EVOLUTION OF INTERNAL VARIABLES

MOST GENERALLY WE CAN WRITE THE EVOLUTION EQNS OF THE FORM

$$\dot{\underset{\sim}{\xi}}_i = \underset{\sim}{g}_i(\underset{\sim}{\sigma}, T, \underset{\sim}{f}_j)$$

MATL TIME DERIVATIVE; USUALLY COROTATIONAL RATE (e.g. JAUMANN RATE)

THERMODYNAMIC CONJUGATE FORCES

THOUGH QUITE GENERAL, SEVERAL PROBLEMS EXIST

1) THE 'INTRINSIC DISSIPATION INEQUALITY' MUST BE CHECKED

$$(\rho \gamma_{LOC} \geq 0)$$

<SOME CODES CHECK IT AT EVERY STEP.

2) SUITABLE MULTIAXIAL GENERALIZATIONS ARE SOMEWHAT DIFFICULT TO CONSTRUCT

GENERALIZED NORMALITY USED FOR DISSIPATION IDEAS.

3) MATERIAL SYMMETRY MUST BE OBEYED.

- ONE APPROACH WHICH PROVIDES SOME SIMPLIFYING GUIDELINES FOR GENERALIZING MODELS TO THREE DIMENSIONS IS TO INTRODUCE SOME GOVERNING PRINCIPLES FOR RELATING THERMODYNAMIC FORCES, $\underset{\sim}{f}_i$, TO THEIR FLUXES, $\dot{\underset{\sim}{\xi}}_i$,

ONSAGER

WE KNOW ALREADY THAT

$$\underset{\sim}{f}_i = \rho \frac{\partial \psi}{\partial \underset{\sim}{\xi}_i}$$

→ GIVES INFO TO CURRENT STATE, BUT NOT EVOLUTION.

ONSAGER & CASIMIR ASSUMED (1930'S)

$$\underset{\sim}{F}_i = -\underset{\sim}{f}_i = \sum_{j=1}^{N} (\underset{\sim}{L})_{ij} \dot{\underset{\sim}{\xi}}_j$$

$\underset{\sim}{F}_i$ (FORCE) ; $\dot{\underset{\sim}{\xi}}_j$ FLUX ; $(\underset{\sim}{L})_{ij}$ A LINEAR OPERATOR

RELATES TO INTRINSIC DISSIPATION OF INEQUALITY

THE OPERATOR $(\underset{\sim}{L})_{ij}$ IS SYMMETRIC & POSITIVE DEFINITE AND MAY DEPEND ON THE $\underset{\sim}{\xi}$'s & T.

HENCE, $\sum_{i=1}^{N} \underbrace{\left(\sum_{j=1}^{N} (\underset{\sim}{L})_{ij} \dot{\underset{\sim}{\xi}}_j\right)}_{\underset{\sim}{f}_i} \dot{\underset{\sim}{\xi}}_i \geq 0$

- ~~THIS IS DESCRIPTIVE IN ITS APPLICABILITY. HEAT CONDUCTION~~ OR FICK'S LAW OF DIFFUSION COULD BE USED.

- ZIEGLER (1940'S) SHOWED THAT THIS WORKS w/ MAX AMT OF ENTROPY DISSIPATION — GENERALIZED NORMALITY.

DISSIPATION POTENTIAL OR GENERALIZED POTENTIAL Ω

- THIS IS BEYOND REALM OF PHYSICS & INTO REALM OF MATHEMATICS
- NOW HELMHOLTZ FREE ENERGY CAN BE BASED ON FIRST PRINCIPLES, BUT DISSIPOTENTIAL IS JUST A MATHEMATICAL CONSTRUCTION.
- Ω FROM ITS NONLINEARITY OF $\underset{\sim}{F}_i$ VERSUS $\dot{\underset{\sim}{\xi}}_i$ RELATIONSHIP. WE ASSUME

$$\Omega = \Omega(\underset{\sim}{F}_i, T, \underset{\sim}{\xi}_i)$$

WE FURTHER ASSUME THAT

THERMODYNAMIC DISP. RATE $\dot{\underset{\sim}{\xi}}_i = \dfrac{\partial \Omega}{\partial \underset{\sim}{F}_i} \Rightarrow$ GENERALIZED NORMALITY HYPOTHESIS

EVOLUTION EQTN, NOT MUCH GUIDANCE FROM FIRST PRINCIPLES TO DEVELOP.

LIKEWISE WE MAY DEFINE THE COMPLEMENTARY FREE ENERGY (GIBBS) AS

$$G = \frac{1}{\rho}\underset{\sim}{\Omega} : \underset{\sim}{\epsilon} - \psi$$

LEADING TO

$$\underset{\sim}{F}_i = \rho \frac{\partial G}{\partial \underset{\sim}{\xi}_i}, \quad \underset{\sim}{\epsilon} = \rho \frac{\partial G}{\partial \underset{\sim}{\Omega}}$$

GIBBS → GIVEN $\underset{\sim}{\Omega}$ GET $\underset{\sim}{\epsilon}$.
HELMHOLTZ → " $\underset{\sim}{\epsilon}$ GET $\underset{\sim}{\Omega}$

< GENERALIZED NORMALITY IS CONSISTENT WITH MAXIMIZATION OF THE ENTROPY PRODUCTION RATE (ZIEGLER 1963)

- IT'S USED AT A CONVENIENCE, BUT IT HAS A MAX ENTROPY PRODUCTION RATE. IF THIS IS TRUE THEN GENERALIZED NORMALITY CAN BE ARGUED TO BE PHYSICALLY BASED.

