CONT. MECHANICS REVIEW

DERIVATIVE OF A TENSOR FIELD:

		EXAMPLE
$\varphi = \varphi(\underset{\sim}{x}, t)$	SCALAR	TEMP.
$\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{x}, t)$	VECTOR	VELOCITY
$\underset{\sim}{A} = \underset{\sim}{A}(\underset{\sim}{x}, t)$	2^{nd} RANK TENSOR	STRESS
$\underset{\approx}{C} = \underset{\approx}{C}(x, t)$	4^{th} RANK TENSOR	ELASTIC MODULI

$$\underset{\sim}{\nabla} = \vec{\underset{\sim}{\nabla}} = \underset{\sim}{e}_i \partial_i = e_i \frac{\partial}{\partial x_i} \quad (\text{SUM ON } i)$$ - LIMITED TO CARTESIAN COORDINATES

NOTE: MOST CONSTITUTIVE EQTNS ARE EMBODIED IN CARTESIAN SYSTEMS

$$\text{GRAD } \varphi = \underset{\sim}{\nabla} \varphi = \underset{\sim}{e}_i \varphi_{,i} = \underset{\sim}{e}_i \frac{\partial \varphi}{\partial x_i}$$ (INCREASES RANK)

$$\text{DIV } \underset{\sim}{v} = \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \frac{\partial v_k}{\partial x_k} = v_{k,k}$$ (DECREASES RANK)

$$\text{CURL } \underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{v} = v_{j,i} \, \epsilon_{ijk} \, \underset{\sim}{e}_k$$ (MAINTAINS RANK)

$$\text{LAPLACIAN } \varphi = \nabla^2 \varphi = \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} \varphi = \varphi_{,ii}$$

FOR A SECOND RANK TENSOR

$$\text{DIV } \underset{\sim}{A} = \underset{\sim}{\nabla} \cdot \underset{\sim}{A} = A_{im,i} \, \underset{\sim}{e}_m$$

$$\text{CURL } \underset{\sim}{A} = \underset{\sim}{\nabla} \times \underset{\sim}{A} = \frac{\partial A_{mn}}{\partial x_i} (\underset{\sim}{e}_i \times \underset{\sim}{e}_m) \underset{\sim}{e}_n$$

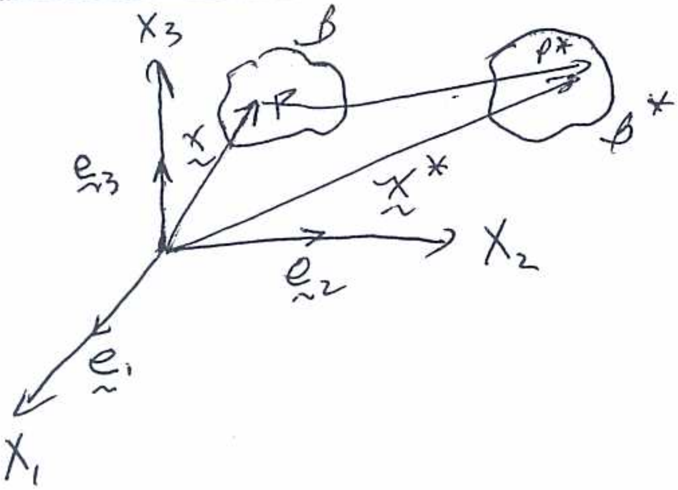
$$\text{GRAD } \underset{\sim}{A} = \underset{\sim}{\nabla} \underset{\sim}{A} = \underset{\sim}{e}_i \frac{\partial A_{mn}}{\partial x_i} \underset{\sim}{e}_m \underset{\sim}{e}_n$$

DIVERGENCE THRM (GENERAL)

$$\int_{\forall} A_{ijkl\cdots,p}\, d\forall = \int_{S} A_{ijkl\cdots}\, n_p\, dS$$

(USED IN BALANCE LAWS: MASS, MOMENTUM, ENERGY)

STRAIN AND KINEMATICS OF DEFORMATION



$\mathcal{B}$ → UNDEFORMED CONFIGURATION

$\mathcal{B}^*$ → CURRENT CONFIGURATION

$\underset{\sim}{x}$ → LAGRANGIAN COORDS

$\underset{\sim}{x}^*$ → EULERIAN OR SPATIAL COORDS

$\underset{\sim}{x}^* = \underset{\sim}{x}^*(\underset{\sim}{x}, t)$ IS THE MOTION OF POINT P

