FOR THE C-D INEQUALITY WITH THAT OBTAINED USING $\rho\dot{u} - \sigma : D = \rho r - \nabla\cdot q = \rho\frac{d}{dt}(\psi + \eta\theta) - \sigma : D = \rho[\dot{\psi} + \dot{\eta}\theta + \eta\dot{\theta}] - \sigma : D$

$$\underbrace{-\rho\dot{\psi} - \rho\eta\dot{\theta} + \sigma : D}_{\substack{\text{INTERNAL DISSIPATION} \\ = \theta\rho\gamma_{LOCAL}}} \quad \underbrace{-\frac{1}{\theta} q\cdot\nabla\theta}_{\substack{\text{DISSIPATION DUE} \\ \text{TO HEAT CONDUCTION} \\ = \theta\rho\gamma_{CONDUCTION}}} \geq 0$$

LOCAL DISSIPATION INEQUALITY

THEN

$$\boxed{\begin{array}{ll} -\rho\dot{\psi} - \rho\eta\dot{\theta} + \sigma : D & \geq 0 \\ -\frac{1}{\theta} q\cdot\nabla\theta & \geq 0 \end{array}}$$

INTRINSIC DISSIPATION

DISSIPATION DUE TO CONDUCTION

STRONG FORM OF C-D INEQUALITY

⇒ FOR AN ELASTIC (REVERSIBLE) PROCESS, THE EQUALITY (=) HOLDS.

FOR CONVENIENCE, ONE MAY DEFINE THE TEMPERATURE GRADIENT AS

$$\boxed{\underset{\sim}{g} = \underset{\sim}{\nabla}\theta} \quad \text{OR} \quad \boxed{g_i = \theta_{,i}} \quad \underline{\text{TEMPERATURE}}\ \underline{\text{GRADIENT}}$$

AND

$$\eta = \hat{\eta}(\underset{\sim}{F}, \theta, \underset{\sim}{\nabla}\theta) = \hat{\eta}(\underset{\sim}{F}, \theta, \underset{\sim}{g})$$

$$\underset{\sim}{q} = \hat{\underset{\sim}{q}}(\underset{\sim}{F}, \theta, \underset{\sim}{\nabla}\theta) = \hat{q}(\underset{\sim}{F}, \theta, \underset{\sim}{g})$$

$$\underset{\sim}{\sigma} = \hat{\underset{\sim}{\sigma}}(\underset{\sim}{F}, \theta, \underset{\sim}{\nabla}\theta) = \hat{\underset{\sim}{\sigma}}(\underset{\sim}{F}, \theta, \underset{\sim}{g})$$

$$\psi = \hat{\psi}(\underset{\sim}{F}, \theta, \underset{\sim}{\nabla}\theta) = \hat{\psi}(\underset{\sim}{F}, \theta, \underset{\sim}{g})$$

⇨ WRITE RESPONSE FUNCTIONS IN TERMS OF STATE VARIABLES, INVOKING PRINCIPLE OF EQUIPRESENCE

THEN THE <u>STRONG</u> <u>FORM</u> OF THE <u>LOCAL</u> <u>DISSIPATION</u> <u>INEQUALITY</u> FOR AN <u>ELASTIC</u>* (<u>REVERSIBLE</u>) <u>PROCESS</u> MAY BE EXPRESSED AS

⇨ EQUALITY HOLDS FOR ELASTIC SOLID (NO INTRINSIC DISSIPATION)

$$\boxed{\begin{array}{l} -\rho\dot{\psi} - \rho\eta\dot{\theta} + \underbrace{\underset{\sim}{\sigma} : \underset{\sim}{L}}_{= \underset{\sim}{\sigma}:\underset{\sim}{D}} = 0 \\ \\ -\dfrac{1}{\theta}\underset{\sim}{q}\cdot\underset{\sim}{\nabla}\theta = -\dfrac{1}{\theta}\underset{\sim}{q}\cdot\underset{\sim}{g} \geq 0 \end{array}}$$

<u>STRONG</u> <u>FORM</u> <u>OF</u> <u>DISSIPATION</u> <u>INEQUALITY</u>

BUT

$$\dot{\psi} = \frac{d}{dt}\hat{\psi}(\underset{\sim}{F}, \theta, \underset{\sim}{\nabla}\theta) = \frac{d}{dt}\hat{\psi}(\underset{\sim}{F}, \theta, \underset{\sim}{g}) = \overbrace{\frac{\partial\psi}{\partial F_{iK}}\dot{F}_{iK} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial g_i}\dot{g}_i}^{\text{USING CHAIN RULE}}$$

$$= \frac{\partial\psi}{\partial F_{iK}}\underbrace{L_{ij}F_{jK}}_{\dot{F}_{iK}} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial g_i}\dot{g}_i$$

FROM $\dot{\underset{\sim}{F}} = \underset{\sim}{L}\cdot\underset{\sim}{F}$

THEN THE EQUALITY FOR THE <u>INTRINSIC DISSIPATION</u> MAY BE RE-WRITTEN AS

$$-\rho\dot{\psi} - \rho\eta\dot{\theta} + \underset{\sim}{\sigma}:\underset{\sim}{L} = 0$$

$$-\rho\left[\frac{\partial\psi}{\partial F_{iK}}F_{jK}L_{ij} + \frac{\partial\psi}{\partial\theta}\dot{\theta} + \frac{\partial\psi}{\partial g_i}\dot{g}_i\right] - \rho\eta\dot{\theta} + \sigma_{ij}L_{ij} = 0$$

<u>NO INSTRINSIC DISSIPATION</u> (<u>REVERSIBLE</u> <u>PROCESS</u>)

COLLECTING LIKE TERMS

$$\left[\sigma_{ij} - \rho \frac{\partial \psi}{\partial F_{iK}} F_{jK}\right] L_{ij} - \rho\left[\eta + \frac{\partial \psi}{\partial \theta}\right]\dot{\theta} - \rho \frac{\partial \psi}{\partial g_i}\dot{g}_i = 0$$

NO INTRINSIC DISSIPATION FOR ELASTIC PROCESS.

THIS EQUALITY MUST HOLD TRUE FOR ARBITRARY $\underset{\sim}{L}$, $\dot{\theta}$, AND $\dot{\underset{\sim}{g}}$. THIS SUGGESTS THE COEFFICIENTS OF $\underset{\sim}{L}$, $\dot{\theta}$, AND $\dot{\underset{\sim}{g}}$ MUST BE INDEPENDENTLY EQUAL TO ZERO, i.e.,

$$\boxed{\begin{aligned} \sigma_{ij} &= \rho \frac{\partial \psi}{\partial F_{iK}} F_{jK} \\ \rho \eta &= -\rho \frac{\partial \psi}{\partial \theta} \\ \rho \frac{\partial \psi}{\partial g_i} &= 0 \end{aligned}}$$

⇨ CAUCHY STRESS IS DERIVABLE FROM HELMHOLTZ FREE ENERGY

⇨ ENTROPY IS DERIVABLE FROM HELMHOLTZ FREE ENERGY

⇨ *HELMHOLTZ FREE ENERGY IS <u>NOT</u> A FUNCTION OF TEMPERATURE GRADIENT, $\underset{\sim}{g} = \underset{\sim}{\nabla}\theta$

THEN FROM THE 2^{ND} LAW OF THERMODYNAMICS FOR AN ELASTIC SOLID

$$\psi = \hat{\psi}(\underset{\sim}{F}, \theta)$$

HELMHOLTZ FREE ENERGY DOES <u>NOT</u> DEPEND ON TEMPERATURE GRADIENT

AND

$$-\underset{\sim}{q} \cdot \underset{\sim}{g} = -\underset{\sim}{q} \cdot \underset{\sim}{\nabla}\theta \geq 0$$

DISSIPATION DUE TO HEAT CONDUCTION

ASSUMING FOURIER'S LAW OF HEAT CONDUCTION $\underset{\sim}{q} = -\kappa \underset{\sim}{\nabla}\theta$, THE LATTER INEQUALITY MAY BE EXPRESSED AS

$$-(-\kappa \underset{\sim}{\nabla}\theta \cdot \underset{\sim}{\nabla}\theta) \geq 0$$

$$\kappa\, \theta_{,i}\theta_{,i} \geq 0$$

ALWAYS POSITIVE FOR NON-ZERO κ

$$\underset{\sim}{q} = \hat{\underset{\sim}{q}}(\underset{\sim}{\nabla}\theta)$$

HEAT FLUX DEPENDS ON TEMPERATURE GRADIENT BUT NOT $\underset{\sim}{F}$

⇨ PRINCIPLE OF EQUIPRESENCE VIOLATED FOR ASSUMED CONSTITUTIVE RELATION

LINEAR ELASTICITY

SUPPOSE THAT THE HELMHOLTZ FREE ENERGY DEPENDS ON GREEN'S STRAIN AND TEMPERATURE, i.e.,

$$\psi = \hat{\psi}(\underbrace{v_i, \theta}_{\text{STATE VARIABLES}}) = \hat{\psi}(\underset{\sim}{E}, \theta) \qquad \underline{\text{HELMHOLTZ}}\ \underline{\text{FREE}}\ \underline{\text{ENERGY}}$$

IN ADDITION, ASSUME THAT THERE IS <u>NO</u> <u>INTERNAL</u> <u>DISSIPATION</u>, i.e., THE PROCESS IS <u>REVERSIBLE</u>. FROM THE $2^{\underline{ND}}$ LAW OF THERMODYNAMICS THIS SUGGESTS THAT $\rho\gamma_{LOCAL} = 0$, WHERE

$$\underbrace{\rho\dot{\eta} - \frac{\rho r}{\theta} + \frac{1}{\theta}\underset{\sim}{\nabla}\cdot\underset{\sim}{q}}_{\substack{\rho\gamma_{LOCAL} \\ =\text{ INTRINSIC ENTROPY} \\ \text{PRODUCTION}}} \underbrace{- \frac{1}{\theta^2}\underset{\sim}{q}\cdot\underset{\sim}{\nabla}\theta}_{\substack{\rho\gamma_{CONDUCTION} \\ =\text{ ENTROPY PRODUCTION} \\ \text{DUE TO CONDUCTION}}} \geq 0 \qquad \text{C-D INEQUALITY}$$