HOWEVER, GEN'L NORMALITY DOESN'T NECESSARILY HOLD, eg MMCs PLASTIC STRAIN RATE IS NOT NORMAL TO YIELD SURFACE. IN TERFACE LEADS TO NON NORMALITY ISSUES. ANOTHER EXAMPLE IS SOILS — LOCAL COMPACTION → SHEAR → NONASSOCIATIVE OR NON NORMAL FLOW.

NEXT, WE CONSIDER THAT THE STRAIN MAY BE DECOMPOSED INTO COMPONENTS BASED ON PHYSICAL ARGUMENTS.

DEFINE $\dot{\underset{\sim}{\xi}}^{IN}$ AS THE INELASTIC STRAIN RATE.

$$\dot{\underset{\sim}{\epsilon}}^{IN} = \underset{\sim}{q}(\underset{\sim}{\epsilon}, T, \underset{\sim}{\xi}_i) = \sum_{i=1}^{N} \frac{\partial \underset{\sim}{\epsilon}}{\partial \underset{\sim}{\xi}_i} \dot{\underset{\sim}{\xi}}_i$$

CREEP, PLASTICITY, ANELASTICITY

TOTAL STRAIN

A CHANGE OF INTERNAL STRUCTURE

$$= \sum_{i=1}^{N} \frac{\partial \left(\rho \frac{\partial \Omega}{\partial \underset{\sim}{\sigma}}\right)}{\partial \underset{\sim}{\xi}_i} \dot{\underset{\sim}{\xi}}_i$$

$$= \rho \sum_{i=1}^{N} \frac{\partial^2 \Omega}{\partial \underset{\sim}{\sigma} \partial \underset{\sim}{\xi}_i} \dot{\underset{\sim}{\xi}}_i = -\sum_{i=1}^{N} \frac{\partial \hat{\underset{\sim}{\epsilon}}}{\partial \underset{\sim}{\sigma}} \dot{\underset{\sim}{\xi}}_i$$

$$= \sum_{i=1}^{N} \frac{\partial \hat{F}_i}{\partial \Omega} \dot{\underset{\sim}{\xi}}_i = \sum_{i=1}^{N} \frac{\partial \Omega}{\partial \hat{\underset{\sim}{F}}_i} \frac{\partial \underset{\sim}{F}}{\partial \underset{\sim}{\sigma}} = \frac{\partial \Omega}{\partial \underset{\sim}{\sigma}}$$

- RICE (1971) SHOWED THAT A NECESSARY, BUT NOT SUFFICIENT CONDITION FOR THE EXISTENCE OF Ω IS THAT EACH FLUX $\dot{\underset{\sim}{\xi}}_i$ DEPENDS ONLY ON ITS CONJUGATE FORCE, $\underset{\sim}{F}_i$.

$$\dot{\underset{\sim}{\xi}}_i = \frac{\partial \Omega_i(\underset{\sim}{F}_i, T, \underset{\sim}{\xi}_K)}{\partial \underset{\sim}{F}_i}$$

IMPLICIT DEPENDENCE

IN OTHER WORDS, THE ISV IS JUST DEPENDENT ONLY ON ITS CORRELATED FORCE, NOT ON OTHERS BUT T IS STILL COUPLED BECAUSE OF IMPLICIT INFLUENCE OF OTHERS. KIND OF LIKE

eg $\sigma_{SCHMID} = \tau : M_\alpha$ BUT LATENT HARDENING CAN OCCUR IN HARDENING AS A COUPLING EFFECT.

IN COMPOSITES INTERFACE DEBONDING COMES TO DO AWAY W/ NON ASSOCIATIVE BEHAVIOR. IN THIS SENSE, GENERAL MATERIAL IS NOT USED-

WHERE $\Omega_i(\underset{\sim}{F}_i, T, \underset{\sim}{\xi}_K) \equiv \int_0^{\underset{\sim}{F}_i} \Lambda_i(\underset{\sim}{F}_i, T, \underset{\sim}{\xi}_K) d\underset{\sim}{F}_i$

SO THIS INCLUDES HISTORY DEPENDENCE.

AND

$$\dot{\underset{\sim}{\xi}}_i = \Lambda_i(\underset{\sim}{F}_i, T, \underset{\sim}{\xi}_K)$$

$$\Omega(\underset{\sim}{F}_i, T, \underset{\sim}{\xi}_K) = \sum_{i=1}^{N} \Omega_i(\underset{\sim}{F}_i, T, \underset{\sim}{\xi}_K)$$

NONASSOCIATIVE → NONNORMALITY
SOILS → NON ASSOCIATIVE

YIELD SURFACE CAN BE REGARDED AS A POTENTIAL → ASSOCIATIVE FLOW RULE

IF WE ASSUME THAT $\Omega(\underset{\sim}{Q}, T; \xi_k) = 0$
AND THAT Ω IS A NON NEGATIVE, CONVEX CLOSED FUNCTION, s.t.

$$\sum_{i=1}^{N} (\underset{\sim}{F}_i - \underset{\sim}{F}_i^*) \dot{\underset{\sim}{\xi}}_i \geq 0 \quad \text{FOR ALL } \underset{\sim}{F}_i^* \text{ s.t.}$$

$$\Omega(\underset{\sim}{F}_i^*, T; \xi_k) \leq \Omega(\underset{\sim}{F}_i, T; \xi_k)$$

SO $\underset{\sim}{F}_i^*$ IS ON OR INSIDE OF SURFACE Ω; THEN THE INTRINSIC DISSIPATION INEQUALITY IS MET TRIVIALLY SINCE $\underset{\sim}{F}_i^* = \underset{\sim}{0}$ MAY BE ASSIGNED ARBITRARILY.

