The Plastic Spin

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A macroscopic formulation of large deformations elastoplasticity with tensorial structure variables is presented. The novel features are the effect of constitutive relations for the plastic spin and, to a lesser degree of importance, of elastically embedding the structure variables. The plastic spin constitutive relations are obtained for different kinds of initial and induced anisotropies on the basis of the representation theorems for isotropic second-order antisymmetric tensor-valued functions, and their role is illustrated by the analysis of several examples at large homogeneous deformations. In particular the analysis of simple shear with nonlinear kinematic hardening of the evanescent memory type provides, on the basis of the second Liapunov method for stability, conditions on the material constants for the occurrence or not of stress oscillations with monotonically increasing shear strain.

1 Introduction

The title of this work emphasizes the basic proposition, made by Mandel [1] and Kratochvil [2], that constitutive relations must be provided not only for the plastic rate of deformation but also for the plastic spin within a macroscopic formulation of elastoplasticity. This was motivated by the kinematics of single crystal plasticity where the plastic spin is routinely specified by microscopic analysis. Such constitutive relations, however, have not been systematically considered, derived, and applied within the framework of a macroscopic approach for large elastic and plastic deformations, where the concept of tensorial structure (or internal) variables is utilized to describe the structure of an anisotropic continuum. This is the principal objective of the present work. The proposed general formulation will clearly demonstrate not only the role of the constitutive relations for the plastic spin, but also the consequence of elastically embedding (or convecting) the structure variables. Based on invariance requirements and the representation theorems for isotropic functions [3], specific constitutive relations for the plastic spin will be formulated for different kinds of initial and induced anisotropies, and their effect illustrated by the analysis of several examples at large homogeneous deformations.

Tensors will be denoted usually by boldface characters in direct notation. With the summation convention over repeated indices implied, the following symbolic operations apply: $\mathbf{a}\boldsymbol{\sigma} = a_{ij}\sigma_{jk}$, $\mathbf{a}:\boldsymbol{\sigma} = a_{ij}\sigma_{ji}$, $\mathbf{a}\cdot\boldsymbol{\sigma} = a_{ij}\sigma_{ij}$, $\mathbf{a}\otimes\boldsymbol{\sigma} = a_{ij}\sigma_{kl}$, with proper extension to different order tensors. The prefix *tr* indicates the trace, a superscript T the transpose, subscripts s and a the symmetric and antisymmetric parts, and a superposed dot, the material time derivative or rate. Under an orthogonal transformation $\mathbf{Q}$, the notation $\mathbf{Q}[\mathbf{a}]$ implies the corresponding transformation of the tensor $\mathbf{a}$, e.g., $\mathbf{Q}[\mathbf{a}] = \mathbf{Q}\mathbf{a}\mathbf{Q}^T$ if $\mathbf{a}$ is of second order.

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2 Kinematics of Structured Media

In a macroscopic approach the collection of microscopic entities that define what can be called the substructure of a continuum, can be described by means of tensorial structure variables defined macroscopically at the current configuration κ. With a relative loss of generality for the sake of simplicity, the structure variables will be restricted to second-order tensors $\mathbf{a}$, vectors $\mathbf{m}$, and scalars k, collectively denoted by s, although higher-order tensors can, and must sometimes, be considered [4]. The state variables, therefore, can be defined as the Cauchy stress $\boldsymbol{\sigma}$ (temperature is omitted for simplicity) and the s.

The multiplicative decomposition of the deformation gradient $\mathbf{F}$ into elastic and plastic parts $\mathbf{V}$ and $\mathbf{P}$ [5] such that $\mathbf{V} = \mathbf{V}^T$ [4], yields

$$\mathbf{F} = \mathbf{V}\mathbf{P} \tag{1}$$

The $\mathbf{V}$ defines a macroscopically unstressed configuration κ_0, which can be visualized as being obtained from κ by actual or virtual (if the stress origin is outside the current yield surface) unloading without rotation. Motivated by corresponding concepts of crystal plasticity [6–9], it can be assumed that the s are elastically embedded so that during the unloading they are transported at κ_0 as second-order tensors $\mathbf{A}$, vectors $\mathbf{M}$, and scalars K, collectively denoted by $\mathbf{S}$. Depending on the particular physical meaning attributed to each s, the transport operation can be expressed by different weighted convected transformations of relative tensors, such as contravariant $\mathbf{A} = |\mathbf{V}|^w\mathbf{V}^{-1}\mathbf{a}\mathbf{V}^{-1}$, $\mathbf{M} = |\mathbf{V}|^w\mathbf{V}^{-1}\mathbf{m}$, covariant $\mathbf{A} = |\mathbf{V}|^w\mathbf{V}\mathbf{a}\mathbf{V}$, $\mathbf{M} = |\mathbf{V}|^w\mathbf{m}\mathbf{V}$, and scalar $K = |\mathbf{V}|^w k$, where $|\mathbf{V}|$ denotes the determinant of $\mathbf{V}$ and w is the weight. Mixed transformations can also be considered, if appropriate. The foregoing transport operations will be symbolized collectively by $\mathbf{S} = \mathbf{V}[\mathrm{s}]$. The transport of $\boldsymbol{\sigma}$ is defined here as the first Piola-Kirchhoff stress in reference to κ_0, $\boldsymbol{\Pi} = |\mathbf{V}|\mathbf{V}^{-1}\boldsymbol{\sigma}\mathbf{V}^{-1}$. Thus, the state variables are defined equivalently either as $\boldsymbol{\sigma}$ and s at κ, or $\boldsymbol{\Pi}$ and $\mathbf{S}$ at κ_0 since $\mathbf{V}$ will be shown to be a function of them.