THEN THE STRONG FORM OF THE $2^{\underline{ND}}$ LAW OF THERMODYNAMICS FOR A <u>REVERSIBLE</u> PROCESS BECOMES

EQUALITY HOLDS FOR REVERSIBLE PROCESS

$$\left.\begin{aligned} \rho\dot{\eta} - \frac{\rho r}{\theta} + \frac{1}{\theta}\underset{\sim}{\nabla}\cdot\underset{\sim}{q} &= \rho\gamma_{LOCAL} = \underline{\underline{0}} \\ -\frac{1}{\theta^2}\underset{\sim}{q}\cdot\underset{\sim}{\nabla}\theta &= \rho\gamma_{CONDUCTION} \geq 0 \end{aligned}\right\} \quad \underline{\text{STRONG}}\ \underline{\text{FORM}}\ \underline{\text{OF}}\ \underline{\text{C-D}}\ \underline{\text{INEQUALITY}}$$

CAN STILL HAVE ENTROPY PRODUCTION DUE TO CONDUCTION

NOW CONSIDER THE ENERGY EQUATION (1ST LAW OF THERMODYNAMICS)

$$\rho \dot{u} = \underbrace{\rho r - \nabla \cdot q}_{= \rho \theta \dot{\eta}} + \sigma : D \qquad \text{ENERGY EQUATION}$$

FROM $\rho \gamma_{LOCAL} = \rho \dot{\eta} - \frac{\rho r}{\theta} + \frac{1}{\theta} \nabla \cdot q \overset{\text{REVERSIBLE}}{=} 0$

OR

$$\sigma : D = \rho \dot{u} - \rho \theta \dot{\eta}$$

ENERGY EQUATION FOR A REVERSIBLE PROCESS

BUT $u = \psi + \eta \theta$

AND $\dot{u} = \dot{\psi} + \eta \dot{\theta} + \theta \dot{\eta}$

u = INTERNAL ENERGY

ψ = HELMHOLTZ FREE ENERGY

η = ENTROPY

$$\sigma : D = \underbrace{\rho \dot{\psi} + \rho \eta \dot{\theta} + \rho \theta \dot{\eta}}_{\rho \dot{u}} - \rho \theta \dot{\eta}$$

THEN

$$\sigma : D = \rho \dot{\psi} + \rho \eta \dot{\theta}$$

ENERGY EQUATION FOR A REVERSIBLE PROCESS

NOW INTEGRATE THE PRECEDING EQUATION OVER AN ARBITRARY SPATIAL (EULERIAN) VOLUME, V.

$$\int_V \sigma : D \, dV = \int_V (\rho \dot{\psi} + \rho \eta \dot{\theta}) \, dV$$

ENERGY EQUATION (GLOBAL FORM) FOR A REVERSIBLE PROCESS

NOTE: THE GLOBAL FORM OF THE ENERGY EQUATION FOR A REVERSIBLE PROCESS MAY BE EXPRESSED IN LAGRANGIAN COORDINATES, i.e.,

$$\underbrace{\int_V \sigma : D \, dV}_{\substack{\text{STRESS}\\ \text{POWER}\\ \text{(EULERIAN)}}} = \underbrace{\int_{V_0} \overset{PK(2)}{\sigma} : \dot{E} \, dV_0}_{\substack{\text{STRESS POWER}\\ \text{(LAGRANGIAN)}}} = \int_{V_0} \rho (\dot{\psi} + \eta \dot{\theta}) \underbrace{J \, dV_0}_{\equiv dV}$$

BUT $\rho_0 = J\rho$ LAGRANGIAN DENSITY! THEN IN TERMS OF LAGRANGIAN COORDINATES

$$\int_{V_0} \underset{\sim}{\sigma}^{PK(2)} : \dot{\underset{\sim}{E}}\, dV_0 - \int_{V_0} \overbrace{\rho_0}^{= J\rho} [\dot{\psi} + \eta\dot{\theta}]\, dV_0 = 0 \qquad \text{ENERGY EQUATION}$$

NOTING THAT V_0 IS ARBITRARY

$$\underset{\sim}{\sigma}^{PK(2)} : \dot{\underset{\sim}{E}} - \rho_0\dot{\psi} - \rho_0\eta\dot{\theta} = 0$$

<u>LOCAL</u> <u>FORM</u> <u>OF</u> <u>ENERGY</u> <u>EQUATION</u> <u>FOR</u> <u>REVERSIBLE</u> <u>PROCESS</u>

NOW, IF $\psi = \hat{\psi}(\underset{\sim}{E}, \theta)$, THEN

$$\dot{\psi} = \frac{\partial \psi}{\partial \underset{\sim}{E}} : \dot{\underset{\sim}{E}} + \frac{\partial \psi}{\partial \theta}\dot{\theta}$$

<u>TIME</u> <u>RATE</u> <u>OF</u> <u>CHANGE</u> <u>OF</u> <u>HELMHOLTZ</u> <u>FREE</u> <u>ENERGY</u>

AND THE ENERGY EQUATION BECOMES

$$\underset{\sim}{\sigma}^{PK(2)} : \dot{\underset{\sim}{E}} - \rho_0 \frac{\partial \psi}{\partial \underset{\sim}{E}} : \dot{\underset{\sim}{E}} - \rho_0 \frac{\partial \psi}{\partial \theta}\dot{\theta} - \rho_0 \eta \dot{\theta} = 0$$

SIMPLIFYING

$$\left[\underset{\sim}{\sigma}^{PK(2)} - \rho_0 \frac{\partial \psi}{\partial \underset{\sim}{E}}\right] : \dot{\underset{\sim}{E}} - \rho_0 \left[\frac{\partial \psi}{\partial \theta} + \eta\right]\dot{\theta} = 0$$

SPECIAL FORM OF ENERGY EQUATION FOR A REVERSIBLE PROCESS

FOR ARBITRARY $\dot{\underset{\sim}{E}}$ AND $\dot{\theta}$, THE TERMS IN BRACKETS $[\cdot]$ MUST BE INDEPENDENTLY EQUAL TO ZERO, i.e.,

$$\boxed{\underset{\sim}{\sigma}^{PK(2)} = \rho_0 \frac{\partial \psi}{\partial \underset{\sim}{E}}} \qquad \boxed{\eta = -\frac{\partial \psi}{\partial \theta}}$$

COMPARE TO $\psi = \hat{\psi}(\underset{\sim}{F}, \theta)$ WITH

$$\sigma_{ij} = \rho \frac{\partial \psi}{\partial F_{iK}} F_{jK}$$

{ONE POSSIBLE EULERIAN FORM}

⇨ <u>HELMHOLTZ</u> <u>FREE</u> <u>ENERGY</u> MAY BE REGARDED AS <u>ELASTIC</u> <u>POTENTIAL</u> <u>ENERGY</u>!

IMPORTANT: FOR AN ISOTHERMAL REVERSIBLE PROCESS, THE HELMHOLTZ FREE ENERGY MAY BE EXPRESSED SOLELY AS A FUNCTION OF GREEN'S STRAIN, i.e.,

$$\psi = \hat{\psi}(\underset{\sim}{E})$$

HELMHOLTZ FREE ENERGY FOR ISOTHERMAL ELASTICITY (REVERSIBLE PROCESS)

ONE MAY EXPAND THE FREE ENERGY ABOUT $\underset{\sim}{E} = \underset{\sim}{0}$ (UNSTRAINED CONFIGURATION) USING A TAYLOR SERIES EXPANSION, i.e., $f(x+\Delta x) = f(x) + f'(x)\,\Delta x + \frac{1}{2} f''(x)\,\Delta x^2 + \ldots$

$$\psi = \psi_0 + \left.\frac{\partial \psi}{\partial \underset{\sim}{E}}\right|_{\underset{\sim}{E}=\underset{\sim}{0}} : \underset{\sim}{E} + \frac{1}{2}\left(\left.\frac{\partial^2 \psi}{\partial \underset{\sim}{E}^2}\right|_{\underset{\sim}{E}=\underset{\sim}{0}} : \underset{\sim}{E}\right) : \underset{\sim}{E} + \text{H.O. TERMS}$$

OR USING INDEX NOTATION

$$\psi = \psi_0 + \left.\frac{\partial \psi}{\partial E_{ij}}\right|_{\underset{\sim}{E}=\underset{\sim}{0}} E_{ij} + \frac{1}{2}\left(\left.\frac{\partial^2 \psi}{\partial E_{ij}\,\partial E_{kl}}\right|_{\underset{\sim}{E}=0} E_{kl}\right) E_{ij} + \text{H.O. TERMS}$$

BUT $\overset{PK(2)}{\underset{\sim}{\sigma}} = \rho_0 \dfrac{\partial \psi}{\partial \underset{\sim}{E}}$ OR $\overset{PK(2)}{\sigma_{ij}} = \rho_0 \dfrac{\partial \psi}{\partial E_{ij}}$

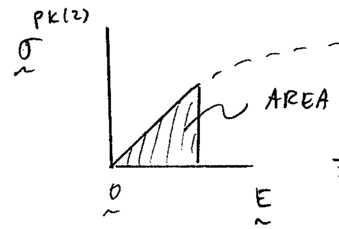
THEN

$$\rho_0 \psi = \rho_0 \psi_0 + \left.\overset{PK(2)}{\underset{\sim}{\sigma}}\right|_{\underset{\sim}{E}=0} : \underset{\sim}{E} + \frac{1}{2}\left(\left.\frac{\partial \overset{PK(2)}{\underset{\sim}{\sigma}}}{\partial \underset{\sim}{E}}\right|_{\underset{\sim}{E}=\underset{\sim}{0}} : \underset{\sim}{E}\right) : \underset{\sim}{E} + \text{H.O. TERMS}$$