- DRUCKER'S NORMALITY FALLS OUT FROM THIS.
- THIS IS BASIS FOR SO-CALLED "STANDARD MATERIAL" OR "STANDARD MODEL"
- NON-STANDARD MATERIALS DO OCCUR ~ NON NORMALITY OF ONE ASPECT BUT NORMALITY USED IN ANOTHER.

NOTES:

1. THE POSTULATE OF GENERALIZED NORMALITY IS NOT REQUIRED. IT DOES SEEM TO CONFORM TO FIRST ORDER IN MANY RELEVANT RESULTS.
2. ONE MAY SELECT Ω TO BE QUADRATIC IN F_i's AS A SPECIAL CASE → VERY COMMON (ELLIPTIC FUNCTION)
3. ASSUMING $\dot{\underset{\sim}{\xi}}_i = \underset{\sim}{g}_i(\underset{\sim}{F}, T, \underset{\sim}{\xi}_j)$, WE MAY PROCEED BY CHECKING $\sum_{i=1}^{N} \underset{\sim}{F}_i \dot{\underset{\sim}{\xi}}_i \geq 0$
4. PLASTICITY & VISCOPLASTICITY, VISCOELASTICITY, ETC ALL FALL IN THIS FRAMEWORK, i.e. CLASSICAL MODELS.

LUBLINER REFLECTS THIS IDEA OF GENERALIZED NORMALITY.

BRING 2-3 BANANA PAPERS | WED 5:30 - THAT DEBT | 4/28/94 | 1/4

DECOMPOSITION OF STRAIN ~ AT LARGE STRAIN MAKES SENSE.

ALTHOUGH A LAGRANGIAN MEASURE OF STRAIN DOESN'T IN GEN'L RELATE TO A PHYSICAL PROCESS AT LARGE STRAIN, INCREMENTS OF STRAIN DO; KIND OF LIKE UPDATING TIMESTEPS IN FEM.

HENCE, $\dot{\underset{\sim}{\varepsilon}} = \dot{\underset{\sim}{\varepsilon}}^{e} + \dot{\underset{\sim}{\varepsilon}}^{in} \Rightarrow \underset{\sim}{D} = \underset{\sim}{D}^{e} + \underset{\sim}{D}^{in}$

INFINITESIMAL — LARGE STRAIN

PHYSICALLY DIRECT INTERPRETATION

$\underset{\sim}{F} = \underset{\sim}{F}^{e}\underset{\sim}{F}^{p}$

$\dot{\underset{\sim}{F}} = \dot{\underset{\sim}{F}}^{e}\underset{\sim}{F}^{p} + \underset{\sim}{F}^{e}\dot{\underset{\sim}{F}}^{p}$

$\underset{\sim}{L} = \dot{\underset{\sim}{F}}\underset{\sim}{F}^{-1} = \dot{\underset{\sim}{F}}^{e}\underset{\sim}{F}^{e-1} + \underset{\sim}{F}^{e}\dot{\underset{\sim}{F}}^{p}\underset{\sim}{F}^{p-1}\underset{\sim}{F}^{e-1}$

LUBLINER (1972) SHOWED THAT THE ADDITIVE DECOMPOSITION OF STRAIN IS COMPATIBLE W/ THE EXISTENCE OF $\psi(\underset{\sim}{\varepsilon}, T, \xi_i)$ IFF

$$\psi(\underset{\sim}{\varepsilon}, T, \xi_i) = \psi^{e}(\underbrace{\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}^{in}}_{\underset{\sim}{\varepsilon}^{e}}, T) + \psi^{in}(\xi_k, T)$$

THERMOELASTIC — INELASTIC PART

ENERGY RELATED TO REARRANGEMENT OF STRUCTURE

AT LARGE STRAIN WE MAY PROPOSE THE DECOMPOSITION:

$$\psi(\underset{\sim}{E}, T, \xi_i) = \psi^{e}(\underset{\sim}{E}^{e}, T) + \psi^{in}(\xi_k, T)$$

GREEN STRAIN — 2nd ORDER ELASTIC EFFECTS INCLUDED.

STRESS FREE CONFIGURATION USED HERE. - UNLOADED STATE ~ FICTITIOUS INTERMEDIATE CONFIGURATION - SERVES AS REFERENCE FOR THIS STRAIN. IT HAS NO MACRO LEVELS OF STRAIN.

NAGDI USED REFERENCE CONFIGURATION IN 1960's. vs.

2 SOURCES OF ELASTIC STRAIN: 1) MICRO HETEROGENEITIES - APPLIED STRESS
2) PLASTIC EFFECTS, LOCKING IN ELASTICITY - FROM GEOMETRIC ASPECTS.

ψ^{e} = THERMOELASTIC FREE ENERGY ASSOCIATED W/ LATTICE STRETCHING UNDER APPLICATION OF EXTERNAL FORCES

THE PRECISE DEFINITION OF THE FICTITIOUS ELASTICALLY UNLOADED CONFIGURATION IS MOTIVATED BY THE PHYSICS OF NONELASTIC DEFORMATION FOR EACH DIFFERENT MATERIAL. (POLYMERS ARE DIFFERENT THAN METALS).

