

Noting that:

$$\operatorname{div} \frac{\bar{q}}{T} = \frac{\operatorname{div} \bar{q}}{T} - \frac{\bar{q} \cdot \overrightarrow{\operatorname{grad}} T}{T^2}$$

and multiplying by $T > 0$, we obtain:

$$\rho \left(T \frac{ds}{dt} - \frac{de}{dt} \right) + \sigma : \mathbf{D} - \bar{q} \cdot \frac{\overrightarrow{\operatorname{grad}} T}{T} \geq 0.$$

The Clausius–Duhem inequality is obtained by introducing a new variable, the specific free energy Ψ defined by:

$$\Psi = e - Ts.$$

Differentiating this, we obtain:

$$\frac{d\Psi}{dt} = \frac{de}{dt} - T \frac{ds}{dt} - s \frac{dT}{dt} \quad \text{or} \quad T \frac{ds}{dt} - \frac{de}{dt} = - \left(\frac{d\Psi}{dt} + s \frac{dT}{dt} \right)$$

which, when substituted in the first term of the fundamental inequality, yields:

$$\sigma : \mathbf{D} - \rho \left(\frac{d\Psi}{dt} + s \frac{dT}{dt} \right) - \bar{q} \cdot \frac{\overrightarrow{\operatorname{grad}} T}{T} \geq 0.$$

For small perturbations, the above may be written as:

$$\bullet \quad \sigma : \dot{\epsilon} - \rho (\dot{\Psi} + s \dot{T}) - \bar{q} \cdot \frac{\overrightarrow{\operatorname{grad}} T}{T} \geq 0$$

or

$$\sigma_{ij} \dot{\epsilon}_{ij} - \rho (\dot{\Psi} + s \dot{T}) - q_i \frac{T_{,i}}{T} \geq 0.$$

2.4 Method of local state

In order to avoid any confusion in the reader's mind, it is time to summarize the variables used to describe the thermomechanical behaviour of solids. The concept of thermodynamic potential will clarify everything! However, before giving its definition, a choice must be made with regard to the nature of the variables. In this choice lie both the weakness and the richness of the method of phenomenological thermodynamics: weakness because the choice is partly subjective and results in different models depending on the

inclination of the authors, and richness because it allows the formulation of the theories to be adopted to the study of one or more phenomena, either coupled or uncoupled, depending on the intended use.

2.4.1 State variables

The method of local state postulates that the thermodynamic state of a material medium at a given point and instant is completely defined by the knowledge of the values of a certain number of variables at that instant, which depend only upon the point considered. Since the time derivatives of these variables are not involved in the definition of the state, this hypothesis implies that any evolution can be considered as a succession of equilibrium states. Therefore, ultrarapid phenomena for which the time scales of the evolutions are of the same order as the relaxation time for a return to thermodynamic equilibrium (atomic vibrations) are excluded from this theory's field of application. Physical phenomena can be described with a precision which depends on the choice of the nature and the number of state variables. The processes defined in this way will be thermodynamically admissible if, at any instant of the evolution, the Clausius–Duhem inequality is satisfied. The state variables, also called thermodynamic or independent variables, are the observable variables and the internal variables.

Observable variables

The formalism of continuum mechanics and thermodynamics as developed above requires the existence of a certain number of state variables; these are the observable variables:

the temperature T

the total strain ϵ (assuming small strains).

We limit ourselves to the two observable variables as they are the only ones which occur in elasticity, viscoelasticity, plasticity, viscoplasticity, damage and fracture phenomena. For reversible (or elastic) phenomena, at every instant of time, the state depends uniquely on these variables. For example, the reversible power is defined with the help of the associated stress σ as:

$$\Phi_e = \sigma : \dot{\epsilon}.$$

Internal variables

For dissipative phenomena, the current state also depends on the past history which is represented, in the method of local state, by the values at each instant of other variables called internal variables.

Plasticity and viscoplasticity require the introduction of the plastic (or viscoplastic) strain as a variable. For small strains, the plastic strain ϵ^p is the permanent strain associated with the relaxed configuration. This configuration is obtained by 'elastic unloading', leading to the additive strain decomposition:

$$\epsilon = \epsilon^p + \epsilon^e.$$

The two internal variables related to the above decomposition may formally be defined as: the plastic strain ϵ^p , and the thermoelastic strain ϵ^e (including, as well, the possibility of thermal dilatation).