OR

$$\rho_0 \psi = \underbrace{\rho_0 \psi_0}_{\text{DATUM}} + \underbrace{\left.\overset{PK(2)}{\sigma_{ij}}\right|_{\underset{\sim}{E}=\underset{\sim}{0}} E_{ij}}_{\text{ENERGY ASSOCIATED WITH RESIDUAL STRESSES}} + \underbrace{\frac{1}{2}\left(\left.\frac{\partial \overset{PK(2)}{\sigma_{ij}}}{\partial E_{kl}}\right|_{\underset{\sim}{E}=\underset{\sim}{0}} E_{kl}\right) E_{ij}}_{\text{STORED ENERGY ASSOCIATED WITH LINEAR STRAINS}} + \underbrace{\text{H.O. TERMS}}_{\text{STORED ENERGY ASSOCIATED WITH LARGE STRAINS}}$$

<u>IMPORTANT</u>: THE TERM $\left.\dfrac{\partial \boldsymbol{\sigma}^{PK(2)}}{\partial \boldsymbol{E}}\right|_{\boldsymbol{E}=\boldsymbol{0}}$ GIVES THE TANGENT STIFFNESS AT $\boldsymbol{E} = \boldsymbol{0}$



AREA UNDER CURVE:

$$\frac{1}{2}\boldsymbol{\sigma}^{PK(2)} : \boldsymbol{E} = \frac{1}{2}\boldsymbol{C} : \boldsymbol{E} : \boldsymbol{E}$$

AT $\boldsymbol{E} = \boldsymbol{0}$,

$$\boldsymbol{\sigma}^{PK(2)} = \boldsymbol{C} : \boldsymbol{E} \quad \text{OR} \quad \sigma_{ij}^{PK(2)} = C_{ijkl} E_{kl}$$

WHERE $\boldsymbol{C}$ (C_{ijkl}) IS THE 4^{TH} RANK <u>TANGENT</u> <u>STIFFNESS</u> <u>TENSOR</u> AT $\boldsymbol{E} = \boldsymbol{0}$

THEN THE FREE ENERGY MAY BE EXPRESSED AS

$$\rho_0 \psi = \underbrace{\rho_0 \psi_0}_{\text{DATUM}} + \underbrace{\left.\boldsymbol{\sigma}^{PK(2)}\right|_{\boldsymbol{E}=\boldsymbol{0}} : \boldsymbol{E}}_{\substack{\text{ENERGY ASSOCIATED} \\ \text{WITH RESIDUAL STRESSES}}} + \underbrace{\frac{1}{2}\Big(\overbrace{\left.\boldsymbol{C}\right|_{\boldsymbol{E}=\boldsymbol{0}} : \boldsymbol{E}}^{\Rightarrow \text{ QUADRATIC FUNCTION OF STRAINS}}\Big) : \boldsymbol{E}}_{\substack{\text{STORED STRAIN ENERGY ASSOCIATED} \\ \text{WITH LINEAR STRAINS}}} + \text{H.O. TERMS}$$

$$\boldsymbol{C} = \frac{\partial^2 \psi}{\partial \boldsymbol{E}^2} = \frac{\partial \boldsymbol{\sigma}^{PK(2)}}{\partial \boldsymbol{E}} \qquad \text{FOURTH RANK TANGENT STIFFNESS FOR } \boldsymbol{E} = \boldsymbol{0}$$

CLEARY, THE LAGRANGIAN STRESS IS GIVEN BY

$$\boldsymbol{\sigma}^{PK(2)} \equiv \rho_0 \frac{\partial \psi}{\partial \boldsymbol{E}} = \left.\boldsymbol{\sigma}^{PK(2)}\right|_{\boldsymbol{E}=\boldsymbol{0}} + \boldsymbol{C} : \boldsymbol{E} + \text{H.O. TERMS}$$

OR

$$\sigma_{ij}^{PK(2)} = \rho_0 \frac{\partial \psi}{\partial E_{ij}} = \underbrace{\left.\sigma_{ij}^{PK(2)}\right|_{\boldsymbol{E}=\boldsymbol{0}}}_{\substack{\text{RESIDUAL} \\ \text{STRESSES}}} + \underbrace{C_{ijkl} E_{kl}}_{\substack{\text{STRESS ASSOCIATED} \\ \text{WITH LINEAR STRAINS}}} + \text{H.O. TERMS}$$

IF THERE ARE <u>NO</u> <u>RESIDUAL</u> <u>STRESSES</u>, i.e., $\underset{\sim}{\sigma}^{PK(2)}\Big|_{\underset{\sim}{E}=\underset{\sim}{0}} = \underset{\sim}{0}$, THEN

$$\underset{\sim}{\sigma}^{PK(2)} = \underset{\approx}{C} : \underset{\sim}{E} + \text{H.O. TERMS}$$

NOW IMPOSE THE RESTRICTION THAT THE <u>DISPLACEMENTS</u> $|u_i|$ AND <u>DISPLACEMENT</u> <u>GRADIENTS</u> $|u_{i,j}|$ ARE <u>INFINITESIMAL</u>. ONE MAY NEGLECT THE EFFECT OF THE HIGHER ORDER TERMS AND NOTE

FOR "SMALL" $|u_i|$ AND $|u_{i,j}|$

$$\left\{ \begin{array}{ll} \underset{\sim}{\sigma}^{PK(2)} = \underset{\sim}{\sigma} & \text{2}^{\text{ND}} \text{ PIOLA-KIRCHHOFF STRESS AND CAUCHY STRESS ARE EQUIVALENT} \\ \underset{\sim}{E} = \underset{\sim}{\epsilon} & \text{GREEN'S STRAIN AND THE INFINITESIMAL STRAIN TENSOR ARE EQUIVALENT} \end{array} \right.$$

THEN THE CONSTITUTIVE EQUATION $\underset{\sim}{\sigma}^{PK(2)} = \underset{\approx}{C} : \underset{\sim}{E}$ + H.O. TERMS MAY BE EXPRESSED AS

$$\boxed{\begin{array}{l} \underset{\sim}{\sigma} = \underset{\approx}{C} : \underset{\sim}{\epsilon} \\[2ex] \sigma_{ij} = C_{ijkl}\, \epsilon_{kl} \end{array}}$$

<u>HOOKE'S</u> <u>LAW</u> <u>FOR</u> <u>ISOTHERMAL</u> <u>INFINITESIMAL</u> <u>LINEAR</u> <u>ELASTICITY</u>

HERE $\psi = \psi(\underset{\sim}{\epsilon})$ FOR <u>ISOTHERMAL</u> <u>INFINITESIMAL</u> <u>ELASTICITY</u>

$$\sigma_{ij} = \rho_0 \frac{\partial \psi}{\partial \epsilon_{ij}} = C_{ijkl}\, \epsilon_{kl}$$ <u>HOOKE'S</u> <u>LAW</u> (NO RESIDUAL STRESSES)

$$C_{ijkl} = \rho_0 \frac{\partial^2 \psi}{\partial \epsilon_{ij}\, \partial \epsilon_{kl}} = \frac{\partial \sigma_{ij}}{\partial \epsilon_{kl}}$$ <u>STIFFNESS</u> <u>TENSOR</u>

DEFINE THE STRAIN ENERGY DENSITY FUNCTION

$$\boxed{W(\underset{\sim}{E}) = \rho_0 \psi(\underset{\sim}{E})}$$

STRAIN ENERGY DENSITY FOR ISOTHERMAL ELASTICITY

⇨ GOOD FOR NON-LINEAR ELASTIC AND LINEAR ELASTIC BEHAVIOR

THEN

$$\boxed{\underset{\sim}{\sigma}^{PK(2)} = \frac{\partial W(\underset{\sim}{E})}{\partial \underset{\sim}{E}}}$$

2ND PIOLA KIRCHHOFF STRESS IS DERIVABLE FROM THE STRAIN ENERGY DENSITY

A GIVEN MATERIAL IS "HYPER-ELASTIC" IF $W(\underset{\sim}{E})$ EXISTS. THIS FORMS THE BASIS OF THE PURELY MECHANICAL THEORY OF ELASTICITY (NO TEMPERATURE EFFECTS)

FOR INFINITESIMAL ISOTHERMAL ELASTICITY:

$$W(\underset{\sim}{\epsilon}) = \underset{\sim}{\sigma}\Big|_{\underset{\sim}{\epsilon}=\underset{\sim}{0}} : \underset{\sim}{\epsilon} + \frac{1}{2}\underset{\sim}{\sigma} : \underset{\sim}{\epsilon} = \underset{\sim}{\sigma}\Big|_{\underset{\sim}{\epsilon}=\underset{\sim}{0}} : \underset{\sim}{\epsilon} + \frac{1}{2}\underset{\approx}{C} : \underset{\sim}{\epsilon} : \underset{\sim}{\epsilon}$$

$$W(\epsilon_{ij}) = \sigma_{ij}\Big|_{\underset{\sim}{\epsilon}=\underset{\sim}{0}} \epsilon_{ij} + \frac{1}{2} C_{ijkl}\,\epsilon_{kl}\,\epsilon_{ij}$$