NAGHDI & GREEN (1965) PROPOSED THAT

$$\psi(\underset{\sim}{E}, T, \xi_i) = \psi^{e}(\underset{\sim}{E}, \underset{\sim}{E}^{in}, T) + \psi^{in}(\xi_k, T)$$

THEY WANTED NO — INELASTIC GREEN STRAIN PLAYS A

THIS G-N FORM MAY BE SPECIALIZED FURTHER BY ASSUMING THAT

$\underset{\sim}{E}^e = \underset{\sim}{E} - \underset{\sim}{E}^{IN}$, BUT THIS WAS NOT THEIR INTENT OF THEIR WORK.

FOR NON PROPORTIONAL LOADINGS, THIS WON'T WORK.

LETS PROCEED w/ INFINITESIMAL STRAIN CASE,

$\underset{\sim}{\epsilon}^e = \underset{\sim}{\epsilon} - \underset{\sim}{\epsilon}^{IN}$ (ITS UNAMBIGUOUS HERE)

FROM EARLIER DEVELOPMENTS

$$\underset{\sim}{T} = \rho \frac{\partial \psi}{\partial \underset{\sim}{\epsilon}} = \rho \frac{\partial \psi^e}{\partial \epsilon} = \rho \frac{\partial \psi^e}{\partial \underset{\sim}{\epsilon}^e}$$

SINCE $\underset{\sim}{\epsilon}^{IN} = \underset{\sim}{\epsilon}^{IN}(\underset{\sim}{\xi}_i, T)$

FROM THE SECOND LAW,

$$\underset{\sim}{\sigma} : \dot{\underset{\sim}{\epsilon}} - \rho \dot{\psi} - \rho \eta \dot{T} \geq 0 \quad \text{OR}$$

$$\underset{\sim}{\sigma} : \dot{\underset{\sim}{\epsilon}}^{IN} \geq -\underset{\sim}{\sigma} : \dot{\underset{\sim}{\epsilon}}^e + \rho \left(\frac{\partial \psi^e}{\partial \underset{\sim}{\epsilon}^e} : \dot{\underset{\sim}{\epsilon}}^e + \frac{\partial \psi}{\partial T} \dot{T} + \sum_{i=1}^{M} \frac{\partial \psi^{IN}}{\partial \xi_i} \dot{\underset{\sim}{\xi}}_i \right) + \rho \eta \dot{T} \geq \left(\frac{\partial \psi^e}{\rho \partial \underset{\sim}{\epsilon}^e} - \underset{\sim}{\sigma} \right) : \dot{\underset{\sim}{\epsilon}}^e + \rho \left(\frac{\partial \psi}{\partial T} + \eta \right) \dot{T} + \sum_{i=1}^{M} \underset{\sim}{f}_i \dot{\underset{\sim}{\xi}}_i$$

EXTERNAL EFFECT | INTERNAL EFFECT

$$\Rightarrow \boxed{\underset{\sim}{\sigma} : \dot{\underset{\sim}{\epsilon}}^{IN} \geq \sum_{i=1}^{M} \underset{\sim}{f}_i \dot{\underset{\sim}{\xi}}_i}$$

RECALL: $\rho \frac{\partial \psi^e}{\partial \underset{\sim}{\epsilon}^e} = \underset{\sim}{\sigma}$ & $\frac{\partial \psi}{\partial T} = -\eta$

WORK RATE / VERY GEN'L.

↑ FLUX + FORCE ANALOGY. ITS OK TO SAY THAT FOR INFINITESIMAL STRAIN BUT IT DOESN'T HAVE PHYSICAL MEANING.

BASED ON ADDITIVE DECOMPOSITION OF FREE ENERGY,

90% OF $\underset{\sim}{\sigma} : \dot{\underset{\sim}{\epsilon}}^{IN}$ GETS DISSIPATED IN MOST METALS + SOLIDS.

THE DISSIPATION IN 1ST LAW IS THE DIFFERENCE BETWEEN

$\underset{\sim}{\sigma} : \dot{\underset{\sim}{\epsilon}}^{IN}$ & $\underset{\sim}{f}_i \dot{\underset{\sim}{\xi}}_i$.

(D) (2)

THIS SECOND LAW RELATION DOES NOT RELY ON IDENTIFICATION OF $\underline{\underline{\varepsilon}}^{IN}$ AS A STATE VARIABLE.
TWO PRINCIPLE OBJECTIONS TO IDENTIFYING $\underline{\underline{\varepsilon}}^{IN}$ AS AN ISV.

1. IT IS NOT A UNIQUE FUNCTION OF STATE AS POINTED OUT BY ONAT (1972)
2. INELASTIC STRAIN HAS NO CLEAR PHYSICAL INTERPRETATION AT LARGE STRAINS. BASICALLY YOU HAVE PLASTIC ROTATIONS AS WELL.

- GREEN-ELASTIC STRAIN = 0 AT INTERMEDIATE CONFIGURATION WHICH MEANS IT HAD STRAIN BEFORE THAT STEP.