Other phenomena such as hardening, damage, fracture, require the introduction of other internal variables of a less obvious nature. These represent the internal state of matter (density of dislocations, crystalline microstructure, configuration of microcracks and cavities, etc.) and there are no means of measuring them by direct observation. They do not appear explicitly either in the conservation laws or in the statement of the second principle of thermodynamics. They are called internal variables, but in fact, they are state variables which will be treated as observable ones.

There is no objective way of choosing the nature of the internal variables best suited to the study of a phenomenon. The choice is dictated by experience, physical feeling and very often by the type of application. They will be defined in the different chapters as the need arises. For their general study, they will be denoted by $V_1, V_2, \dots, V_k, \dots$; V_k representing either a scalar or a tensorial variable.

2.4.2 *Thermodynamic potential, state laws*

Once the state variables have been defined, we postulate the existence of a thermodynamic potential from which the state laws can be derived. Without entering into the details, let us say that the specification of a function with a scalar value, concave with respect to T , and convex with respect to other variables, allows us to satisfy *a priori* the conditions of thermodynamic stability imposed by the inequalities that can be derived from the second principle. It is possible to work in an equivalent way with different potentials. Here we choose the free specific energy potential Ψ ,

which depends on observable state variables and internal variables:

$$\Psi = \Psi(\epsilon, T, \epsilon^e, \epsilon^p, V_k).$$

In elastoplasticity (or viscoplasticity) the strains appear only in the form of their additive decomposition $\epsilon - \epsilon^p = \epsilon^e$, so that:

$$\Psi = \Psi((\epsilon - \epsilon^p), T, V_k) = \Psi(\epsilon^e, T, V_k)$$

which shows that:

$$\partial\Psi/\partial\epsilon^e = \partial\Psi/\partial\epsilon = -\partial\Psi/\partial\epsilon^p.$$

We now use the Clausius–Duhem inequality with:

$$\dot{\Psi} = \frac{\partial\Psi}{\partial\epsilon^e} : \dot{\epsilon}^e + \frac{\partial\Psi}{\partial T} \dot{T} + \frac{\partial\Psi}{\partial V_k} \dot{V}_k.$$

to obtain:

$$\left(\sigma - \rho \frac{\partial\Psi}{\partial\epsilon^e} \right) : \dot{\epsilon}^e + \sigma : \dot{\epsilon}^p - \rho \left(s + \frac{\partial\Psi}{\partial T} \right) \dot{T} - \rho \frac{\partial\Psi}{\partial V_k} \dot{V}_k - \frac{\bar{q}}{T} \cdot \overrightarrow{\text{grad}} T \geq 0.$$

A classical hypothesis permits us to cancel some terms in this inequality independently. We may imagine, first of all, an elastic deformation taking place at constant ($\dot{T} = 0$) and uniform ($\overrightarrow{\text{grad}} T = 0$) temperature which alters neither the plastic strain ($\dot{\epsilon}^p = 0$) nor the internal variables ($\dot{V}_k = 0$). For this to happen, it is necessary to consider that the elastic deformations can occur at a time scale higher than those which would question the validity of the hypothesis of local state, and lower than those of dissipative phenomena. Since the Clausius–Duhem inequality holds regardless of any particular $\dot{\epsilon}^e$, it necessarily follows that:

$$\sigma - \rho(\partial\Psi/\partial\epsilon^e) = 0.$$

Assuming this equality to hold, we now imagine a thermal deformation in which $\dot{\epsilon}^p = 0$, $\dot{V}_k = 0$, $\overrightarrow{\text{grad}} T = 0$. Then, since T is arbitrary, it follows that:

$$s + \partial\Psi/\partial T = 0.$$

These expressions define the thermoelastic laws:

- $\sigma = \rho(\partial\Psi/\partial\epsilon^e),$
- $s = -\partial\Psi/\partial T.$

We note that:

$$\sigma = \rho(\partial\Psi/\partial\epsilon^e) = \rho(\partial\Psi/\partial\epsilon) = -\rho(\partial\Psi/\partial\epsilon^p)$$

which shows that the stress is a variable associated with the elastic strain, with the total strain, and with the plastic strain (with a minus sign).

In an analogous manner, we define the thermodynamic forces associated with the internal variables by:

$$A_k = \rho(\partial\Psi/\partial V_k).$$

These relations constitute the state laws:

the entropy s and the stress tensor σ having been defined elsewhere, the specification of the thermodynamic potential $\Psi(\varepsilon^e, T, V_k)$ furnishes the coupled or uncoupled theories of thermo-elasticity;

in contrast, the variables A_k associated with the internal variables, which have not yet been introduced, are defined by the specification of the thermodynamic potential $\Psi(\dots, V_k)$;

s, σ and A_1, A_2, \dots, A_k constitute the associated variables. The vector formed by these variables is the gradient of the function Ψ in the space of the variables T, ε^e, V_k . This vector is normal to the surface $\Psi = \text{constant}$.