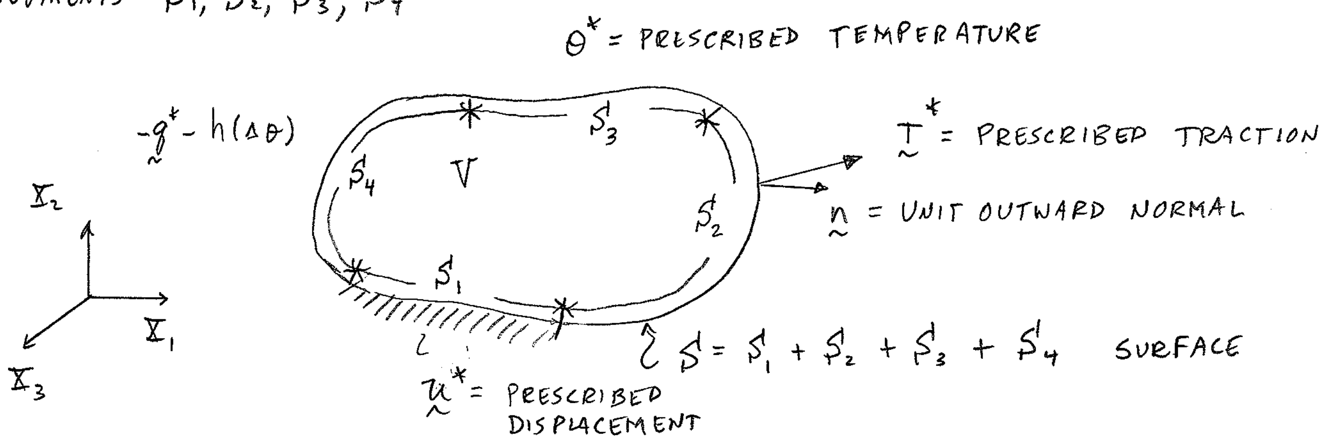
⇨ STRAIN ENERGY DENSITY IS A QUADRATIC FUNCTION OF STRAIN

$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = \underbrace{\sigma_{ij}\Big|_{\epsilon=0}}_{\text{RESIDUAL STRESS}} + \underbrace{C_{ijkl}\,\epsilon_{kl}}_{\text{MECHANICAL STRESS}}$$

CAUCHY STRESS

COMMON BOUNDARY CONDITIONS FOR SOLIDS

THE BOUNDARY CONDITIONS EMPLOYED IN SOLID MECHANICS PROBLEMS DEPEND ON THE CHOICE OF DEPENDENT VARIABLES IN THE FIELD EQUATIONS. FOR A SOLID WITH VOLUME, V, LET THE SURFACE OF THE BODY BE DECOMPOSED INTO SEGMENTS S_1, S_2, S_3, S_4



DIRICHLET BOUNDARY CONDITIONS (BCs):

$\underset{\sim}{u} = \underset{\sim}{u}^*$ ON S_1 DIPLACEMENT BC

OR $\underset{\sim}{v} = \underset{\sim}{v}^*$ ON S_1 VELOCITY BC

NEUMANN BC:

$\underset{\sim}{n} \cdot \underset{\sim}{\sigma} = \underset{\sim}{T}^*$ ON S_2 TRACTION BC

DIRICHLET BC:

$\theta = \theta^*$ ON S_3 TEMPERATURE BC

NEUMANN BC:

CONVECTION COEFFICIENT

$$\kappa \frac{\partial \theta}{\partial n} = \kappa \theta_{,i} n_i = -q - h(\theta - \theta_{AMBIENT}) \quad \text{ON } S_4$$

HEAT FLOW

HOOKE'S LAW FOR ISOTHERMAL INFINITESIMAL LINEAR ELASTICITY

IN GENERAL,

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl} \qquad \text{HOOKE'S LAW}$$

WHERE $\epsilon_{ij} = u_{[i,j]} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ SMALL STRAIN TENSOR

σ_{ij} CAUCHY STRESS TENSOR

NOTE: THE 4^{TH} RANK STIFFNESS TENSOR HAS 81 COMPONENTS, i.e.

$$3^4 = 81 \quad \{ = (\text{RANGE})^{\text{RANK}} \}$$

BUT FROM A BALANCE OF ANGULAR MOMENTUM, $\sigma_{ij} = \sigma_{ji}$. IN ADDITION, $\epsilon_{ij} = \epsilon_{ji}$.

THIS MEANS

$$C_{ijkl} = C_{jikl} = C_{jilk} = C_{ijlk} \quad \Rightarrow \text{36 INDEPENDENT COMPONENTS.}$$

ALSO, FOR A HYPER-ELASTIC MATERIAL WITH NO RESIDUAL STRESSES

$$W(\underset{\sim}{\epsilon}) = \tfrac{1}{2}\,\sigma_{kl}\,\epsilon_{kl} = \tfrac{1}{2}\,C_{ijkl}\,\epsilon_{kl}\,\epsilon_{ij}$$

(½ AREA UNDER LINEAR STRESS-STRAIN CURVE)

AND
$$\sigma_{ij} = \frac{\partial W}{\partial \epsilon_{ij}} = \frac{\partial W}{\partial \epsilon_{ji}} = \sigma_{ji}$$

$$\sigma_{kl} = \frac{\partial W}{\partial \epsilon_{kl}} = \frac{\partial W}{\epsilon_{kl}} = \sigma_{kl}$$

$$\Rightarrow \quad C_{ijkl} = C_{klij}$$

$\therefore$ AT MOST THERE ARE 21 INDEPENDENT COMPONENTS OF C_{ijkl}

ASIDE: FOR THE PURELY ANISOTROPIC CASE, FULL COUPLING EXISTS BETWEEN THE COMPONENTS OF STRESS AND STRAIN, $\underset{\sim}{\sigma} = \underset{\approx}{C} : \underset{\sim}{\epsilon}$.

$\Rightarrow$ NORMAL STRESS WILL PRODUCE SHEARING STRAIN

THIS IS EASILY SEEN IF VOIGT NOTATION IS USED TO EXPRESS HOOKE'S LAW.

USING VOIGT NOTATION (i.e., $\underset{\sim}{\sigma}$ AND $\underset{\sim}{\epsilon}$ ARE EXPRESSED AS 6×1 VECTORS) HOOKES LAW MAY BE EXPRESSED IN MATRIX FORM, i.e.,

$$
\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{13} \end{Bmatrix}
=
\begin{bmatrix}
C_{1111} & C_{1122} & C_{1133} & C_{1112} & C_{1123} & C_{1113} \\
 & C_{2222} & C_{2233} & C_{2212} & C_{2223} & C_{2213} \\
 & & C_{3333} & C_{3312} & C_{3323} & C_{3313} \\
 & (SYM) & & C_{1212} & C_{1223} & C_{1213} \\
 & & & & C_{2323} & C_{2313} \\
 & & & & & C_{1313}
\end{bmatrix}
\begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{23} \\ 2\epsilon_{31} \end{Bmatrix}
$$

HOOKE'S LAW FOR GENERAL ANISOTROPIC CASE (21 CONSTANTS)

NOTE FACTOR TWO ON SHEAR STRAINS

<u>IMPORTANT</u>: THE <u>MATRIX</u> OF STIFFNESS COEFFICIENTS (i.e., THE COMPONENTS OF THE 4^{TH} RANK STIFFNESS TENSOR $\underset{\approx}{C}$ EXPRESSED AS A 6×6 MATRIX) DOES NOT SATISFY THE TRANSFORMATION LAW FOR TENSORS

$$C'_{ijkl} = a_{ip}\, a_{jq}\, a_{kr}\, a_{ls}\, C_{pqrs}$$

$\underset{\sim}{a}$ = TRANSFORMATION TENSOR

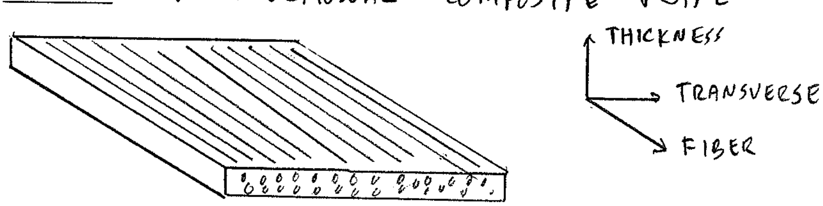
THIS NOTATION IS ADOPTED PURELY AS A MATTER OF CONVENIENCE

<u>SPECIAL</u> <u>CASES</u>:

1) <u>ORTHOTROPIC</u> <u>MATERIAL</u> (THREE MUTUALLY ORTHOGONAL PLANES OF MATERIAL SYMMETRY)

$\Rightarrow$ $\underset{\approx}{C}$ HAS <u>NINE</u> INDEPENDENDENT MATERIAL CONSTANTS

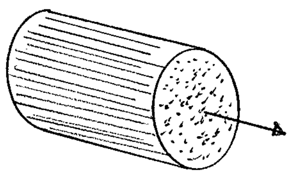
<u>EXAMPLE</u>: UNIDIRECTIONAL COMPOSITE PLATE



2) TRANSVERSELY ISOTROPIC MATERIAL (ONE AXIS OF ROTATIONAL SYMMETRY THAT ALSO DEFINES A PLANE OF REFLECTION SYMMETRY)

⇨ $\underset{\approx}{C}$ HAS FIVE INDEPENDENT MATERIAL CONSTANTS

EXAMPLE: UNI-DIRECTIONAL FIBER BUNDLE (TOW)



PLANE OF SYMMETRY AND AXIS OF ROTATIONAL SYMMETRY

3) ISOTROPIC MATERIAL BEHAVIOR (THE COMPONENTS OF $\underset{\approx}{C}$ ARE INDEPENDENT OF COORDINATE SYSTEM, i.e., $\underset{\approx}{C}' = \underset{\approx}{C}$ FOR ANY TRANSFORMATION DEFINED BY $\underset{\sim}{a}$)

⇨ $\underset{\approx}{C}$ HAS TWO INDEPENDENT MATERIAL CONSTANTS

⇨ THE ISOTROPIC MATERIAL ASSUMPTION WORKS WELL FOR MANY ENGINEERING MATERIALS UNDER LINEAR ELASTIC CONDITIONS

⇨ IMPORTANT: ALL REAL MATERIALS ARE HIGHLY ANISOTROPIC WHEN VIEWED AT THE MICROSCALE.