FINITE STRAIN FORMULATION

ASSUME THAT $\underline{\underline{E}}^e$ EXISTS, REFERENCED TO SOME ELASTICALLY UNLOADED CONFIGURATION, i.e. $\underline{\underline{E}}^e = \underline{\underline{f}}(\hat{\underline{\underline{T}}}^{PK(2)})$

$\hat{\underline{\underline{T}}}^{PK(2)} = \rho_0 \frac{\partial \psi^e}{\partial \underline{\underline{E}}^e}$, $\underline{\underline{T}} = \rho \underline{\underline{F}} \cdot \frac{\partial \psi^e}{\partial \underline{\underline{E}}^e} \cdot \underline{\underline{F}}^T$ ← REF. TO INTERMEDIATE CONFIG

2nd LAW

$\underline{\underline{T}} : \underline{\underline{D}}^{IN} \geq \sum_{i=1}^{N} \underline{\underline{f}}_i \cdot \dot{\underline{\xi}}_i$ and $\underline{\underline{D}} = \underline{\underline{D}}^e + \underline{\underline{D}}^{IN}$

$\psi = \psi^e(\underline{\underline{F}}^e, T) + \psi^{IN}(\underline{\xi}_i, T)$

$\underline{\underline{F}}^e$ = THERMOELASTIC DEF. GRADIENT.

$\underline{\underline{E}}^e = \frac{1}{2}(\underline{\underline{F}}^{eT} \cdot \underline{\underline{F}}^e - \underline{\underline{I}})$

$\underline{\underline{F}} \equiv$ DEFORMATION GRADIENT (LAGRANGIAN)

$\underline{\underline{F}}^{IN} \equiv$ INELASTIC DEFORMATION GRADIENT

CLEARLY $\underline{\underline{F}}^e$ DEPENDS IMPLICITLY ON THE $\underline{\xi}_i$

RECALL, POLAR DECOMPOSITION

$$\underset{\sim}{F} = \underset{\sim}{R} \cdot \underset{\sim}{U} \qquad \text{(RIGHT P.D.)}$$

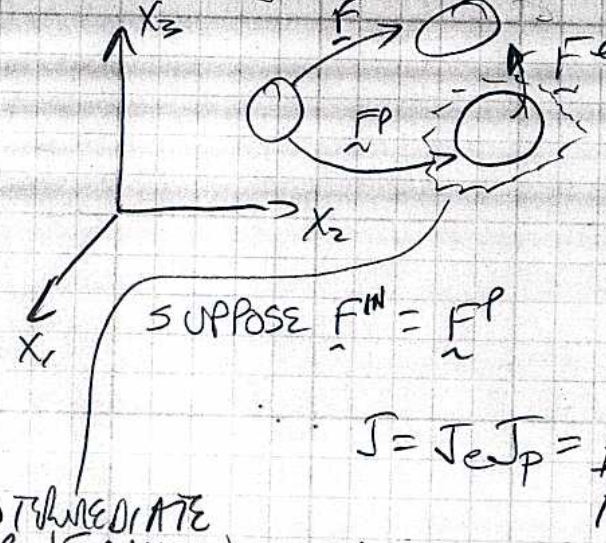
- BASED ON SPHERES TURNING TO ELLIPSES, THIS HOLDS UNAMBIGUOUSLY (IT REALLY IS A CONSTITUTIVE EQUATION). THIS REPRESENTS A STRETCH FOLLOWED BY ROTATION.

LIKEWISE, WE MAY POSTULATE THAT

MULTIPLICATIVE DECOMPOSITION OF DEF. GRAD. $\rightarrow \underset{\sim}{F} = \underset{\sim}{F}^{e} \cdot \underset{\sim}{F}^{IN}$ ← ITS A CONSTITUTIVE LAW.

- ALTERNATIVELY, $\underset{\sim}{F}^{IN} = \underset{\sim}{F}^{e-1} \cdot \underset{\sim}{F}$ REPRESENTS A DEFORMATION $\underset{\sim}{F}$ FOLLOWED BY ELASTIC UNLOADING TO AN INTERMEDIATE CONFIGURATION

$\underset{\sim}{F}^{e-1}$ - REPRESENTS ELASTIC UNLOADING.



ELASTICALLY UNLOADING $\underset{\sim}{F} \rightarrow \underset{\sim}{F}^{e}$

$$J_e \equiv \det \underset{\sim}{F}_e = \frac{\rho_{\text{INTERMEDIATE CONFIG}}}{\rho_{\text{CURRENT}}}$$

(JACOBIAN OF ELASTIC PART)

SUPPOSE $\underset{\sim}{F}^{IN} = \underset{\sim}{F}^{p}$

$$J_p = \det \underset{\sim}{F}_p = \frac{\rho_0}{\rho_{\text{INT.}}}$$

$$J = J_e J_p = \frac{\rho_0}{\rho_{\text{CURRENT}}} = 1 \text{ IF INCOMPRESSIBLE.}$$

INTERMEDIATE CONFIGURATION - IN TERMS OF GEOMETRY (CONFIGURATION) + NOT STRAIN.

$$\hat{\underset{\sim}{\sigma}} = J_e \underset{\sim}{F}^p \cdot \underset{\sim}{L}^{-1} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{F}^{-T} \cdot \underset{\sim}{F}^{pT}$$

$$= J_e \underset{\sim}{F}^{e-1} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{F}^{e-T}$$

↑ PULLBACK INCLUDES J_e ~ ELASTIC JACOBIAN.

BUT $J = J_e J_p = J_e$ BECAUSE OF INCOMPRESS $J_p = 1$

AT INTERMEDIATE CONFIGURATION,

$$\psi = \psi^e(\hat{\underset{\sim}{E}}^e, T) + \psi^{IN}(\hat{\underset{\sim}{\xi}}_J, T)$$

↑ USED TO REARRANGE STRUCTURE SO THE HELMHOLTZ FREE ENERGY FUNCTION IS DIFFERENT AT INT. CONFIG. THAN REF. BASED ON ITS INTERNAL REARRANGEMENT WHICH IS IRREVERSIBLE.

PLASTIC STRAIN IS NOT A STATE VARIABLE.