The elastic and plastic rate of transformation (deformation and rotation) and the rate of $\mathbf{S}$ can be thought as taking place relatively to the substructure defined at κ_0 by the current values of $\mathbf{S}$, including orientation [4], followed by a simultaneously occurring rigid body spin ω in order to accommodate the total material spin $\mathbf{W}$. The set of directions created by the tensorial nature of $\mathbf{S}$ (e.g., the eigenvectors of second-order tensors), can be considered as the macroscopic conceptual counterpart of the lattice in single crystals, with the difference that these directions evolve in general with $\mathbf{S}$. For easier interpretation, but without being necessary for the development, the spin ω can be associated with the spin of Mandel's director vectors [1], which at any given instance can be thought as being attached to the substructure. The concept of the spin ω is of cardinal importance because in reference to any fixed cartesian coordinate system the aforementioned rates must be corotational with ω. Defining in general for later use the corotational rates of a tensor $\mathbf{a}$ and a vector $\mathbf{m}$ with respect to an antisymmetric tensor $\boldsymbol{\Omega}$ by

$$D\mathbf{a}/Dt = \dot{\mathbf{a}} - \boldsymbol{\Omega}\mathbf{a} + \mathbf{a}\boldsymbol{\Omega}, \quad D\mathbf{m}/Dt = \dot{\mathbf{m}} - \boldsymbol{\Omega}\mathbf{m} \tag{2}$$

and denoting by a superposed $\circ$ the corotational rates with $\boldsymbol{\Omega} = \omega$, the following basic kinematical relations and definitions can be written

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\dot{\mathbf{P}}\mathbf{P}^{-1}\mathbf{V}^{-1} = \omega + \mathring{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\mathring{\mathbf{P}}\mathbf{P}^{-1}\mathbf{V}^{-1} \tag{3}$$

$$\mathbf{D}^e = (\mathring{\mathbf{V}}\mathbf{V}^{-1})_s, \quad \mathbf{D}^p_0 = (\mathring{\mathbf{P}}\mathbf{P}^{-1})_s, \quad \mathbf{D}^p = (\mathbf{V}\mathring{\mathbf{P}}\mathbf{P}^{-1}\mathbf{V}^{-1})_s \tag{4}$$

$$\mathbf{W}^e = (\mathring{\mathbf{V}}\mathbf{V}^{-1})_a, \quad \mathbf{W}^p_0 = (\mathring{\mathbf{P}}\mathbf{P}^{-1})_a, \quad \mathbf{W}^p = (\mathbf{V}\mathring{\mathbf{P}}\mathbf{P}^{-1}\mathbf{V}^{-1})_a \tag{5}$$

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p, \quad \mathbf{W} = \mathbf{W}^* + \mathbf{W}^p, \quad \mathbf{W}^* = \omega + \mathbf{W}^e \tag{6}$$

where $\mathring{\mathbf{V}}$ and $\mathring{\mathbf{P}}$ are defined according to equations $(2)_1$ and $(2)_2$ with $\boldsymbol{\Omega} = \omega$, respectively; equation $(2)_2$ is used for $\mathring{\mathbf{P}}$ as if it were a vector because $\mathbf{P}$ is attached to κ_0 by its first index only [1]. The superscripts e and p refer to elastic and plastic. The name plastic rate of deformation and plastic spin can be associated with either $\mathbf{D}^p_0$ and $\mathbf{W}^p_0$ at κ_0, or $\mathbf{D}^p$ and $\mathbf{W}^p$ at κ, [10]. It is clear from the foregoing development that the plastic spin expresses the rate of rotation of the continuum with respect to its substructure. Hence, it becomes meaningful for anisotropy where the substructure is characterized by preferred directions.

Anticipating the formulation of the rate constitutive equations and denoting by superposed $*$ and $\triangledown$ the corotational rates with respect to $\mathbf{W}^*$ and $\mathbf{W}$ (Jaumann rates), the following kinematical relations can be written for a representative tensor on the basis of the foregoing kinematics

$$\mathbf{A} = |\mathbf{V}|^w\mathbf{V}^{-1}\mathbf{a}\mathbf{V}^{-1}, \quad \mathring{\mathbf{A}} = |\mathbf{V}|^w\mathbf{V}^{-1}\overset{\square}{\mathbf{a}}\mathbf{V}^{-1} \tag{7}$$

$$\overset{\square}{\mathbf{a}} = \mathring{\mathbf{a}} - \mathbf{a}\mathbf{V}^{-1}\mathring{\mathbf{V}} - \mathring{\mathbf{V}}\mathbf{V}^{-1}\mathbf{a} + w\mathbf{a}\,tr\mathbf{D}^e = \overset{*}{\mathbf{a}} - \mathbf{a}\mathbf{D}^e - \mathbf{D}^e\mathbf{a} + w\mathbf{a}\,tr\mathbf{D}^e \tag{8}$$

$$\overset{\vee}{\mathbf{a}} = \dot{\mathbf{a}} - \mathbf{a}\mathbf{F}^{-T}\dot{\mathbf{F}}^T - \dot{\mathbf{F}}\mathbf{F}^{-1}\mathbf{a} + w\mathbf{a}\,tr\mathbf{D} = \overset{\triangledown}{\mathbf{a}} - \mathbf{a}\mathbf{D} - \mathbf{D}\mathbf{a} + w\mathbf{a}\,tr\mathbf{D} \tag{9}$$

$$\overset{\vee}{\mathbf{a}} - \overset{\square}{\mathbf{a}} = -\mathbf{a}\mathbf{V}^{-1}\mathbf{P}^{-T}\mathring{\mathbf{P}}^T\mathbf{V} - \mathbf{V}\mathring{\mathbf{P}}\mathbf{P}^{-1}\mathbf{V}^{-1}\mathbf{a} + w\mathbf{a}\,tr\mathbf{D}^p \tag{10}$$