$\underset{\sim}{u}(\underset{\sim}{x}, t) = \underset{\sim}{x}^* - \underset{\sim}{x} \equiv$ DISPLACEMENT

IN THE NEIGHBORHOOD OF P,

$$d\underset{\sim}{x}^* = \frac{\partial \underset{\sim}{x}^*}{\partial \underset{\sim}{x}} \cdot d\underset{\sim}{x} = \underset{\sim}{F} \cdot d\underset{\sim}{x}$$

$\underset{\sim}{F}$ = DEFORMATION GRADIENT (LAGRANGIAN)

$$F_{ij} = \frac{\partial x_i^*}{\partial x_j}$$

$J = \det(F_{ij})$ = JACOBIAN OF DEF. GRADIENT

$d\forall = J \, d\forall_0$ ⇒ RELATES LAGRANGIAN WITH EULERIAN SYSTEMS.

DEFORMED VOLUME ($d\forall$), REFERENCE VOLUME ($d\forall_0$)

NOTE: J IS RELATED TO VOLUME CHANGE

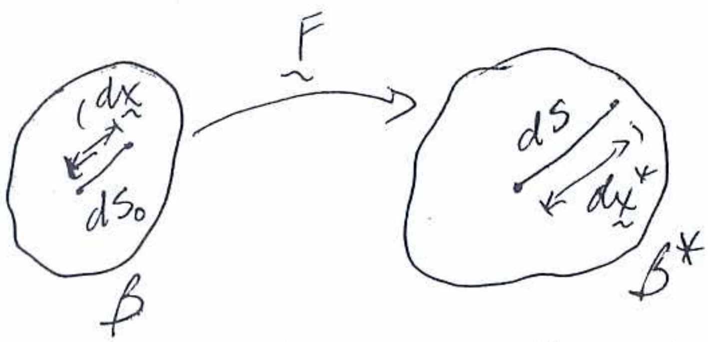
BY CONSERVATION OF MASS.

$$\rho \, d\forall = dm = \rho_0 \, d\forall_0 = dm_0$$

$$\Rightarrow J = \frac{d\forall}{d\forall_0} = \frac{\rho_0}{\rho} \quad \text{LAGRANGIAN STATEMENT}$$

MEASURES OF STRAIN

$$\left.\begin{array}{l} \underset{\sim}{C} = \underset{\sim}{F}^T \cdot \underset{\sim}{F} \\ C_{ij} = \dfrac{\partial x_m^*}{\partial x_i} \dfrac{\partial x_m^*}{\partial x_j} \; (= C_{ji}) \end{array}\right\} = \text{RIGHT CAUCHY-GREEN TENSOR}$$



$$(dS)^2 = d\underset{\sim}{x} \cdot \underset{\sim}{C} \cdot d\underset{\sim}{x}$$

$$(dS)^2 - (dS_0)^2 = 2 \, d\underset{\sim}{x} \cdot \underset{\sim}{E} \cdot d\underset{\sim}{x}$$

$$\underset{\sim}{E} = \tfrac{1}{2}(\underset{\sim}{C} - \underset{\sim}{I}) \equiv \text{GREEN - ST. VENANT STRAIN TENSOR}$$

$$\left.\begin{aligned} C_{ij} &= \delta_{ij} + u_{j,i} + u_{i,j} + u_{k,i}\, u_{k,j} \\ E_{ij} &= \tfrac{1}{2}\left[u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j} \right. \end{aligned}\right\} \text{LAGRANGIAN}$$

$\underset{\sim}{U} = \underset{\sim}{C}^{1/2} \equiv$ RIGHT STRETCH TENSOR

↳ USED IN POLAR DECOMPOSITION

$\underset{\sim}{F} = \underset{\sim}{R}\,\underset{\sim}{U}$

$$\lambda(i) = \sqrt{\frac{ds_{(i)}^2}{dS_0^2(i)}} \equiv \text{STRETCH RATIO IN } d\underset{\sim}{x}_{(i)} \text{ DIRECTION}$$

EIGENVALUES (PRINCIPAL VALUES) OF $\underset{\sim}{U}$ ARE THE PRINCIPAL STRETCH RATIOS

FOR INFINITESIMAL STRAIN,

$$|u_i| < \epsilon_1, \quad |u_{i,j}| < \epsilon_2$$

$$\Rightarrow \epsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

NOTE: ϵ_{ij} IS WITHOUT DISTINCTION OF DIFFERENTION wrt $\underset{\sim}{X}$ OR $\underset{\sim}{X}^*$ SINCE $\underset{\sim}{X} \cong \underset{\sim}{X}^*$.