THE LEGENDRE TRANSFORM OF ψ IS THE COMPLEMENTARY FREE ENERGY, OR GIBBS FUNCTION

$$\hat{G} = \frac{1}{\hat{\rho}} \hat{\underset{\sim}{E}}^e : \hat{\underset{\sim}{\sigma}}^{PK(2)} - \hat{\psi}$$

(i) $\hat{G}^e = \hat{G}^e(\hat{\underset{\sim}{\sigma}}^{PK(2)}, T) + \hat{G}^{IN}(\hat{\underset{\sim}{\xi}}_J, T)$

(ii) $\hat{\underset{\sim}{E}}^e = \hat{\rho} \frac{\partial \hat{G}}{\partial \underset{\sim}{\sigma}^{PK(2)}}$

- TOTAL ROTATION = RIGID BODY + ELASTIC + PLASTIC
  (↳ IN METALS ITS SMALL; IN POLYMERS ITS LARGE)

BY VIRTUE OF (i) + (ii) THE CORROTATIONAL DERIVATIVE OF $\hat{\underset{\sim}{E}}^e$ IS $\mathring{\hat{\underset{\sim}{E}}}^e \equiv \hat{\rho} \frac{\partial^2 G^e}{\partial \hat{\underset{\sim}{\sigma}}^{PK(2)} \partial \hat{\underset{\sim}{\sigma}}^{PK(2)}} : \mathring{\hat{\underset{\sim}{\sigma}}}^{PK(2)}$

- IT MUST SATISFY OBJECTIVITY + IT MUST BE DONE ⇝ OBSERVER IN SUBSTRUCTURE (i.e. AT INTERMEDIATE CONFIGURATION). THEN EVERYTHING I SEE IS ONLY ELASTIC RESPONSE.

IN THE CURRENT CONFIGURATION, THE ELASTIC RATE OF DEFORMATION

$$\underset{\sim}{D}^e = \underset{\sim}{F}^{e-T} \cdot \mathring{\underset{\sim}{E}}^e \cdot \underset{\sim}{F}^{e-1} \quad \text{SINCE} \quad \mathring{\underset{\sim}{E}}^e = \underset{\sim}{F}^{eT} \cdot \underset{\sim}{D}^e \cdot \underset{\sim}{F}^e$$

$$= \underset{\sim}{F}^{e-T} \cdot \frac{1}{2}\left(\dot{\underset{\sim}{F}}^{eT} \cdot \underset{\sim}{F}^{eT} + \underset{\sim}{F}^{eT} \cdot \dot{\underset{\sim}{F}}^e\right) \cdot \underset{\sim}{F}^{e-1} = \left(\dot{\underset{\sim}{E}}^e \cdot \underset{\sim}{F}^{e-1}\right)^{sy}$$

By noting that

$$W = W^e + W^r + W$$

$$\overset{\square}{\sigma} = \dot{\sigma} - (\underbrace{D^e + W - W^p}_{L^e}) \cdot \sigma + \sigma \cdot (\underbrace{-D^e + W - W^p}_{L^e})$$

$$\overset{\wedge}{\sigma} = \overset{\nabla}{\sigma} + (-D^e + W^p) \cdot \sigma - \sigma \cdot (D^e + W^r)$$

since

$$\overset{\nabla}{\sigma} = \dot{\sigma} - W \cdot \sigma + \sigma \cdot W$$

Jaumann rate treated all spin as a rigid body spin. Does not diff. between intermediate config. & there. Not based on physical admissibility.

Next, we must push forward the rate of change of stress from the intermediate to the current config.

(ii) $$\overset{\vee}{\sigma} = J_e^{-1} F^e \cdot \hat{\sigma}^{PK(2)} \cdot F^{e^T}$$

↑ co-rodformational rate

This rate of change is called co-rodformational because both F^e and σ are co-rotated with W.

Combining (i) & (ii) in current config.

$$\overset{\vee}{\sigma} = \overset{\square}{\sigma} + \frac{\dot{J}_e}{J_e}\sigma = \overset{\square}{\sigma} - \frac{\dot{\rho}}{\rho}\sigma = \overset{\square}{\sigma} + \frac{\dot{V}}{V}\sigma = \overset{\square}{\sigma} + \sigma \operatorname{tr} D^e$$

Likewise, the fluxes are specified in the int config.

i.e. $$\hat{\underset{\sim}{\xi}}_j = \frac{\partial \Omega(\hat{\underset{\sim}{F}}_L, T, \hat{\underset{\sim}{\xi}}_K)}{\partial \hat{\underset{\sim}{F}}_j}$$

or $$\hat{\underset{\sim}{\xi}}_j = g_j(\hat{\underset{\sim}{\sigma}}^{PK(2)}, T, \hat{\underset{\sim}{\xi}}_K) \text{ or } \hat{\underset{\sim}{g}}_j(\hat{\underset{\sim}{F}}_K, T)$$