The associated variables form a set of normal variables in duality with the observable and internal state variables. Table 2.2 summarizes the set of variables introduced in this way.

2.4.3 Dissipation, complementary laws

As we have seen, the thermodynamic potential allows us to write relations between observable state variables and associated variables. However, for internal variables it allows only the definition of their associated variables. In order to describe the dissipation process, mainly the evolution of the internal variables, a complementary formalism is needed. This is precisely the objective of the dissipation potentials.

Intrinsic dissipation, thermal dissipation

Taking into account the state laws and putting $\vec{g} = \overrightarrow{\text{grad}} T$, the Clausius-Duhem inequality can be reduced to express the fact that dissipation is necessarily positive:

$$\bullet \quad \Phi = \sigma : \dot{\varepsilon}^p - A_k \dot{V}_k - \vec{g} \cdot \vec{q} / T \geq 0.$$

We note that Φ is a sum of the products of the force variables or dual

Method of local state

Table 2.2. Thermodynamic variables

| State variables | | |
|-----------------|-----------------|----------------------|
| Observable | Internal | Associated variables |
| ε | | σ |
| T | | s |
| | ε^e | σ |
| | ε^p | $-\sigma$ |
| | V_k | A_k |

variables σ, A_k, \vec{g} with the respective flux variables $\dot{\varepsilon}^p, -\dot{V}_k, -\vec{q}/T$. The sum of the first two terms:

$$\Phi_1 = \sigma : \dot{\varepsilon}^p - A_k \dot{V}_k$$

is called the intrinsic dissipation (or mechanical dissipation). It consists of plastic dissipation plus the dissipation associated with the evolution of the other internal variables; it is generally dissipated by the volume element in the form of heat. The last term:

$$\Phi_2 = -\vec{g} \cdot \vec{q} / T = -\frac{\vec{q}}{T} \cdot \overrightarrow{\text{grad}} T$$

is the thermal dissipation due to the conduction of heat.

Dissipation potential

In order to define the complementary laws related to the dissipation process, we postulate the existence of a dissipation potential (or pseudo-potential) expressed as a continuous and convex scalar valued function of the flux variables, wherein the state variables may appear as parameters:

$$\varphi(\dot{\varepsilon}^p, \dot{V}_k, \vec{q}/T).$$

This potential is a positive convex function with a zero value at the origin of the space of the flux variables, $\dot{\varepsilon}^p, \dot{V}_k, \vec{q}/T$. The complementary laws are then expressed by the normality property (or normal dissipativity):

$$\sigma = \frac{\partial \varphi}{\partial \dot{\varepsilon}^p} \quad A_k = -\frac{\partial \varphi}{\partial \dot{V}_k} \quad \vec{g} = -\frac{\partial \varphi}{\partial (\vec{q}/T)}.$$

The thermodynamic forces are the components of the vector $\overrightarrow{\text{grad}} \varphi$ normal to the $\varphi = \text{constant}$ surfaces in the space of the flux variables.

In fact, the complementary laws are more easily expressed in the form of the evolution laws of flux variables as functions of dual variables. The Legendre–Fenchel transformation enables us to define the corresponding potential $\varphi^*(\sigma, A_k, \bar{g})$, the dual of φ with respect to the variables $\dot{\epsilon}^p, \dot{V}_k$ and \bar{q}/T . By definition:

$$\varphi^*(\sigma, A_k, \bar{g}) = \text{Sup}_{(\dot{\epsilon}^p, \dot{V}_k, \bar{q}/T)} ((\sigma \cdot \dot{\epsilon}^p - A_k \dot{V}_k - \bar{g} \cdot \bar{q}/T) - \varphi(\dot{\epsilon}^p, \dot{V}_k, \bar{q}/T)).$$

The transformation, written in a slightly clumsy form, is illustrated graphically in Fig. 2.4 in which only one variable has been retained.