ASSUMING ISOTROPIC MATERIAL BEHAVIOR, THE 4TH RANK STIFFNESS TENSOR MAY BE EXPRESSED IN TERMS OF THE TWO LAMÉ CONSTANTS

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{kj})$$

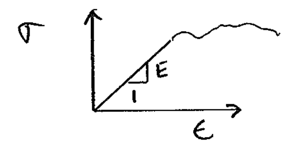
WHERE

$$\boxed{\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}}$$

LAMÉ CONSTANTS

$\mu = G$ SHEAR MODULUS

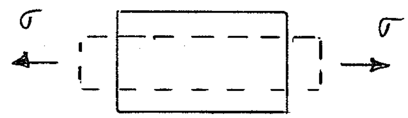
HERE



E = UNIAXIAL YOUNG'S MODULUS

ν = POISSON'S RATIO

$= -\dfrac{\text{TRANSVERSE EXTENSIONAL STRAIN}}{\text{LONGITUDINAL EXTENSIONAL STRAIN}}$



POISSON'S EFFECT: NORMAL STRESS IN THE X_1-DIRECTION CAUSES EXTENSIONAL STRAIN IN THE X_2- AND X_3-DIRECTIONS

ASSUMING THE ISOTROPIC MATERIAL IDEALIZATION, HOOKE'S LAW MAY BE EXPRESSED AS

$$\sigma_{ij} = C_{ijkl}\,\epsilon_{kl}$$

$$= \left[\lambda\,\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}\right)\right]\epsilon_{kl}$$

USE SUBSTITUTION PROPERTY OF $\underset{\sim}{\delta}$

$$\Rightarrow \quad \boxed{\sigma_{ij} = \lambda\,\epsilon_{kk}\,\delta_{ij} + 2\mu\,\epsilon_{ij}}$$

HOOKE'S LAW FOR ISOTHERMAL ISOTROPIC LINEAR ELASTICITY

NOTE: $\epsilon_{kk} = \dfrac{\Delta V}{V_0}$ CUBICLE DILATATION

THE PRECEDING EQUATION MAY BE READILY INVERTED

$$\boxed{\epsilon_{ij} = \frac{\sigma_{ij}}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\,\sigma_{kk}\,\delta_{ij}}$$

STRAIN-STRESS RELATIONSHIP FOR ISOTHERMAL ISOTROPIC LINEAR ELASTICITY

OR IN TERMS OF ν AND E

$$\boxed{\epsilon_{ij} = \frac{1}{E}\left[(1+\nu)\,\sigma_{ij} - \nu\,\sigma_{kk}\,\delta_{ij}\right]}$$

ALTERNATE FORM OF HOOKE'S LAW

$$\epsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}\left(\sigma_{22} + \sigma_{33}\right)$$

etc...

$$\epsilon_{12} = \frac{1+\nu}{E}\,\sigma_{12} = \frac{\sigma_{12}}{2\mu} = \frac{\sigma_{12}}{2G}$$

RECALL THAT THE STRESS TENSOR CAN BE DECOMPOSED INTO HYDROSTATIC AND DEVIATORIC PARTS

$$\sigma_{ij} = \frac{\sigma_{kk}}{3}\delta_{ij} + S_{ij} = \sigma_M \delta_{ij} + S_{ij}$$

σ_M = MEAN NORMAL STRESS
$\underset{\sim}{S}$ = DEVIATORIC STRESS

LIKEWISE, ONE MAY CHARACTERIZE THE HYDROSTATIC AND DEVIATORIC STRESS-STRAIN RESPONSE, i.e.,

$$\sigma_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$\sigma_{ii} = 3\lambda \epsilon_{kk} + 2\mu \epsilon_{ii}$$

$$\boxed{\frac{\sigma_{kk}}{3} = \sigma_M = \left(\lambda + \frac{2}{3}\mu\right)\epsilon_{ii} = K\epsilon_{kk}}$$

<u>HYDROSTATIC</u> <u>STRESS-STRAIN</u> <u>RESPONSE</u>

WHERE

$$\boxed{\begin{aligned} K &= \lambda + \tfrac{2}{3}\mu \\ &= \frac{E}{3(1-2\nu)} \end{aligned}}$$

<u>BULK</u> <u>MODULUS</u>

IF $\left.\begin{aligned} K &= \infty \\ \nu &= \tfrac{1}{2} \end{aligned}\right\} \Rightarrow$ INCOMPRESSIBLE DEFORMATION

<u>ASIDE</u>: $\nu = \tfrac{1}{2}$ IS A COMMON ASSUMPTION IN ELEMENTARY PLASTICITY THEORIES (CONSTANT VOLUME ASSUMPTION)

SIMILARLY

$$\boxed{S_{ij} = 2\mu \eta_{ij} = 2G\eta_{ij}}$$

<u>DEVIATORIC</u> <u>STRESS-STRAIN</u> <u>RESPONSE</u>

WHERE $\epsilon_{ij} = \frac{\epsilon_{kk}}{3}\delta_{ij} + \eta_{ij} = \frac{1}{3}\frac{\Delta V}{V_0}\delta_{ij} + \eta_{ij}$

η_{ij} = DEVIATORIC COMPONENT OF SMALL STRAIN TENSOR, $\underset{\sim}{\epsilon}$.

ASIDE: SINCE THE STRAIN ENERGY DENSITY IS POSITIVE DEFINITE, THEN THE ELASTIC CONSTANTS MUST ASSUME CERTAIN VALUES.

$$W(\underset{\sim}{\epsilon}) = \tfrac{1}{2}\sigma_{ij}\epsilon_{ij} \geq 0 \qquad \text{STRAIN ENERGY DENSITY IS POSITIVE DEFINITE}$$

$$= \tfrac{1}{2}\Big[\underbrace{\tfrac{\sigma_{kk}}{3}\delta_{ij}}_{\text{HYDROSTATIC STRESS}} + \underbrace{s_{ij}}_{\text{DEVIATORIC STRESS}}\Big]\Big[\underbrace{\tfrac{\epsilon_{mm}}{3}\delta_{ij}}_{\sim\text{VOLUMETRIC STRAIN}} + \underbrace{\eta_{ij}}_{\text{DEVIATORIC STRAIN}}\Big] \geq 0$$

$$W(\underset{\sim}{\epsilon}) = \tfrac{1}{2}\Big[\tfrac{\sigma_{kk}}{3}\tfrac{\epsilon_{mm}}{3}\overbrace{\delta_{ii}}^{=3} + \tfrac{\sigma_{kk}}{3}\overbrace{\eta_{ii}}^{\equiv 0} + \overbrace{s_{ii}}^{\equiv 0}\tfrac{\epsilon_{mm}}{3} + s_{ij}\eta_{ij}\Big] \geq 0$$

$$= \tfrac{1}{2}\Big[\tfrac{\sigma_{kk}}{3}\epsilon_{mm} + s_{ij}\eta_{ij}\Big] \geq 0$$

APPLY HOOKE'S LAW FOR HYDROSTATIC AND DEVIATORIC CASES

$$= \tfrac{1}{2}\Big[(\lambda + \tfrac{2}{3}\mu)(\epsilon_{mm})^2 + 2\mu\,\eta_{ij}\eta_{ij}\Big] \geq 0$$

CLEARLY, THE LAMÉ CONSTANTS MUST TAKE ON VALUES SUCH THAT

$\underset{\sim}{\eta} = \underset{\sim}{0} \Rightarrow$ $\qquad \boxed{\lambda + \tfrac{2}{3}\mu \geq 0}$

$\epsilon_{kk} = 0 \Rightarrow$ $\qquad \boxed{\mu \geq 0}$

<u>RESTRICTIONS ON ELASTIC CONSTANTS</u>

λ, μ = LAMÉ CONSTANTS

THIS LEADS TO

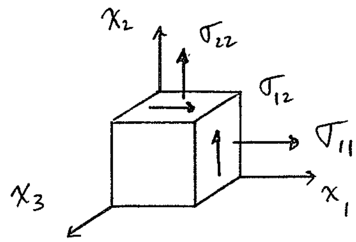
$$\boxed{\begin{array}{c} E > 0 \\ -1 < \nu < \tfrac{1}{2} \end{array}}$$

E = YOUNG'S MODULUS

ν = POISSON'S RATIO

$\mu = G = \dfrac{E}{2(1+\nu)}$ = SHEAR MODULUS

PLANE PROBLEMS IN LINEAR ELASTICITY



PLANE STRESS: $\sigma_{33} = \sigma_{31} = \sigma_{32} = 0$

- NO COMPONENT OF TRACTION IN THE x_3- DIRECTION
- SURFACE $\underset{\sim}{n} = \underset{\sim}{e}_3$ CORRESPONDS TO A FREE SURFACE
- A PLANE STRESS ASSUMPTION WORKS WELL FOR THIN SHEETS, PLATES, AND SHELLS WHERE THE THICKNESS IN THE x_3-DIRECTION IS MUCH SMALLER THAN THE REMAINING TWO IN-PLANE DIMENSIONS.

PLANE STRAIN: $\epsilon_{33} = \epsilon_{31} = \epsilon_{32} = 0$

- MORE PRECISELY, THE x_3 - COMPONENT OF DISPLACEMENT $u_3 \equiv 0$ AND $u_1 = u_1(x_1, x_2)$ AND $u_2 = u_2(x_1, x_2)$.

  $\Rightarrow$ u_1 AND u_2 ARE NOT FUNCTIONS OF x_3
- A PLANE STRAIN ASSUMPTION OFTEN IS APPROPRIATE FOR: THICK PARTS WHERE THE DIMENSION IN ONE COORDINATE DIRECTION (SAY x_3) IS MUCH LARGER THAN THE REMAINING IN-PLANE DIMENSIONS.