and similar relations can be stated for other kinds of transports of tensors, vectors and scalars. In particular equations (7) can be stated in a general symbolic notation as $\mathbf{S} = \mathbf{V}[\mathbf{s}]$, $\mathring{\mathbf{S}} = \mathbf{V}[\overset{\square}{\mathbf{s}}]$. The superposed $\vee$ denotes the convected derivatives of relative tensors (for $w = 1$, equation (9) yields the Truesdell derivative). The superposed $\square$ denotes the so-called corodeformational rate due to the elastic embedding of $\mathbf{a}$ and the simultaneous corotation of both $\mathbf{a}$ and $\mathbf{V}$ with ω, as can be seen from the use of $\mathring{\mathbf{a}}$ and $\mathring{\mathbf{V}}$ in equation (8). A final important point is that under a superposed rigid body rotation/reflection at κ defined by an orthogonal tensor $\mathbf{Q}$, the $\mathbf{P}$, $\mathbf{V}$, and ω become $\mathbf{QP}$, $\mathbf{QVQ}^T$, and $\mathbf{Q}\omega\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T$, correspondingly, because κ_0 must also rotate/reflect by $\mathbf{Q}$ based on the definition of $\mathbf{V}$. Hence, one can show that the σ, $\mathbf{s}$, $\boldsymbol{\Pi}$, $\mathbf{S}$, and all rates defined in equations (4), (5), (7)–(9), transform only by rotation/reflection under $\mathbf{Q}$.

3 Constitutive Formulation

3.1 Rate Equations. With a scalar loading index λ for a state satisfying a yield criterion, both to be defined in the sequel, the following rate equations can be stated

$$\mathbf{D}^p_0 = <\lambda>\mathbf{N}^p_0(\boldsymbol{\Pi},\mathbf{S}), \quad \mathbf{W}^p_0 = <\lambda>\boldsymbol{\Omega}^p_0(\boldsymbol{\Pi},\mathbf{S}) \tag{11}$$

$$\mathring{\mathbf{S}} = <\lambda>\bar{\mathbf{S}}(\boldsymbol{\Pi},\mathbf{S}) \tag{12}$$

with $<\lambda> = \lambda$ if $\lambda > 0$ and $<\lambda> = 0$ if $\lambda \leq 0$. The use of $\mathring{\mathbf{S}}$ in equation (12) follows directly from the discussion preceding equation (2). Based on the transformation law under superposed rigid body rotation/reflection, the corresponding invariance requirements render $\mathbf{N}^p_0$, $\boldsymbol{\Omega}^p_0$ and $\bar{\mathbf{S}}$ isotropic functions of their arguments $\boldsymbol{\Pi}$ and $\mathbf{S}$ (definition in Appendix). Clearly this does not imply in general that the material is isotropic due to the tensorial nature of $\mathbf{S}$ [4]. Introducing the definitions

$$\mathbf{N}^p = (\mathbf{V}(\mathbf{N}^p_0 + \boldsymbol{\Omega}^p_0)\mathbf{V}^{-1})_s, \quad \boldsymbol{\Omega}^p = (\mathbf{V}(\mathbf{N}^p_0 + \boldsymbol{\Omega}^p_0)\mathbf{V}^{-1})_a \tag{13}$$

equations (4), (5), and (11) yield

$$\mathbf{D}^p = <\lambda>\mathbf{N}^p(\boldsymbol{\Pi},\mathbf{S}), \quad \mathbf{W}^p = <\lambda>\boldsymbol{\Omega}^p(\boldsymbol{\Pi},\mathbf{S}) \tag{14}$$

where again $\mathbf{N}^p$ and $\boldsymbol{\Omega}^p$ are isotropic functions of $\boldsymbol{\Pi}$ and $\mathbf{S}$. Similarly one can use relations such as (7)–(9) in conjunction with equations (11) and (14) in order to express equation (12) in terms of rates associated with the values $\mathbf{s}$ at κ. For example, the following set of equivalent forms for the rate equations of $\mathbf{a}$ applies

$$\mathring{\mathbf{A}} = <\lambda>\bar{\mathbf{A}}(\boldsymbol{\Pi},\mathbf{S}) \quad \text{or} \tag{15a}$$

$$\overset{\square}{\mathbf{a}} = <\lambda>\bar{\mathbf{a}} \quad \text{with} \quad \bar{\mathbf{A}} = |\mathbf{V}|^w\mathbf{V}^{-1}\bar{\mathbf{a}}\mathbf{V}^{-1}, \quad \text{or} \tag{15b}$$

$$\overset{\vee}{\mathbf{a}} = <\lambda>(\bar{\mathbf{a}} - (\mathbf{a}\mathbf{N}^p + \mathbf{N}^p\mathbf{a}) + (\mathbf{a}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\mathbf{a}) + w\mathbf{a}\,tr\mathbf{N}^p) \tag{15c}$$