It can be shown that, if the function φ^* is differentiable, the normality property is preserved for the variables $\dot{\epsilon}^p, -\dot{V}_k, -\bar{q}/T$, and the complementary laws of evolution can then be written as:

- $\dot{\epsilon}^p = \partial \varphi^* / \partial \sigma,$
- $-\dot{V}_k = \partial \varphi^* / \partial A_k,$
- $-\frac{\bar{q}}{T} = \partial \varphi^* / \partial \bar{g}.$

Let us note once more the properties that the potentials φ and φ^* must possess for the automatic satisfaction of the second principle of thermodynamics: they must be nonnegative, convex functions, zero at the origin: ($\sigma = A_k = g = 0$). Later, we will generally use the potential φ^* and the rela-

Fig. 2.4. Construction of the graph of a potential $\varphi^*(\sigma)$, dual of $\varphi(\dot{\epsilon}^p)$, by the Legendre–Fenchel transformation.

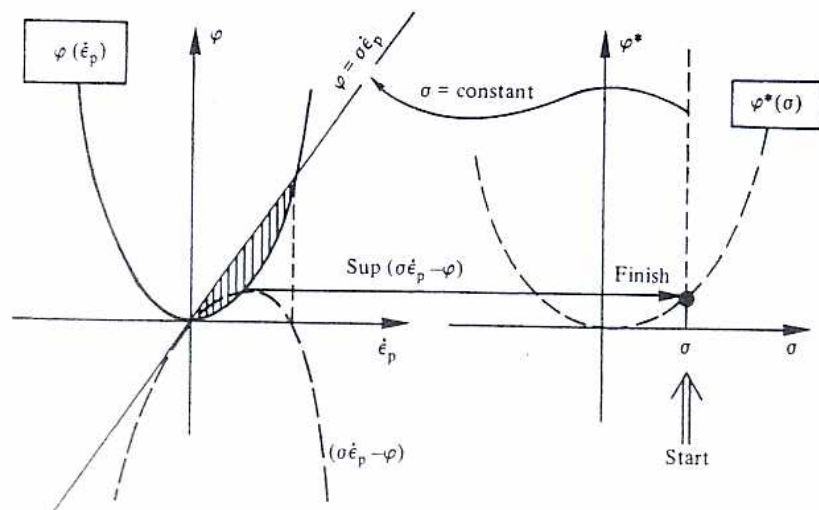


Table 2.3. Dissipation variables

| Flux variables | Dual variables |
|--------------------|----------------------------|
| $\dot{\epsilon}^p$ | σ |
| $-\dot{V}_k$ | A_k |
| $-\bar{q}/T$ | $\bar{g} = \text{grad } T$ |

tions expressing the evolution of the flux variables. It should be noted that the normality rule is sufficient to ensure the satisfaction of the second principle of thermodynamics, but it is not a necessary condition. This rule applies to generalized standard materials. A standard material is defined as that for which only the first of the above three rules, $\dot{\epsilon}^p = \partial \varphi^* / \partial \sigma$, applies. This first relation yields the plasticity or viscoplasticity laws. The second equation expresses the evolution laws of the internal variables, and the third one leads to the Fourier law of thermostatics. Table 2.3 provides a summary of the dissipation variables.

The whole problem of modelling a phenomenon lies in the determination of the analytical expressions for the thermodynamic potential Ψ and for the dissipation potential φ or its dual φ^* , and their identification in characteristic experiments. In fact the values of φ or φ^* are almost impossible to measure as they represent an energy usually dissipated as heat. The flux variables and the dual variables are quite easy to measure and it is on their values that the modelling and identification are based. The complementary laws of evolution are therefore directly identified but the dissipation potential is used as a guideline for writing their analytical expression.

It should be noted that one can generalize the dissipation potentials by including the state variables themselves as parameters. The above development is not modified at all. The dissipation potentials are then written as:

$$\begin{aligned} &\varphi(\dot{\epsilon}^p, \dot{V}_k, \bar{q}/T; \epsilon^c, T, V_k) \\ &\varphi^*(\sigma, A_k, \bar{g}; \epsilon^c, T, V_k). \end{aligned}$$

variable parameters

Onsager's symmetry relations

A first simplification consists in assuming that the function φ^* is of a positive-definite quadratic form in terms of the dual variables. Then denoting the flux variables by \dot{V}_α and the dual variables by A_α the potential

φ^* may be written

$$\varphi^* = \frac{1}{2} C_{\alpha\beta} (\varepsilon^c, T, V_k) A_\alpha A_\beta.$$

Consequently, under such circumstances, every complementary law of evolution is linear with respect to the corresponding dual variables:

$$\dot{V}_\alpha = C_{\alpha\beta} A_\beta.$$

The matrix $C_{\alpha\beta}$ is symmetric. This property is known as Onsager's symmetry relation.