The plastic strain rate, $\underset{\sim}{D}^p$, and plastic spin, W^p, must also be specified

no potential $$\underset{\sim}{D}^p = \underset{\sim}{N}^p(\hat{\underset{\sim}{\sigma}}^{PK(2)}, \hat{\xi}_j) = \left(\underset{\sim}{F}^e \cdot (\hat{\underset{\sim}{D}}^p + \hat{\underset{\sim}{W}}^p) \cdot \underset{\sim}{F}^{e-1}\right)^{sym}$$

where

$$\hat{\underset{\sim}{D}}^p = \hat{h}(\hat{\underset{\sim}{\sigma}}^{PK(2)}, T, \hat{\xi}_i)$$

$$\hat{\underset{\sim}{W}}^p = \left(\underset{\sim}{F}^e \cdot (\hat{\underset{\sim}{D}}^p + \hat{\underset{\sim}{W}}^p) \cdot \underset{\sim}{F}^{e-1}\right)^a$$

where

$$\hat{\underset{\sim}{W}}^p = \hat{\underset{\sim}{\Omega}}^p(\hat{\underset{\sim}{\sigma}}^{PK(2)}, T, \hat{\xi}_j)$$

update in intermediate conf. & push forward to current config.

$$\underset{\sim}{D}^e = \left[\dot{\underset{\sim}{F}}^e \cdot \underset{\sim}{F}^{e-1}\right]^{sym}$$ — F^e is a 2 point tensor

$$\underset{\sim}{D}^p = \left[\underset{\sim}{F}^e \cdot \dot{\underset{\sim}{F}}^p \cdot \underset{\sim}{F}^{p-1} \cdot \underset{\sim}{F}^{e-1}\right]^{sym}$$

$W = (\dot{F} \cdot F^{-1})^a = W^e + W^p + W$ wrt a fixed reference frame

$W^e = \left[\dot{F}^e \cdot F^{e-1}\right]^a$, $W^p = \left[F^e \cdot \dot{F}^p \cdot F^{p-1} \cdot F^{e-1}\right]^a$

For an isothermal process, assuming the complementary free energy, $\hat{G}^e$, is decomposed into elastic & plastic components

$$\dot{\hat{E}}^e = \hat{\rho} \frac{\partial^2 \hat{G}^e}{\partial \hat{\sigma}^{PK(2)} \cdot \partial \hat{\sigma}^{PK(2)}} : \dot{\hat{\sigma}}^{PK(2)} = \hat{\mathcal{L}} : \dot{\hat{\sigma}}^{PK(2)}$$

linear case

For linear elasticity $\dot{\hat{E}}^e = \hat{\mathcal{L}} : \dot{\hat{\sigma}}^{PK(2)}$

$\hat{\mathcal{L}}$ = compliance tensor wrt the intermediate configuration

In the current configuration

$$D^e = F^{e-T} \cdot \dot{\hat{E}}^e \cdot F^{e-1}$$

For stress you need the time rate of change to push forward to current config.

$$\dot{\hat{\sigma}}^{PK(2)} = J_e \left(F^{e-1} \cdot \dot{\sigma} \cdot F^{e-T}\right)$$

Summary:

(A) General case of finite strain with decomposed complementary free energy:

$$\underset{\sim}{D} = \underset{\sim}{D}^e + \underset{\sim}{D}^p = \underset{\sim}{D}^e + \underset{\sim}{D}^{in} \quad ; \quad G = G^e(\hat{\underset{\sim}{\sigma}}^{pk(2)}, T) + G^{in}(\hat{\xi}_i, T)$$

where $\hat{\underset{\sim}{\sigma}}^{pk(2)}$ and $\hat{\xi}_i$ are defined in the isoclinic configuration.

I. Elastic relation:

$$\overset{\nabla}{\underset{\sim}{\sigma}} = \underset{\approx}{\mathcal{L}} : \underset{\sim}{D}^e$$ For hypoelastic, isothermal law neglecting elasto-plastic coupling

or $$\hat{\underset{\sim}{E}}^e = \hat{\rho} \frac{\partial \hat{G}^e}{\partial \hat{\underset{\sim}{\sigma}}^{pk(2)}}$$ in general

with $$\underset{\sim}{D}^e = (\underset{\sim}{F}^{e-1})^T \cdot \hat{\underset{\sim}{E}}^e \cdot \underset{\sim}{F}^{e-1}$$

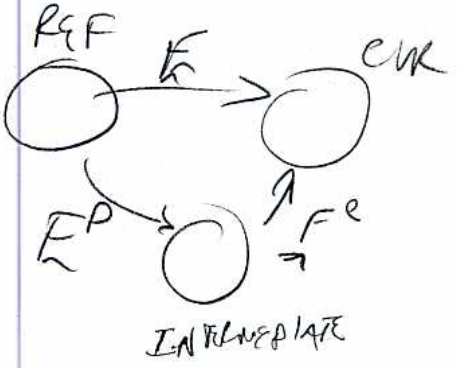
II. Inelastic relations:

(i) flow rule: $$\hat{\underset{\sim}{D}}^p = \hat{\underset{\sim}{h}}(\hat{\underset{\sim}{\sigma}}^{pk(2)}, T, \hat{\xi}_i) \quad \text{or} \quad \hat{\underset{\sim}{D}}^p = \frac{\partial \hat{\Omega}}{\partial \hat{\underset{\sim}{\sigma}}^{pk(2)}}$$

(ii) plastic spin: $$\hat{\underset{\sim}{W}}^p = \hat{\underset{\sim}{\Omega}}^p(\hat{\underset{\sim}{\sigma}}^{pk(2)}, T, \hat{\xi}_i)$$

Isotropic tensor function (so is $\hat{\underset{\sim}{h}}$)

} quantities in isoclinic configuration



In current configuration, $\underset{\sim}{L}^p = \dot{\underset{\sim}{F}}^p \underset{\sim}{F}^{p-1}$

$$\underset{\sim}{D}^p = \left[\underset{\sim}{F}^e \cdot (\hat{\underset{\sim}{D}}^p + \hat{\underset{\sim}{W}}^p) \cdot \underset{\sim}{F}^{e-1} \right]^{sym}$$

$$\underset{\sim}{W}^p = \left[\underset{\sim}{F}^e \cdot (\hat{\underset{\sim}{D}}^p + \hat{\underset{\sim}{W}}^p) \cdot \underset{\sim}{F}^{e-1} \right]^{a}$$