Equation (15*b*) can be written in general for any $\mathbf{s}$ as $\overset{\square}{\mathbf{s}} = <\lambda>\bar{\mathbf{s}}$ with $\bar{\mathbf{S}} = \mathbf{V}[\bar{\mathbf{s}}]$ when $\mathbf{S} = \mathbf{V}[\mathbf{s}]$. For small elastic deformations one has $\mathbf{V} \simeq \mathbf{I}$ (identity tensor), hence $\boldsymbol{\Pi} \simeq \sigma$, $\mathbf{S} \simeq \mathbf{s}$ and equation (12) can be written as

$$\mathring{\mathbf{s}} = <\lambda>\bar{\mathbf{s}}(\sigma,\mathbf{s}), \quad \text{or equivalently} \tag{16a}$$

$$\overset{\triangledown}{\mathbf{a}} = <\lambda>(\bar{\mathbf{a}} + \mathbf{a}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\mathbf{a}), \quad \overset{\triangledown}{\mathbf{m}} = <\lambda>(\bar{\mathbf{m}} - \boldsymbol{\Omega}^p\mathbf{m}), \quad \dot{k} = <\lambda>\bar{k} \tag{16b}$$

with $\bar{\mathbf{s}} = \{\bar{\mathbf{a}}, \bar{\mathbf{m}}, \bar{k}\}$ isotropic functions of σ and $\mathbf{s}$, and where the transition from equation (16*a*) to (16*b*) is based on $\omega = \mathbf{W} - \mathbf{W}^p$, valid for $\mathbf{V} \simeq \mathbf{I}$. In equations (15*c*) and (16*b*) observe the clear distinction between the purely constitutive parts $\bar{\mathbf{a}}$, $\bar{\mathbf{m}}$, and the remaining terms due to the kinematics, where the role of the plastic spin via $\boldsymbol{\Omega}^p$ is clearly demonstrated. If a tensorial structure variable is purely orientational (such as a preferred direction of the substructure), one must set $\bar{\mathbf{S}} \equiv \mathbf{0}$ or equivalently $\bar{\mathbf{a}} \equiv \mathbf{0}$, $\bar{\mathbf{m}} \equiv \mathbf{0}$. The Almansi plastic strain $\mathbf{A}^p = (1/2)\ (\mathbf{I} - \mathbf{P}^{-T}\mathbf{P}^{-1})$ can be used, if appropriate, as one of the $\mathbf{S}$, and can be shown that $\mathring{\mathbf{A}}^p = <\lambda>[\mathbf{N}^p_0 - (\mathbf{A}^p\mathbf{N}^p_0 + \mathbf{N}^p_0\mathbf{A}^p) - (\mathbf{A}^p\boldsymbol{\Omega}^p_0 - \boldsymbol{\Omega}^p_0\mathbf{A}^p)]$, consistent with equation (15*a*). A different plastic deformation measure $\boldsymbol{\Delta}^p$, which is not a true strain measure, could be defined by integration of $\mathring{\boldsymbol{\Delta}}^p = \mathbf{D}^p$ and used as one of the $\mathbf{S}$.

3.2 Elastic Relations With Elastoplastic Coupling and/or Damage. Denoting by $\mathbf{E}^e = (1/2)(\mathbf{V}^2 - \mathbf{I})$ the Green elastic strain, ρ_0 the mass density at κ_0 and $\psi = \bar{\psi}(\boldsymbol{\Pi},\mathbf{S})$ the complementary free energy per unit mass, isotropic function of $\boldsymbol{\Pi}$ and $\mathbf{S}$ (invariance), the elastic relations can be obtained from $\mathbf{E}^e = \rho_0(\partial\bar{\psi}/\partial\boldsymbol{\Pi})$. Since $\mathbf{E}^e$ (and $\mathbf{V}$) is also isotropic function of $\boldsymbol{\Pi}$ and $\mathbf{S}$, the relation (*A*1) of the Appendix can be applied for $\mathring{\mathbf{E}}^e = \mathbf{V}\mathbf{D}^e\mathbf{V}$ which in conjunction with equation (12) and the foregoing kinematics yields

$$\overset{\circ}{\mathbf{E}}{}^e = \overset{\circ}{\mathbf{E}}{}^r + \overset{\circ}{\mathbf{E}}{}^c = \mathfrak{L}^{0-1}:\overset{\circ}{\mathbf{\Pi}} + <\lambda>\mathbf{N}_0^c \tag{17a}$$

$$\mathbf{D}^e = \mathbf{D}^r + \mathbf{D}^c = \mathfrak{L}^{-1}:\overset{\square}{\sigma} + <\lambda>\mathbf{N}^c \tag{17b}$$

$$\mathfrak{L}^0 = (\rho_0 \partial^2 \bar{\psi}/\partial\mathbf{\Pi}\otimes\partial\mathbf{\Pi})^{-1},$$

$$\mathfrak{L}_{ijkl} = |\mathbf{V}|^{-1} V_{i\alpha} V_{j\beta} V_{k\gamma} V_{l\delta} \mathfrak{L}^0_{\alpha\beta\gamma\delta} \tag{17c}$$

$$\mathbf{N}^c = \mathbf{V}^{-1}\mathbf{N}_0^c\mathbf{V}^{-1} = \rho_0\mathbf{V}^{-1}[(\partial^2\bar{\psi}/\partial\mathbf{\Pi}\otimes\partial\mathbf{S})\cdot\bar{\mathbf{S}}]\mathbf{V}^{-1} - \mathbf{A}^e tr\mathbf{N}^p \tag{17d}$$

where $\mathbf{A}^e = \mathbf{V}^{-1}\mathbf{E}^e\mathbf{V}^{-1}$, $\overset{\square}{\sigma}$ is defined by equation (8) with $w=1$ and σ instead of $\mathbf{a}$, and $\mathfrak{L}$ are the incremental elastic moduli. The $\mathbf{D}^r$ and $\mathbf{D}^c$ represent the incrementally reversible and elastoplastic coupling or damage induced components of $\mathbf{D}^e$ at κ, counterparts of $\overset{\circ}{\mathbf{E}}{}^r$ and $\overset{\circ}{\mathbf{E}}{}^c$ at κ_0. The $tr\mathbf{N}^p$ in equation (17d) appears due to the mass conservation relation $\dot{\rho}_0 + \rho_0 tr\mathbf{D}^p = 0$ at κ_0. The evolution of tensorial $\mathbf{S}$ can alter initial elastic symmetries, unless $\bar{\mathbf{S}} \equiv \mathbf{O}$ (purely orientational) in which case one can still have $\mathbf{N}^c \neq \mathbf{O}$ if $tr\mathbf{N}^p \neq 0$, equation (17d).