FOR THE CASE OF PLANE STRESS OR PLANE STRAIN, THE GOVERNING EQUATIONS MAY BE SIMPLIFIED SOMEWHAT.

EXAMPLE: HOOKE'S LAW FOR PLANE STRESS $(\sigma_{33} = \sigma_{31} = \sigma_{32} = 0)$

$$\epsilon_{11} = \frac{\sigma_{11}}{E} - \frac{\nu}{E}\sigma_{22}$$

$$\epsilon_{22} = \frac{\sigma_{22}}{E} - \frac{\nu}{E}\sigma_{11}$$

$$\epsilon_{33} = \frac{-\nu}{E}(\sigma_{11} + \sigma_{22})$$

$\Rightarrow$ THROUGH THE THICKNESS STRAIN IS NON-ZERO!

$$2\epsilon_{12} = \frac{\sigma_{12}}{G} \qquad \epsilon_{13} = \epsilon_{23} = 0$$

FIELD EQUATIONS OF LINEARIZED, INFINITESIMAL, ISOTROPIC, ISOTHERMAL ELASTICITY
(15 EQUATIONS IN 15 UNKNOWNS)

$$\boxed{\begin{aligned} \sigma_{ji,j} + \rho b_i &= \rho \ddot{u}_i \\ \sigma_{ij} &= \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} \\ \epsilon_{ij} &= u_{[i,j]} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \end{aligned}}$$

EQUATION OF MOTION

STRESS - STRAIN LAW

STRAIN - DISPLACEMENT RELATIONSHIP

IF THE STRAIN ENERGY DENSITY FUNCTION, $W(\underset{\sim}{\epsilon})$, IS POSITIVE DEFINITE AND CONTINUOUS, THEN FOR $\ddot{u}_j = 0$ AND INFINITESIMAL STRAIN ($\underset{\sim}{\epsilon}$) THE SOLUTION IS UNIQUE. HENCE, THE FIELD EQUATIONS (AND COMPATIBILITY EQUATIONS) MAY BE EXPRESSED EITHER IN TERMS OF DISPLACEMENTS OR STRESSES.

DISPLACEMENT FORMULATION / NAVIER'S EQUATIONS

ONE MAY SUBSTITUTE THE STRESS-STRAIN RELATIONS INTO THE EQUATIONS OF MOTION TO OBTAIN A PURELY DISPLACEMENT BASED FORMULATION

$$\begin{aligned} \sigma_{ij} &= \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij} = \lambda u_{k,k} \delta_{ij} + 2\mu u_{[i,j]} \\ &= \lambda u_{k,k} \delta_{ij} + \mu (u_{i,j} + u_{j,i}) \end{aligned}$$

HOOKE'S LAW IN TERMS OF DISPLACEMENTS

SUBSTITUTE THE PRECEDING EXPRESSION INTO THE EQUATIONS OF MOTION

$$\overbrace{\lambda u_{k,kj} \delta_{ij} + \mu (u_{i,jj} + \underset{\equiv\, u_{k,ik} = u_{k,ki}}{\underbrace{u_{j,ij}}})}^{\sigma_{ji,j}} + \rho b_i = \rho \ddot{u}_i$$

COMBINE TERMS

$$\boxed{(\lambda + \mu) u_{k,ki} + \mu u_{i,kk} + \rho b_i = \rho \ddot{u}_i}$$

OR

$$\boxed{(\lambda + \mu) \underset{\sim}{\nabla} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \mu \nabla^2 \underset{\sim}{u} + \rho \underset{\sim}{b} = \rho \ddot{\underset{\sim}{u}}}$$

NAVIER'S EQUATIONS OF MOTION

IN ORDER TO EMPLOY NAVIER'S EQUATIONS, ALL BOUNDARY CONDITIONS MUST BE EXPRESSED IN TERMS OF DISPLACEMENTS, u_i.

TRACTION BOUNDARY CONDITIONS (NEUMANN BC):

$$\boxed{T_j^* = n_i \sigma_{ij} = n_i \left\{ \lambda u_{k,k} \delta_{ij} + 2\mu \cdot \tfrac{1}{2} (u_{i,j} + u_{j,i}) \right\}} \quad \text{ON } S_2$$

OR

$$\boxed{\underset{\sim}{T}^* = \underset{\sim}{n} \cdot \underset{\sim}{\sigma} = \lambda (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) \underset{\sim}{n} + \mu (\underset{\sim}{u} \overleftarrow{\underset{\sim}{\nabla}} + \overrightarrow{\underset{\sim}{\nabla}} \underset{\sim}{u}) \cdot \underset{\sim}{n}}$$

WHERE $\underset{\sim}{T}^* = T_j^* \underset{\sim}{e}_j$ PRESCRIBED (KNOWN) TRACTION

DISPLACEMENT BOUNDARY CONDITIONS (DIRICHLET BC)

$$\boxed{u_i^* = u_i} \quad \text{ON } S_1$$

OR

$$\boxed{\underset{\sim}{u}^* = \underset{\sim}{u}}$$

WHERE $\underset{\sim}{u}^* = u_i^* \underset{\sim}{e}_i$ PRESCRIBED (KNOWN) DISPLACEMENT

$S = S_1 + S_2$

NOTE: SINCE THE PROBLEM IS FORMULATED IN TERMS OF DISPLACEMENTS, $\underset{\sim}{u}$, THEN THE COMPATIBILITY CONDITIONS ARE UNNECESSARY.

STRESS FORMULATION / BELTRAMI - MICHELL COMPATIBILITY EQUATIONS

FOR THE ELASTOSTATIC CASE, ONE MUST EMPLOY THE COMPATIBILITY RELATIONSHIPS TO ENSURE A CONTINUOUS SINGLE VALUED DISPLACEMENT FIELD IN THE ABSENCE OF EXPLICIT TREATMENT OF DISPLACEMENTS. THIS APPROACH IS POSSIBLE ONLY IF ALL BOUNDARY CONDITIONS ARE TRACTION (NEUMANN) BOUNDARY CONDITIONS.

COMBINING THE STRAIN COMPATIBILITY EQUATIONS $(\epsilon_{ij,km} + \epsilon_{km,ij} - \epsilon_{ik,jm} - \epsilon_{jm,ik} = 0)$ WITH THE STRESS-STRAIN LAW AND EQUATION OF EQUILIBRIUM LEADS TO

$$\boxed{\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} + \rho(b_{i,j} + b_{j,i}) + \frac{\nu}{1-\nu}\rho\, b_{k,k}\,\delta_{ij} = 0}$$

BELTRAMI - MICHELL COMPATIBILITY EQUATIONS

NOTE: THE PRECEDING RELATIONSHIP ONLY REPRESENTS THREE FUNCTIONALLY INDEPENDENT CONDITIONS. HENCE, ONE MUST ALSO EMPLOY THE EQUILIBRIUM CONDITIONS $(\sigma_{ji,j} + \rho b_i = 0)$ IN ORDER TO SOLVE FOR THE SIX UNKNOWN STRESSES.

⇨ IN PRACTICE, A PURELY STRESS-BASED APPROACH USING THE BELTRAMI - MICHELLE COMPATIBILITY CONDITIONS IS NOT CONVENIENT TO IMPLEMENT. UNDER CERTAIN CONDITIONS, A FORMULATION CAST IN TERMS OF STRESS FUNCTIONS IS POSSIBLE.

EXAMPLE: AIRY STRESS FUNCTION FOR 2-D ELASTICITY

CLASSICAL FLUID MECHANICS

FLUID: A SUBSTANCE THAT CAN NOT SUSTAIN A SHEAR STRESS WHEN AT REST.

IN GENERAL, FOR A FLUID IN MOTION THE STRESS TENSOR MAY BE DECOMPOSED AS FOLLOWS:

$$\boxed{\sigma_{ij} = \underbrace{-p\,\delta_{ij}}_{\text{THERMODYNAMIC PRESSURE}} + \underbrace{\tau_{ij}}_{\text{VISCOUS STRESS TENSOR}}}$$

STRESS TENSOR FOR A FLUID IN MOTION

HERE $p = -\frac{1}{3}\left(\sigma_{kk} - \tau_{kk}\right)$ THERMODYNAMIC PRESSURE

IMPORTANT: FOR A FLUID AT REST THE VISCOUS STRESS $\tau_{ij} \equiv 0$, AND THE THERMODYNAMIC PRESSURE CORRESPONDS TO THE HYDROSTATIC STRESS (MEAN NORMAL STRESS), i.e., $p = -\frac{\sigma_{kk}}{3}$

IN GENERAL, THE VISCOUS STRESS TENSOR τ_{ij} DEPENDS UPON THE RATE OF DEFORMATION TENSOR, i.e.

$$\underset{\sim}{\tau} = \underset{\sim}{f}(\underset{\sim}{D})$$

IF THE VISCOUS STRESS-RATE OF DEFORMATION RELATIONSHIP IS NON-LINEAR, THEN THE FLUID IS STOKESIAN. IF $\underset{\sim}{f}(\underset{\sim}{D})$ IS A LINEAR FUNCTION OF D_{ij}, THEN THE FLUID IS NEWTONIAN, i.e.

$$\tau_{ij} = K_{ijkl} D_{kl}$$

CONSTITUTIVE LAW FOR A NEWTONIAN FLUID.