RIGHT POLAR DECOMPOSITION

WE ASSUME THAT THE 3 ORTHOGONAL PRINCIPAL DIRECTIONS OF THE STRETCH TENSOR <u>REMAIN</u> ORTHOGONAL THROUGHOUT THE DEFORMATION

$$\underset{\sim}{F} = \underset{\sim}{R} \cdot \underset{\sim}{U}$$

$\underset{\sim}{R}$ — ROTATION TENSOR

$$\left.\begin{aligned} \underset{\sim}{R}^T \cdot \underset{\sim}{R} &= \underline{\underline{I}} \\ \text{DET } \underset{\sim}{R} &= 1 \end{aligned}\right\} \underset{\sim}{R} \text{ IS PROPER ORTHOGONAL}$$

MATERIAL TIME DERIVATIVE

$$\dot{(\ )} = \frac{D}{Dt} = \underbrace{\left.\frac{\partial}{\partial t}\right|_{\underset{\sim}{x}^*}}_{\text{LOCAL}} + \underbrace{\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_{\underset{\sim}{x}^*}}_{\text{CONVECTIVE}}$$

WHERE $\underset{\sim}{v} = \left.\frac{\partial \underset{\sim}{x}^*}{\partial t}\right|_{\underset{\sim}{X}} = \frac{d\underset{\sim}{x}^*}{dt} \quad (\underset{\sim}{v} = \underset{\sim}{v}(\underset{\sim}{x}^*, t))$

FOR DIFFERENTIATION IN SPATIAL (EULERIAN) SETTING.

FOR STEADY FLOW: $\left.\frac{\partial}{\partial t}\right|_{\underset{\sim}{x}^*} = 0$

FOR UNIFORM FLOW: $\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_{\underset{\sim}{x}^*} = 0$

EULERIAN RATE OF DEFORMATION + SPIN

$$\frac{D\underset{\sim}{F}}{Dt} = \dot{\underset{\sim}{F}} = \underset{\sim}{L} \cdot \underset{\sim}{F}$$

$$\underset{\sim}{L} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}^*} \rightarrow L_{ij} = \frac{\partial v_i}{\partial x_j^*} \equiv \text{SPATIAL VEL. GRADIENT}$$

$$\left.\begin{aligned} \underset{\sim}{D} &= \tfrac{1}{2}\left(\vec{\nabla}_{\underset{\sim}{x}^*}\underset{\sim}{v} + \underset{\sim}{v}\overleftarrow{\nabla}_{\underset{\sim}{x}^*}\right) \\ D_{ij} &= \tfrac{1}{2}\left(v_{i,j} + v_{j,i}\right) \end{aligned}\right\}$$

$\underset{\sim}{D} \equiv$ RATE OF DEFORMATION TENSOR

$$\underset{\sim}{D} = \left(\vec{\underset{\sim}{\nabla}}\,\underset{\sim}{v}\right)^s$$

$$\underset{\sim}{L} = \underset{\sim}{D} + \underset{\sim}{W}$$

$\underset{\sim}{W}$: ANTI-SYMMETRIC (CONTINUUM SPIN)

$\underset{\sim}{D}$: SYMMETRIC

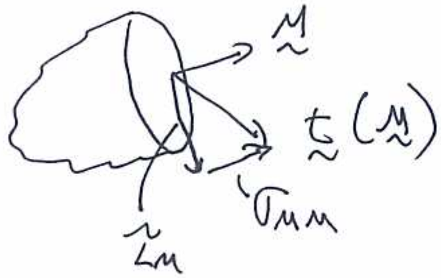
$$\dot{\underset{\sim}{E}} = \underset{\sim}{F}^T \cdot \underset{\sim}{D} \cdot \underset{\sim}{F}$$ MATERIAL TIME RATE OF GREEN - ST. VENANT STRAIN

FOR INFINITESIMAL STRAIN

$$\dot{\underset{\sim}{E}} \cong \underset{\sim}{D} \cong \dot{\underset{\sim}{\epsilon}}$$

STRESS

$$\underset{\sim}{t}(\underset{\sim}{n}) = \frac{d\,\underset{\sim}{f}_n}{dA_n}$$