3.3 Yield Criterion, Loading Index, and Final Form.

A macroscopically smooth yield criterion together with its symmetric stress gradient can be defined by

$$f(\sigma,\mathrm{s}) = 0, \quad \mathbf{N}^n = (\partial f/\partial\sigma)_s \tag{18}$$

with f an isotropic function of σ, s due to invariance requirements. The dependence of f on the current values σ, s at κ is motivated by corresponding formulations in crystal plasticity [6–9] where a resolved shear stress yield threshold τ is defined by $f = \mathbf{m}\sigma\mathbf{n} - \tau = 0$, with $\mathbf{m}$ and $\mathbf{n}$ vectors along the slip and normal to slip plane directions at κ, and which could be considered as being unit vectors at κ_0 [9]. In the same spirit one can express $\mathbf{N}^p$ and $\mathbf{\Omega}^p$ directly as isotropic functions of σ and s. Equivalent expressions for f in terms of quantities defined at κ_0 are possible, but a slight advantage of equation $(18)_1$ is that f can be determined in principle without requiring an elastic unloading to κ_0 (possibly only virtual). Since the s depend on σ due to their elastic embedding, one expects that the loading direction will divert from $\mathbf{N}^n$ in a way that will be precisely defined in the sequel.

Due to the isotropy of f, the consistency condition $\dot{f} = 0$ be expressed according to equation (A2) of the Appendix in terms of the corotational rates $\overset{*}{\sigma}$, $\overset{*}{\mathrm{s}}$, which in turn can be expressed in terms of $\overset{\square}{\sigma}$ and $\overset{\square}{\mathrm{s}}$ by means of equations similar to (8) with $\mathbf{D}^e$ obtained from equation (17b), yielding

$$\dot{f} = \mathbf{N}^n:\overset{*}{\sigma} + (\partial f/\partial \mathrm{s})\cdot\overset{*}{\mathrm{s}} = (\mathbf{N}^n - \mathbf{Z}^n:\mathfrak{L}^{-1}):\overset{\square}{\sigma} + \lambda((\partial f/\partial \mathrm{s})\cdot\bar{\mathrm{s}} - \mathbf{Z}^n:\mathbf{N}^c) = 0 \tag{19}$$

Use of similar equations to (15b) was made for $\overset{\square}{\mathrm{s}}$, and $\mathbf{Z}^n$ is defined in the following equation (25). Equation (19) determines the loading index λ in terms of $\overset{\square}{\sigma}$. On the basis of equations $(6)_1$, (14), (17b), (19), and utilizing equation (10) with $w=1$ and σ instead of $\mathbf{a}$ in order to express $\overset{\square}{\sigma}$ in terms of $\overset{\vee}{\sigma}$ (similarly defined from equation (9) with $w = 1$ and $\mathbf{a} = \sigma$), the final form for a state on $f = 0$ is given by

$$\mathbf{D} = \mathfrak{L}^{-1}:\overset{\square}{\sigma} + <\lambda>(\mathbf{N}^p + \mathbf{N}^c) = \mathfrak{L}^{-1}:\overset{\vee}{\sigma} + <\lambda>\mathbf{N}' = \Lambda^{-1}:\overset{\vee}{\sigma} \tag{20}$$

$$\lambda = \frac{\mathbf{N}:\overset{\square}{\sigma}}{H + \mathbf{Z}^n:\mathbf{N}^c} = \frac{\mathbf{N}:\overset{\vee}{\sigma}}{H + \mathbf{Z}^n:\mathbf{N}^c + \mathbf{N}:\mathbf{Z}^p}$$

$$= \frac{\mathbf{N}:\mathfrak{L}:\mathbf{D}}{H + \mathbf{N}:\mathfrak{L}:\mathbf{N}^p + \mathbf{N}^n:\mathfrak{L}:\mathbf{N}^c} \tag{21}$$

$$\Lambda = \mathfrak{L} - \bar{h}(\lambda)\frac{(\mathfrak{L}:\mathbf{N}')\otimes(\mathbf{N}:\mathfrak{L})}{H + \mathbf{N}:\mathfrak{L}:\mathbf{N}^p + \mathbf{N}^n:\mathfrak{L}:\mathbf{N}^c} \tag{22}$$

$$\mathbf{N}' = \mathbf{N}^p + \mathbf{N}^c - \mathfrak{L}^{-1}:\mathbf{Z}^p, \quad \mathbf{N} = \mathbf{N}^n - \mathbf{Z}^n:\mathfrak{L}^{-1},$$
$$H = -(\partial f/\partial \mathrm{s})\cdot\bar{\mathrm{s}} \tag{23}$$

$$\mathbf{Z}^p = -(\sigma\mathbf{N}^p + \mathbf{N}^p\sigma) + (\sigma\mathbf{\Omega}^p - \mathbf{\Omega}^p\sigma) + \sigma tr\mathbf{N}^p \tag{24}$$

$$\mathbf{Z}^n = -(\sigma\mathbf{N}^n + \mathbf{N}^n\sigma) + (\mathbf{N}^n:\sigma)\mathbf{I} + \mathbf{\Gamma} \tag{25}$$