IMPORTANT: FROM EMPIRICAL OBSERVATIONS, ALL FLUIDS ARE ISOTROPIC. AS A CONSEQUENCE, THE VISCOUS STRESS - RATE OF DEFORMATION CONSTITUTIVE LAW MAY BE EXPRESSED IN TERMS OF TWO VISCOSITY COEFFICIENTS (MATERIAL CONSTANTS), i.e.,

$$\boxed{\Upsilon_{ij} = \lambda^* D_{KK}\,\delta_{ij} + 2\mu^* D_{ij}}$$

CONSTITUTIVE EQUATION FOR A HOMOGENEOUS NEWTONIAN FLUID

WHERE

λ^*, μ^* = VISCOSITY COEFFICIENTS

⇒ DENOTE VISCOUS PROPERTIES OF THE FLUID

⇒ COMPARE TO $\sigma_{ij} = \lambda \epsilon_{KK}\delta_{ij} + 2\mu \epsilon_{ij}$ FOR A LINEAR ELASTIC SOLID

THE TOTAL STRESS IN THE FLUID MAY BE EXPRESSED AS

$$\sigma_{ij} = \underbrace{-p}_{\text{THERMODYNAMIC PRESSURE}}\delta_{ij} + \underbrace{\Upsilon_{ij}}_{\text{VISCOUS STRESS}}$$

$$\boxed{\sigma_{ij} = -p\,\delta_{ij} + \lambda^* D_{KK}\,\delta_{ij} + 2\mu^* D_{ij}}$$

STRESS IN A NEWTONIAN FLUID IN MOTION

CONSIDER THE HYDROSTATIC STRESS (CONTRACT ON INDEX i IN PREVIOUS EXPRESSION)

$$\frac{1}{3}\sigma_{ii} = \frac{1}{3}\left[-p\,\overset{=3}{\delta_{ii}} + \lambda^* D_{KK}\,\overset{=3}{\delta_{ii}} + 2\mu^* D_{ii}\right]$$

$$\frac{\sigma_{ii}}{3} = -p + \left(\lambda^* + \tfrac{2}{3}\mu^*\right)\underbrace{D_{KK}}$$

↳ VOLUMETRIC PART OF RATE OF DEFORMATION

OR $$\frac{\sigma_{ii}}{3} = -p + \mathcal{K}^* D_{KK}$$

WHERE $$\boxed{\mathcal{K}^* = \lambda^* + \tfrac{2}{3}\mu^*}$$

COEFFICIENT OF BULK VISCOSITY

IMPORTANT: STOKE'S CONDITION $(\kappa^* = 0)$ ENSURES THAT FOR A FLUID AT REST THE HYDROSTATIC STRESS (MEAN NORMAL STRESS) CORRESPONDS TO THE NEGATIVE OF THE THERMODYNAMIC PRESSURE, i.e.,

$$\frac{\sigma_{ii}}{3} = \sigma_M = -p + \cancelto{0}{\kappa^*} D_{KK} = -p + \cancelto{0}{\left(\lambda^* + \tfrac{2}{3}\mu^*\right)} D_{KK}$$

$$\boxed{\frac{\sigma_{ii}}{3} = \sigma_M = -p}$$

THERMODYNAMIC PRESSURE FOR A FLUID AT REST CORRESPONDS TO (-) THE HYDROSTATIC STRESS

$$\boxed{\begin{aligned} \kappa^* &= \lambda^* + \tfrac{2}{3}\mu^* = 0 \\ \lambda^* &= -\tfrac{2}{3}\mu^* = 0 \end{aligned}}$$

STOKE'S CONDITION

CONSIDER THE CONSTITUTIVE EQUATION FOR A NEWTONIAN FLUID

$$\sigma_{ij} = -p\,\delta_{ij} + \lambda^* D_{KK}\,\delta_{ij} + 2\mu^* D_{ij}$$

DECOMPOSE $\underset{\sim}{\sigma}$ AND $\underset{\sim}{D}$ INTO SPHERICAL (HYDROSTATIC) AND DEVIATORIC PARTS, i.e.,

$$\boxed{s'_{ij} = \sigma_{ij} - \frac{\sigma_{KK}}{3}\delta_{ij}}$$

DEVIATORIC STRESS

$$\boxed{\beta_{ij} = D_{ij} - \frac{D_{KK}}{3}\delta_{ij}}$$

DEVIATORIC PART OF $\underset{\sim}{D}$

SUBSTITUTE THESE RELATIONSHIPS INTO THE CONSTITUTIVE LAW FOR A NEWTONIAN FLUID,

$$\underbrace{\frac{\sigma_{kk}}{3}\delta_{ij} + s_{ij}}_{\sigma_{ij} = \text{STRESS TENSOR}} = -p\,\delta_{ij} + \lambda^* D_{kk}\delta_{ij} + 2\mu^*\Big(\underbrace{\frac{D_{kk}}{3}\delta_{ij} + \beta_{ij}}_{D_{ij} = \text{RATE OF DEFORMATION TENSOR}}\Big)$$

COLLECTING TERMS

$$\frac{\sigma_{kk}}{3}\delta_{ij} + s_{ij} = \Big[-p + \underbrace{(\lambda^* + \tfrac{2}{3}\mu^*)}_{\kappa^* = \text{BULK VISCOSITY}} D_{kk}\Big]\delta_{ij} + 2\mu^*\beta_{ij}$$

SIMPLIFYING

$$\frac{\sigma_{kk}}{3}\delta_{ij} + s_{ij} = \left(-p + \kappa^* D_{kk}\right)\delta_{ij} + 2\mu^*\beta_{ij}$$

THE CONSTITUTIVE EQUATION MAY BE DECOMPOSED INTO TWO SEPARATE CONSTITUTIVE RELATIONS, i.e.,

$$\boxed{s_{ij} = 2\mu^*\beta_{ij}}$$

⇒ RELATES THE SHEAR EFFECT OF THE MOTION TO THE DEVIATORIC STRESS

$$\boxed{\begin{aligned}\frac{\sigma_{kk}}{3} &= -p + \kappa^* D_{kk}\\ &= -p + \kappa^* \underbrace{v_{k,k}}_{}\end{aligned}}$$

⇒ RELATES MEAN NORMAL STRESS TO THERMODYNAMIC PRESSURE AND BULK VISCOSITY (COMPRESSIBILITY) EFFECTS

FOR INCOMPRESSIBLE FLOWS $v_{k,k} = 0$ AND $\frac{\sigma_{kk}}{3} = -p$!

FUNDAMENTAL EQUATIONS OF VISCOUS FLOW

AN EULERIAN DESCRIPTION IS USED TO EXPRESS THE GOVERNING EQUATIONS FOR VISCOUS FLUID FLOW SINCE A FLUID HAS NO REFERENCE CONFIGURATION OR "NATURAL STATE" THAT IT RETURNS TO UPON REMOVAL OF THE APPLIED LOADS. IN GENERAL, THE THERMOMECHANICAL BEHAVIOR OF A NEWTONIAN FLUID IS GOVERNED BY THE FOLLOWING FIELD EQUATIONS:

(i) $\dot{\rho} + \rho v_{i,i} = 0$ — CONTINUITY EQUATION (EULERIAN STATEMENT OF CONSERVATION OF MASS)

(ii) $\sigma_{ji,j} + \rho b_i = \rho \dot{v}_i$ — EQUATIONS OF MOTION (BALANCE OF LINEAR MOMENTUM)

(iii) $\sigma_{ij} = -p\delta_{ij} + \lambda^* D_{kk}\delta_{ij} + 2\mu^* D_{ij}$ — CONSTITUTIVE EQUATIONS

(iv) $\rho \mathring{u} = \sigma_{ij} D_{ij} - q_{i,i} + \rho r$ — ENERGY EQUATION (1ST LAW OF THERMODYNAMICS)

(v) $p = \hat{p}(\rho, \theta)$ — KINETIC EQUATION OF STATE

(vi) $u = \hat{u}(\rho, \theta)$ — CALORIC EQUATION OF STATE

(vii) $q_i = -\kappa \theta_{,i}$ — HEAT CONDUCTION EQUATIONS (FOURIER'S LAW)

THE PRECEDING FIELD EQUATIONS IN COMBINATION WITH

$$D_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i})$$ RATE OF DEFORMATION TENSOR

PROVIDE 22 EQUATIONS IN 22 UNKNOWNS $(\overset{6}{\underset{\sim}{\sigma}}, \overset{6}{\underset{\sim}{D}}, \overset{3}{\underset{\sim}{v}}, \overset{3}{\underset{\sim}{q}}, \overset{1}{\rho}, \overset{1}{u}, \overset{1}{p}, \overset{1}{\theta})$,

OF COURSE, THE VISCOUS FLUID FLOW MUST ALSO BE CONSISTENT WITH THE 2ND LAW OF THERMODYNAMICS. IF THERMAL EFFECTS ARE NEGLIGIBLE AND A PURELY MECHANICAL PROCESS ENSUES, THEN THE PROBLEM MAY BE CHARACTERIZED USING a) THE CONTINUITY EQUATION, b) THE EQUATIONS OF MOTION, c) THE CONSTITUTIVE EQUATIONS, AND d) A KINETIC EQUATION OF STATE OF THE FORM

$$p = \hat{p}(\rho)$$ KINETIC EQUATION OF STATE FOR A PURELY MECHANICAL PROCESS (TEMPERATURE INDEPENDENT)

PURELY MECHANICAL VISCOUS FLOW INVOLVES 17 EQUATIONS IN 17 UNKNOWNS, i.e.,

$$\left.\begin{aligned} &\dot{\rho} + \rho v_{i,i} = 0 \\ &\sigma_{ji,j} + \rho b_i = \rho \mathring{v}_i \\ &\sigma_{ij} = -p\,\delta_{ij} + \lambda^* D_{kk}\,\delta_{ij} + 2\mu^* D_{ij} \\ &D_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) = v_{[i,j]} \\ &p = \hat{p}(\rho) \end{aligned}\right\}$$

FIELD EQUATIONS FOR PURELY MECHANICAL VISCOUS FLOW

UNKNOWNS: $\overset{1}{\rho}, \overset{1}{p}, \overset{3}{\underset{\sim}{v}}, \overset{6}{\underset{\sim}{\sigma}}, \overset{6}{\underset{\sim}{D}}$