CAUCHY STRESS: $\sigma_{nn} = \underset{\sim}{t}(\underset{\sim}{n}) \cdot \underset{\sim}{n}$

$$\tau_n = \sqrt{\underset{\sim}{t} \cdot \underset{\sim}{t} - \sigma_{nn}^2}$$

CAUCHY'S FIRST LAW

$$\underset{\sim}{t} = \underset{\sim}{n} \cdot \underset{\sim}{\sigma}$$

$$t_i = n_j \cdot \sigma_{ji}$$

EQTN OF MOTION (EULERIAN) - LINEAR MOMENTUM BALANCE

$$\underset{\sim}{\nabla}_{x^*} \cdot \underset{\sim}{\sigma} + \rho\,\underset{\sim}{b} = \rho\,\underset{\sim}{a}$$

$$\sigma_{ij,i} + \rho b_j = \rho a_j = \rho \frac{Dv_j}{Dt} = \rho \frac{\partial v_j}{\partial t} + \rho v_k \frac{\partial v_j}{\partial x_k^*}$$

QUASI-STATIC EQUILIBRIUM : $\underset{\sim}{a} = \underset{\sim}{0}$

ANGULAR MOMENTUM: $\sigma_{ij} = \sigma_{ji}$

PIOLA - KIRCHOFF STRESSES (LAGRANGIAN)

$$\underset{\sim}{\sigma}^{PK(1)} = J \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \quad \} \text{ NOT SYMMETRIC}$$

$$\underset{\sim}{\sigma}^{PK(2)} = J \underset{\sim}{F}^{-1} \cdot \underset{\sim}{\sigma} \cdot (\underset{\sim}{F}^{-1})^T \quad \} \text{ SYMMETRIC}$$

EQTN OF MOTION IN REF. CONFIGURATION

$$\underset{\sim}{\nabla}_{\underset{\sim}{X}} \cdot (\underset{\sim}{\sigma}^{PK(2)} \cdot \underset{\sim}{F}^T) + \rho_0 b_0 = \rho_0 \ddot{\underset{\sim}{u}}$$

CONTINUITY EQTN - CONSERVATION OF MASS

$$\rho_0 = \rho J \qquad \text{LAGRANGIAN}$$

$$\frac{D\rho}{Dt} + \rho \frac{\partial v_i}{\partial x_i} = \dot{\rho} + \rho v_{i,i} = 0 \qquad \text{EULERIAN}$$

STRESS POWER

$$P_d = \int_{V} \underset{\sim}{\sigma} : \underset{\sim}{D} \, dV = \int_{V} \sigma_{ij} D_{ij} \, dV$$

$$= \int_{V_0} \underset{\sim}{\sigma}^{PK(2)} : \dot{\underset{\sim}{E}} \, dV_0$$

WORK CONJUGATES (HILL)

$\sigma \leftrightarrow D$

DEVIATORIC + HYDROSTATIC STRESS

$$\underset{\sim}{\sigma} = \underset{\sim}{\sigma}' + \underset{\sim}{\sigma}^{\forall}$$

$$\underset{\sim}{\sigma}' = \underset{\sim}{\sigma} - \underset{\sim}{\sigma}^{\forall} = \underset{\sim}{\sigma} - \underbrace{\tfrac{1}{3}\,\sigma_{kk}\,\underset{\approx}{I}}_{\underset{\sim}{\sigma}^{\forall}}$$

↑ DEVIATORIC STRESS

REYNOLDS TRANSPORT THRM

- MATERIAL TIME DERIVATIVE OF A VOLUME INTEGRAL ($\underset{\sim}{M}$ IS ARBITRARY RANK)

$$\frac{D}{Dt}\int_{\forall} \underset{\sim}{M}\, d\forall = \int_{\forall} \frac{D}{Dt}\left(\underset{\sim}{M}\, d\forall\right)$$

DIFF wrt FIXED SET OF PARTICLES

↑ FIXED CONTROL VOLUME

$$= \int_{\forall} \frac{D}{Dt}\underset{\sim}{M}\, d\forall + \int_{\forall} \underset{\sim}{M}\, \underset{\sim}{\nabla}_{\underset{\sim}{x}^*} \cdot \underset{\sim}{v}\, d\forall$$

$$= \int_{\forall} \frac{\partial \underset{\sim}{M}}{\partial t}\, d\forall + \int_{S} \left(\underset{\sim}{n} \cdot \underset{\sim}{v}\right) \underset{\sim}{M}\, ds$$