$$\mathbf{\Gamma} = \pm\left(\mathbf{a}^T\frac{\partial f}{\partial \mathbf{a}} + \frac{\partial f}{\partial \mathbf{a}}\mathbf{a}^T\right)_s \pm \left(\frac{\partial f}{\partial \mathbf{m}}\otimes\mathbf{m}\right)_s + w\left(\frac{\partial f}{\partial \mathrm{s}}\cdot\mathrm{s}\right)\mathbf{I} \tag{26}$$

with $\bar{h}(\lambda)$ the heavyside step function defined as zero at $\lambda = 0$, and + or − in equation (26) when $\mathbf{a}$, $\mathbf{m}$ are transported from κ to κ_0 by a covariant or contravariant transformation, respectively, of weight w. The $\mathbf{\Gamma}$ encompasses the effect of elastically embedding the s. Many of the features observed in equations (20)–(26) are the macroscopic counterparts of well-established results in microscopic crystal plasticity theories, but appear to be novel in a macroscopic formulation with tensorial structure variables. For example, the $\mathbf{Z}^p$ and $\mathbf{Z}^n$ divert the irreversible rate of deformation and loading directions from $\mathbf{N}^p + \mathbf{N}^c$ and $\mathbf{N}^n$, to $\mathbf{N}'$ and $\mathbf{N}$, respectively, the diversions being of the order stress/elastic moduli. In fact one could broadly associate the $\mathbf{N}'$, $\mathbf{N}$, and $\mathbf{N}^p + \mathbf{N}^c$ with the quantities $\mathbf{P}'$, $\mathbf{Q}$, and $\mathbf{P}$ defined in [7]. With H the plastic modulus, the denominator of the third member of equation (21) corresponds to what is called the effective hardening modulus [11] due to geometrical and elastoplastic coupling (and/or damage) effects expressed by $\mathbf{N}:\mathbf{Z}^p$ and $\mathbf{Z}^n:\mathbf{N}^c$, respectively. The elastoplastic coupling effect has not been considered in [7, 11]. It is possible to have $\mathbf{N}^c \neq \mathbf{0}$ with $\mathbf{N}^p = \mathbf{\Omega}^p = \mathbf{0}$ in which case $f = 0$ acts only as a damage criterion. Certain differences in relation to the aforementioned references are due to the use of different stress rates. In fact one can change the stress rate by properly modifying the elastic moduli and the values of $\mathbf{Z}^p$ and $\mathbf{Z}^n$. For example, assuming for simplicity that $\mathbf{N}^c = \mathbf{0}$, the Jaumann rate $\overset{\nabla}{\tau} = \overset{\nabla}{\sigma} + \sigma tr\mathbf{D}$ of the Kirchhoff stress $\tau = \sigma$, but $\dot{\tau} = \dot{\sigma} + \sigma tr\mathbf{D}$, can be used instead of $\overset{\vee}{\sigma}$ by setting $\mathbf{Z}^p = \sigma\mathbf{\Omega}^p - \mathbf{\Omega}^p\sigma + \sigma tr\mathbf{N}^p$, $\mathbf{Z}^n = (\mathbf{N}^n:\sigma)\ \mathbf{I} + \mathbf{\Gamma}$ and employing elastic moduli obtained by adding the quantity $(1/2)(\delta_{ik}\sigma_{lj} + \delta_{il}\sigma_{kj} + \delta_{jl}\sigma_{ik} + \delta_{jk}\sigma_{il})$ to $\mathfrak{L}_{ijkl}$, δ_{ij} being the Kronecker delta. For small elastic deformations $\mathbf{V} \simeq \mathbf{I}$ and the effect of elastic embedding is negligible. The corresponding equations can be obtained from equations (20)–(23) by substituting $\overset{\circ}{\sigma}$ and $\overset{\nabla}{\sigma}$ for $\overset{\square}{\sigma}$ and $\overset{\vee}{\sigma}$, respectively, and setting $\mathbf{Z}^n = \mathbf{0}$, $\mathbf{Z}^p = \sigma\mathbf{\Omega}^p - \mathbf{\Omega}^p\sigma$, as obtained directly in [12]. If only the volumetric elastic deformation is finite, one has $\mathbf{V} \simeq |\mathbf{V}|\mathbf{I}$ and all the foregoing relations must be modified accordingly.

4 The Plastic Spin

In a sequence of recent papers [12–14] the author was the first to use the representation theorems for isotropic functions [3] in conjunction with the concept of tensorial structure variables to provide explicit forms of constitutive relations for the plastic spin. Independently Loret [15] obtained similar results. Subsequently the main results presented in [12–14] will be briefly summarized and supplemented by a number of new observations. For simplicity small elastic deformations will be considered, i.e., $\mathbf{\Pi} \simeq \sigma$ and $\mathbf{S} \simeq \mathrm{s}$.

With λ defined from equation (21), the constitutive relation for $\mathbf{W}^p$ depends on $\mathbf{\Omega}^p$, equation (14). In the case of one symmetric tensor structure variable $\mathbf{a}$, the $\mathbf{\Omega}^p$, being an isotropic function of σ and $\mathbf{a}$, can be represented by [3]

$$\Omega^p = \eta_1(\mathbf{a}\sigma - \sigma\mathbf{a}) + \eta_2(\mathbf{a}^2\sigma - \sigma\mathbf{a}^2) + \eta_3(\mathbf{a}\sigma^2 - \sigma^2\mathbf{a}) + \eta_4(\mathbf{a}\sigma\mathbf{a}^2 - \mathbf{a}^2\sigma\mathbf{a}) + \eta_5(\sigma\mathbf{a}\sigma^2 - \sigma^2\mathbf{a}\sigma) \quad (27)$$