FOR THE GENERAL CASE OF THEMOMECHANICAL VISCOUS FLUID FLOW, CERTAIN FIELD EQUATIONS MAY BE COMBINED IN ORDER TO EXPRESS THE PROBLEM IN MORE COMPACT FORM. SPECIFICALLY, ONE MAY SUBSTITUTE THE CONSTITUTIVE LAW

$$\sigma_{ij} = -p\,\delta_{ij} + \lambda^* D_{kk}\,\delta_{ij} + 2\mu^* D_{ij}$$

INTO THE EQUATION OF MOTION

$$\sigma_{ji,j} + \rho b_i = \rho \mathring{v}_i$$

WHILE NOTING

$$D_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}) = v_{[i,j]}$$

THE RESULTING EXPRESSION IS THE NAVIER-STOKES EQUATION FOR VISCOUS FLUID FLOW.

$$\boxed{\begin{aligned} \rho \dot{v}_i &= \rho b_i - p_{,i} + (\lambda^* + \mu^*) v_{j,ji} + \mu^* v_{i,jj} \\ \rho \dot{\underset{\sim}{v}} &= \rho \underset{\sim}{b} - \nabla p + (\lambda^* + \mu^*) \underset{\sim}{\nabla}(\underset{\sim}{\nabla} \cdot \underset{\sim}{v}) + \mu^* \nabla^2 \underset{\sim}{v} \end{aligned}}$$

NAVIER - STOKES EQUATION FOR A NEWTONIAN - FLUID

SIMILARLY, THE CONSTITUTIVE EQUATION $\sigma_{ij} = -p\delta_{ij} + \lambda^* D_{KK} \delta_{ij} + 2\mu^* D_{ij}$

$$= (-p + \kappa^* D_{KK})\delta_{ij} + 2\mu^* \beta_{ij}$$

AND FOURIER'S LAW $q_i = -\kappa\, \theta_{,i}$ MAY BE SUBSTITUTED INTO THE ENERGY EQUATION, i.e.,

$$\rho \dot{u} = \underset{\sim}{\sigma} : \underset{\sim}{D} - \underset{\sim}{\nabla} \cdot \underset{\sim}{q} + \rho r$$

OR

$$\boxed{\rho \dot{u} = -p \underbrace{\underset{\sim}{\nabla} \cdot \underset{\sim}{v}}_{D_{KK}} + \kappa^* (\underset{\sim}{\nabla} \cdot \underset{\sim}{v})^2 + 2\mu^* \underset{\sim}{\beta} : \underset{\sim}{\beta} + \underbrace{\underset{\sim}{\nabla} \cdot (\kappa \underset{\sim}{\nabla} \theta)}_{-q_{i,i}} + \rho r}$$

ENERGY EQUATION

WHERE κ = CONDUCTIVITY COEFFICIENT

$\kappa^* = (\lambda^* + \frac{2}{3}\mu^*)$ BULK VISCOSITY

$\beta_{ij} = D_{ij} - \frac{D_{KK}}{3}\delta_{ij}$ DEVIATORIC PART OF RATE OF DEFORMATION

THE NAVIER - STOKES EQUATIONS, ENERGY EQUATION, AND

$\dot{\rho} + \rho v_{i,i} = 0$ CONTINUITY EQUATION

$p = \hat{p}(\rho, \theta)$ KINETIC EQUATION OF STATE

$u = \hat{u}(\rho, \theta)$ CALORIC EQUATION OF STATE

PROVIDE 7 EQUATIONS IN 7 UNKNOWNS $(\rho, p, \theta, u, \underset{\sim}{v})$.

NOTE: IF STOKES' CONDITION IS SATISFIED $\kappa^* = \lambda^* + \frac{2}{3}\mu^* \equiv 0$ (NO BULK VISCOSITY; THIS ENSURES THAT THE THERMODYNAMIC PRESSURE $p = \sigma_{kk}/3$ FOR FLUID STATICS) THEN THE NAVIER-STOKES EQUATIONS MAY BE EXPRESSED AS

$$\boxed{\begin{aligned} \rho \dot{v}_i &= \rho b_i - p_{,i} + \tfrac{1}{3}\mu^*\left(v_{j,ji} + 3 v_{i,jj}\right) \\ \rho \dot{\underset{\sim}{v}} &= \rho \underset{\sim}{b} - \underset{\sim}{\nabla} p + \frac{\mu^*}{3}\left(\underset{\sim}{\nabla}(\underset{\sim}{\nabla}\cdot\underset{\sim}{v}) + 3\nabla^2 \underset{\sim}{v}\right) \end{aligned}}$$

NAVIER-STOKES EQUATIONS FOR NO BULK VISCOSITY (STOKE'S CONDITION)

$\Rightarrow$ NO VISCOUS BEHAVIOR ASSOCIATED WITH VOLUMETRIC CHANGES

IMPORTANT: ALTHOUGH A LINEAR VISCOUS CONSTITUTIVE RESPONSE (i.e., $\underset{\sim}{\sigma} = (-p + \lambda^* D_{kk})\underset{\sim}{\delta} + 2\mu^* \underset{\sim}{D}$) IS ASSUMED FOR A NEWTONIAN FLUID, THE NAVIER-STOKES EQUATIONS MAY BE HIGHLY NON-LINEAR. THIS IS BECAUSE IN AN EULERIAN FORMULATION, THE MATERIAL TIME DERIVATIVE $\dot{\underset{\sim}{v}}$ MUST BE EXPRESSED AS

$$\dot{v}_i = \frac{\partial v_i}{\partial t} + v_j v_{i,j}$$

$$\dot{\underset{\sim}{v}} = \frac{\partial \underset{\sim}{v}}{\partial t} + \underbrace{\underset{\sim}{v}\cdot\underset{\sim}{\nabla} v}_{\text{CONVECTIVE TERM INTRODUCES NON-LINEARITY}}$$

EULERIAN ACCELERATION

THE NAVIER-STOKES EQUATIONS ARE VALID FOR COMPRESSIBLE LAMINAR FLOW OF NEWTONIAN FLUIDS. ADDITIONAL SIMPLIFYING ASSUMPTIONS MAY BE EMPLOYED WHEN DESCRIBING A NUMBER OF SPECIALIZED FLUIDS.

SPECIALIZED FLUIDS

i) BAROTROPIC FLUIDS: EQUATION OF STATE IS INDEPENDENT OF TEMPERATURE. THIS MAY RESULT FROM ISOTHERMAL CONDITIONS (CONSTANT TEMPERATURE) AND ADIABATIC FLOW (NO HEAT ENTERS OR LEAVES THE FLUID).

e.g., $$p = \hat{p}(\rho)$$

ii) INCOMPRESSIBLE FLUIDS: DENSITY OF A FLUID IS CONSTANT. THEN FROM THE CONTINUITY EQUATION $\overset{0}{\cancel{\dot{\rho}}} + \rho v_{i,i} = 0$ AND $v_{i,i} = 0$. THE NAVIER-STOKES EQUATION FOR AN IDEALIZED INCOMPRESSIBLE FLUID IS

$$\rho \dot{v}_i = \rho b_i - p_{,i} + \mu^* v_{i,jj}$$ GOOD FOR OIL, H_2O

iii) INVISCID (FRICTIONLESS) FLUIDS OR "PERFECT" FLUIDS: A FLUID THAT CANNOT SUSTAIN A SHEAR STRESS EVEN WHEN IN MOTION IS AN INVISCID OR PERFECT FLUID. IN THIS CASE, THE VISCOSITY COEFFICIENTS $\lambda^* = \mu^* \equiv 0$. FOR A PERFECT FLUID, THE NAVIER-STOKES EQUATIONS REDUCE TO

$$\rho \dot{v}_i = \rho b_i - p_{,i}$$ ⇨ "EULER EQUATIONS OF MOTION"

ASIDE: AN IDEAL GAS IS A PERFECT FLUID THAT OBEYS THE IDEAL GAS LAW

$$p = \hat{p}(\rho, \theta) = \rho R \theta$$ KINETIC EQUATION OF STATE

IMPORTANT: ALL REAL FLUIDS ARE BOTH VISCOUS AND COMPRESSIBLE TO A CERTAIN DEGREE

BOUNDARY CONDITIONS FOR VISCOUS FLOWS

$(\;)^* =$ SPECIFIED VALUE

DIRICHLET BOUNDARY CONDITIONS

FIXED BOUNDARY: $\underset{\sim}{v}^* = \underset{\sim}{v}$ ON S_1 (NO SLIP CONDITION)

ENTRANCE OR EXIT PRESSURE: $p^* = \frac{1}{2}\,\underset{\sim}{v}\cdot\underset{\sim}{v}$ ON S_2

TEMPERATURE: $\theta^* = \theta$ ON S_3

NEUMANN BOUNDARY CONDITIONS

TRACTION BOUNDARY: $$\underbrace{\underset{\sim}{T}^*}_{\text{TYPICALLY ZERO}} = \underset{\sim}{n}\cdot\underset{\sim}{\sigma} = \underbrace{\underset{\sim}{n}}_{\text{GENERALLY UNKNOWN}}\cdot\Big(\underbrace{\underset{\sim}{\sigma}^{REV}}_{\text{REVERSIBLE}} + \underbrace{\underset{\sim}{\sigma}^{DISS}}_{\text{DISSIPATIVE}}\Big) \quad \text{ON } S_4$$

CONDUCTION BOUNDARY: $$\kappa\,\underset{\sim}{\nabla}\theta\cdot\underset{\sim}{n} = \kappa\frac{\partial\theta}{\partial n} = -q^* = -\underbrace{h}_{\text{CONVECTION COEFFICIENT}}\left(\theta - \theta_{AMBIENT}\right)$$

⇨ CAN ALSO INCLUDE FORCED CONVECTION, RADIATION, ETC.