Decoupling of intrinsic and thermal dissipation

A second simplification consists in assuming a decoupling of intrinsic and thermal dissipations. This does not mean that the corresponding physical mechanisms are decoupled. This assumption amounts to considering the dissipation potential as the sum of two terms, one dependent on the dual variables σ, A_k , and the other on the variable \bar{g} :

$$\varphi^* = \varphi_1^*(\sigma, A_k) + \varphi_2^*(\bar{g})$$

and the second principle of thermodynamics is satisfied by the following inequalities respectively:

$$\Phi_1 = \sigma : \dot{\varepsilon}^p - A_k \dot{V}_k = \sigma : \frac{\partial \varphi_1^*}{\partial \sigma} + A_k \frac{\partial \varphi_1^*}{\partial A_k} \geq 0$$

$$\Phi_2 = -\bar{g} \cdot \frac{\bar{q}}{T} = \bar{g} \cdot \frac{\partial \varphi_2^*}{\partial \bar{g}} \geq 0.$$

Note that, since φ^* is convex in σ and A_k and passes through the origin, the first of the above inequalities is automatically satisfied. We then have:

$$\sigma : \frac{\partial \varphi_1^*}{\partial \sigma} + A_k \frac{\partial \varphi_1^*}{\partial A_k} \geq \varphi_1^* \geq 0.$$

The phenomena of instantaneous dissipation

When the behaviour is independent of the velocities, the function $\varphi(\dot{\varepsilon}^p, \dot{V}_k)$ is a positive, homogeneous function of degree 1 and its dual function φ^* is nondifferentiable. By extension, we write that $\dot{\varepsilon}^p$ belongs to the subdifferential of φ^* defined by:

$$\partial \varphi_{(\sigma_0)}^* = \{ \dot{\varepsilon}^p / \varphi_{(\sigma_0)}^* \geq \varphi_{(\sigma_0)}^* + \dot{\varepsilon}^p : (\sigma - \sigma_0), \forall \sigma \}.$$

Elements of heat

In addition, we take the convex function of the criterion $f(\sigma, A_k)$ with $f = 0$, has φ^* as an indicator function.

$$\varphi^* = 0 \text{ if } f < 0 \rightarrow \dot{\varepsilon}^p = 0$$

$$\varphi^* = +\infty \text{ if } f = 0 \rightarrow \dot{\varepsilon}^p \neq 0.$$

A proof, not given here, allows us to assert that it is equivalent to write

$$\dot{\varepsilon}^p \in \partial \varphi_{(\sigma)}^* \text{ and } \dot{\varepsilon}^p = \frac{\partial F}{\partial \sigma} \dot{\lambda} \text{ if } \begin{cases} f = 0 \\ \dot{f} = 0 \end{cases}$$

where F is a potential function equal to f in the case of 'associated' theory and $\dot{\lambda}$ is a multiplier determined by the consistency condition $\dot{f} = 0$.

The equations describing normality have to be replaced by:

$$\dot{\varepsilon}^p = \dot{\lambda} (\partial F / \partial \sigma), \quad -\dot{V}_k = \dot{\lambda} (\partial F / \partial A_k)$$

or

$$\bullet \quad \dot{\varepsilon}^p = \dot{\lambda} \partial f / \partial \sigma, \quad -V_k = \dot{\lambda} \partial f / \partial A_k$$

2.5 Elements of heat

2.5.1 Fourier's law

The law of heat diffusion, or Fourier's law, expresses a linear relation between the heat flux vector \bar{q} and its dual variable \bar{g} . This is a direct consequence of the two simplifications introduced regarding dissipative potentials. In fact, we let:

$$\varphi_2^* = \frac{1}{2} \mathbf{C} \cdot \bar{g} \cdot \bar{g}$$

and

$$-\frac{\bar{q}}{T} = \frac{\partial \varphi_2^*}{\partial \bar{g}} = \mathbf{C} \cdot \bar{g} = \mathbf{C} \cdot \overrightarrow{\text{grad}} T.$$

If we now make the hypothesis that diffusion properties are isotropic for the material under consideration, then the tensor \mathbf{C} is reduced to a scalar tensor. Moreover, this scalar is considered to vary inversely with respect to the temperature so that it is possible to write:

$$\bullet \quad \bar{q} = -k \overrightarrow{\text{grad}} T$$

or

$$q_i = -k T, i$$