where the η_i's are scalar functions (not necessarily polynomial) of the well-known isotropic invariants of σ, $\mathbf{a}$ and any other scalar structure variable k (e.g., equivalent plastic strain). Clearly, the deviatoric parts σ' and $\mathbf{a}'$ can substitute σ and $\mathbf{a}$ in equation (27), by redefining the η_i. If $\mathbf{a}\sigma = \sigma\mathbf{a}$ (i.e., σ and $\mathbf{a}$ are coaxial), which includes as a special case the isotropic one with $\mathbf{a}' = \mathbf{0}$, it follows that $\Omega^p = \mathbf{0}$ even for an anisotropic material. This conclusion can be extended to finite elastic deformations; with $\mathbf{A\Pi} = \mathbf{\Pi A}$ it follows that $\Omega^p_0 = \mathbf{0}$ and that $\mathbf{V}$ and $\mathbf{N}^p_0$ commute (same principal directions as $\mathbf{A}$ and $\mathbf{\Pi}$). Hence, from equations (13) one has $\mathbf{N}^p = \mathbf{N}^p_0$ and $\Omega^p = \mathbf{0}$. The requirement of continuous transition from anisotropy to isotropy and vice-versa is satisfied if the sufficient conditions $\lim \eta_i(\mathbf{a}')^{y_i} = \mathbf{0}$ as $\mathbf{a}' \to \mathbf{0}$ are imposed, with y_i the corresponding exponents in equation (27). For a single generator this is also a necessary condition. A discontinuous Ω^p at $\mathbf{a}' = \mathbf{0}$ will cause a corresponding discontinuity in $\dot{\sigma}$ since it appears in the definition of Λ, equation (22), and of λ in terms of $\overset{\vee}{\sigma}$, equation (21), via the terms involving $\mathbf{Z}^p$. It was suggested in [12–14] to use the first only generator of equation (27) as a first approximation. The "intensity" of the plastic spin for a given σ on $f = 0$ depends on the η_i, the norm of $\mathbf{a}$ and the degree of noncoaxiality between σ and $\mathbf{a}$, as it can easily be seen by employing the spectral representation for σ and $\mathbf{a}$ in equation (27).

For orthotropic symmetries along three orthonormal vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ the structure variables (purely orientational) can be defined by $\mathbf{a}_1 = \mathbf{n}_1 \otimes \mathbf{n}_1$ and $\mathbf{a}_2 = \mathbf{n}_2 \otimes \mathbf{n}_2$, [16, 17]. On the basis of [3] the Ω^p can be represented by

$$\Omega^p = \eta_1(\mathbf{a}_1\sigma - \sigma\mathbf{a}_1) + \eta_2(\mathbf{a}_2\sigma - \sigma\mathbf{a}_2) + \eta_3(\mathbf{a}_1\sigma\mathbf{a}_2 - \mathbf{a}_2\sigma\mathbf{a}_1) + \eta_4(\mathbf{a}_1\sigma^2 - \sigma^2\mathbf{a}_1) + \eta_5(\mathbf{a}_2\sigma^2 - \sigma^2\mathbf{a}_2) + \eta_6(\sigma\mathbf{a}_1\sigma^2 - \sigma^2\mathbf{a}_1\sigma) + \eta_7(\sigma\mathbf{a}_2\sigma^2 - \sigma^2\mathbf{a}_2\sigma) \quad (28)$$

with η_i functions of $tr\sigma$, $tr\sigma^2$, $tr\sigma^3$, $tr\mathbf{a}_1\sigma$, $tr\mathbf{a}_2\sigma$, $tr\mathbf{a}_1\sigma^2$, $tr\mathbf{a}_2\sigma^2$, [16], and any other scalar structure variable. Denoting by a superposed $\hat{}$ the tensor components in reference to a cartesian coordinate system $\hat{\mathbf{x}} = \{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ along the axes of orthotropy, and assuming that Ω^p is represented by the first three generators only in equation (28) which are linear in σ, one obtains $\hat{\Omega}^p_{12} = \eta'_3\hat{\sigma}_{12}$, $\hat{\Omega}^p_{13} = \eta'_2\hat{\sigma}_{13}$, $\hat{\Omega}^p_{23} = \eta'_1\hat{\sigma}_{23}$ with $\eta'_1 = \eta_2$, $\eta'_2 = \eta_1$ and $\eta'_3 = \eta_1 - \eta_2 + \eta_3$. Defining $\mathbf{N}^p$ from the associated flow rule $\mathbf{N}^p = (\partial f/\partial\sigma)$ with f given by Hill's orthotropic yield criterion [18]

$$f = A(\hat{\sigma}_{11} - \hat{\sigma}_{22})^2 + B(\hat{\sigma}_{22} - \hat{\sigma}_{33})^2 + C(\hat{\sigma}_{33} - \hat{\sigma}_{11})^2 + D\hat{\sigma}^2_{23} + E\hat{\sigma}^2_{13} + F\hat{\sigma}^2_{12} - k^2 = 0 \quad (29)$$

it follows based on the foregoing that

$$\hat{W}^p_{12} = \frac{\eta'_3}{2F}\hat{D}^p_{12},\ \hat{W}^p_{13} = \frac{\eta'_2}{2E}\hat{D}^p_{13},\ \hat{W}^p_{23} = \frac{\eta'_1}{2D}\hat{D}^p_{23} \quad (30)$$