Alternatively,

$$\mathcal{R}^p = \underset{\sim}{h}\,(\underset{\sim}{\sigma}, T, \xi_j) \quad \text{or} \quad \mathcal{R}^p = \frac{\partial \Omega}{\partial \underset{\sim}{\sigma}}$$

$$\underset{\sim}{W}^p = \underset{\sim}{\Omega}^p(\underset{\sim}{\sigma}, T, \xi_j)$$

where the ξ_j are either updated in the isoclinic configuration and then transported into the reference configuration or are expressed via evolution equations in the current configuration, i.e.

INTERMEDIATE CONFIG.

$$\overset{\circ}{\hat{\xi}}_j = \hat{H}_j(\hat{\underset{\sim}{\sigma}}^{pk(2)}, T, \hat{\xi}_j) \quad \text{or} \quad \overset{\circ}{\xi}_j = \frac{\partial \hat{\Omega}}{\partial \underset{\sim}{F}_j}$$

$$\text{or} \quad \overset{\nabla}{\xi}_j = H_j(\underset{\sim}{\sigma}, T, \xi_j) \quad \text{or} \quad \overset{\nabla}{\xi}_j = \frac{\partial \Omega}{\partial \underset{\sim}{F}_j}$$

where $\hat{\xi}_j = J e^w$

Note: Use of $\hat{\underset{\sim}{E}}^e = \hat{\rho} \frac{\partial \hat{G}^e}{\partial \hat{\underset{\sim}{\sigma}}^{pk(2)}}$ necessitates tracking of intermediate configuration.

(B) Case of Small Elastic Strain (Common for metals)

current $\leftrightarrow$ intermediate (after rigid rotation)

$$\underset{\sim}{F}^e \cong \underset{\sim}{I}\ ,\ J^e \cong 1\ ,\ \overset{\circ}{\hat{\underset{\sim}{E}}}{}^e \to \underset{\sim}{D}^e\ ,\ \underset{\sim}{\sigma}^{PK(2)} \cong \underset{\sim}{\sigma}\ ,\ \hat{\xi}_j \cong \xi_j$$

SMALL STRAIN

$$\underset{\sim}{\hat{D}}^P \cong \underset{\sim}{D}^P\ ,\ \underset{\sim}{\hat{W}}^P \cong \underset{\sim}{W}^P\ ,\ \underset{\sim}{E}^e \cong \underset{\sim}{\hat{E}}^e\ ,\ \underset{\sim}{\mathcal{L}} \cong \hat{\underset{\sim}{\mathcal{L}}}\ ,\ \tau \cong \sigma$$

ONLY DIFFERENCE BETWEEN CURRENT & INTERMEDIATE CONFIG. IS RIGID BODY ROTATION (LIKE IN CODES)

$$\sigma\, tr\, \underset{\sim}{D}^e \cong 0$$

Thus,

I. Elastic relation:

$$\underset{\sim}{D}^e \cong \underset{\sim}{\mathcal{L}} : \overset{\circ}{\underset{\sim}{\sigma}}$$

for hypoelastic, isothermal law neglecting elasto-plastic coupling

$= F^e F^P$

since $\overset{\circ}{\underset{\sim}{\sigma}} \cong \overset{\circ}{\underset{\sim}{\sigma}} = \dot{\underset{\sim}{\sigma}} - \underset{\sim}{\omega}\cdot\underset{\sim}{\sigma} + \underset{\sim}{\sigma}\cdot\underset{\sim}{\omega}$

COROTATIONAL w.r.t. LATTICE FOR STRESSES AND ISV's:

$$= \dot{\underset{\sim}{\sigma}} - (\underset{\sim}{W} - \underset{\sim}{W}^P)\cdot\underset{\sim}{\sigma} + \underset{\sim}{\sigma}\cdot(\underset{\sim}{W} - \underset{\sim}{W}^P)$$

$$= \overset{\Delta}{\underset{\sim}{\sigma}} + \underset{\sim}{W}^P\cdot\underset{\sim}{\sigma} - \underset{\sim}{\sigma}\cdot\underset{\sim}{W}^P$$

since $\underset{\sim}{W}^e \cong \underset{\sim}{0}$

or

$$\underset{\sim}{E}^e \cong \rho\,\frac{\partial G^e}{\partial \underset{\sim}{\sigma}}$$

II. Inelastic relations:

(i) Flow rule: $\underset{\sim}{D}^P = \underset{\sim}{h}(\underset{\sim}{\sigma}, T, \xi_i)$ or $\underset{\sim}{D}^P = \frac{\partial \Omega}{\partial \underset{\sim}{\sigma}}$

(ii) plastic spin: $\underset{\sim}{W}^P = \underset{\sim}{\Omega}^P(\underset{\sim}{\sigma}, T, \xi_i)$

Internal variables :

$$\dot{\underset{\sim}{\xi}}_j = \underset{\sim}{H}_j(\underset{\sim}{\sigma}, T, \underset{\sim}{\xi}_k) \quad \text{or} \quad \dot{\underset{\sim}{\xi}}_j = \frac{\partial \Omega}{\partial \underset{\sim}{F}_j}$$

→ No need for ~~an~~ explicit consideration of an intermediate configuration.