When the σ' changes sign, and still is on $f = 0$, one expects on physical grounds (sign change of microscopic resolved shear stresses and corresponding shear strain rates) that so does $\mathbf{W}^p$, hence the η_i's must be even functions of σ'. If the orthotropy has the same "intensity" along $\hat{x}_1$ and $\hat{x}_2$ one expects that $\hat{W}^p_{12} = 0$, thus $\eta'_3 = 0$. Similarly for η'_2 and η'_1. For the fifth class of transverse isotropy characterized by $\mathbf{n}_1$ along $\hat{x}_1$, the structure variable is $\mathbf{a}_1 = \mathbf{n}_1 \otimes \mathbf{n}_1$ [16, 17]. The Ω^p is obtained from equation (28) by setting $\eta_2 = \eta_3 = \eta_5 = \eta_7 = 0$, and the yield criterion from equation (29) with $A = C$, $E = F$ and $D = 2(A + 2B)$. Using only generators linear in σ, equation (30) applies with $E = F$, $\eta'_1 = 0$ and $\eta'_2 = \eta'_3 = \eta_1$. For this case observe that the value $\eta_1/2F = 1$ can be associated with a unidirectionally fiber-reinforced material with $\hat{x}_1$ along the fibers' direction, under the kinematical restriction that plastic shear cannot occur by slip on planes perpendicular to $\hat{x}_1$. Similarly the value $\eta_1/2F = -1$ can be visualized as the case of a deck of cards with $\hat{x}_1$ normal to their planes and where slip is not permitted on planes parallel to $\hat{x}_1$.

5 Kinematic Hardening and Simple Shear

Within the framework of the general development, the following kinematic hardening constitutive model can be proposed

$$f = (3/2)(\sigma' - \alpha):(\sigma' - \alpha) - k^2 = 0 \quad (31)$$

$$\mathbf{D}^p = <\lambda>\mathbf{n},\ \mathbf{n} = (3/2)^{1/2}(\sigma' - \alpha)/k,\ \mathbf{n}:\mathbf{n} = 1 \quad (32)$$

$$\mathbf{W}^p = <\lambda>\eta_1(\alpha\sigma' - \sigma'\alpha) = (1/2)\rho(\alpha\mathbf{D}^p - \mathbf{D}^p\alpha) \quad (33)$$

$$\overset{\circ}{\alpha} = <\lambda>((2/3)h_\alpha\mathbf{n} - (2/3)^{1/2}c_r\alpha) = (2/3)h_\alpha\mathbf{D}^p - ((2/3)\mathbf{D}^p:\mathbf{D}^p)^{1/2}c_r\alpha \quad (34)$$

where clearly f, $\mathbf{D}^p$, $\mathbf{W}^p$, and $\overset{\circ}{\alpha}$ are isotropic functions of the state variables σ', α and k, the α represents the usual deviatoric back-stress tensor and k is constant or variable (isotropic hardening). Equation (34) is the finite deformation version of the kinematic hardening rule with evanescent memory originally proposed in [19]. It introduces the positive material constants h_α and c_r which can be calibrated from $\sigma_{11} = \pm[k + (h_\alpha/c_r)\ [1 - \exp(-c_r\,|\epsilon^p|)]]$obtained from equation (31) and integration of equation ɪ34) for uniaxial loading with ϵ^p the corresponding logarithmic plastic strain. The key equation (33) is obtained from equations (27) and (32) using the first generator only, as originally suggested in [14]. It introduces the material parameter ρ, by setting $\eta_1 = (3/2)^{1/2}(\rho/2k)$, which has the dimensions of $(\text{stress})^{-1}$ and is isotropic function of σ' (or $\sigma' - \alpha$), α, k and any other scalar structure variable such as the equivalent plastic strain, so that $\lim(\rho\alpha) = \mathbf{0}$ as $\alpha \to \mathbf{0}$. Expecting that $\mathbf{W}^p$ changes sign when $\sigma' - \alpha$ does, ρ must be an even function of $\sigma' - \alpha$.

The proposed constitutive model will be used for the analysis of simple shear γ defined by the velocity gradient components

$$D_{12} = D_{21} = W_{12} = -W_{21} = \dot{\gamma}/2,\quad D_{ij} = W_{ij} = 0 \text{ for other } i, j \quad (35)$$

The simple shear analysis has been presented qualitatively, numerically, and analytically for a variety of kinematic hardening models and corotational rates for α [12, 14, 15, 20–24]. In particular the phenomenon of stress oscillations with increasing γ, first reported in [20], prompted investigations on the role of model constants that remained largely numerical or qualitative. In the spirit of [12, 14], where the analytical solution for $c_r = 0$ was provided, sufficient and necessary conditions on h_α, c_r, and ρ for the occurrence or not of such oscillations will be rigorously obtained by analytical means, as well as closed-form expressions on the stress limits as $|\gamma| \to \infty$.

Assuming for simplicity a rigid-plastic response, constant k, $\alpha_{11} + \alpha_{22} = 0$, $\alpha_{13} = \alpha_{23} = 0$ and $\sigma_{33} = 0$, equations (31)–(34) and (35) yield [12, 14]

$$\sigma_{11} = \alpha_{11} = -\sigma_{22} = -\alpha_{22},\quad \sigma_{12} = \alpha_{12} + \text{sgn}\dot{\gamma}(k/\sqrt{3}) \quad (36)$$

and $\sigma_{13} = \sigma_{23} = 0$, with $\text{sgn}\dot{\gamma}$ = sign of $\dot{\gamma}$ and the evolution of α_{11}, α_{12} governed by the system of nonlinear differential equations

$$d\alpha_{11}/d\gamma = -\text{sgn}\dot{\gamma}(c_r/\sqrt{3})\alpha_{11} + (1 - \rho\alpha_{11})\alpha_{12} \quad (37a)$$

$$d\alpha_{12}/d\gamma = -\text{sgn}\dot{\gamma}(c_r/\sqrt{3})\alpha_{12} - (1 - \rho\alpha_{11})\alpha_{11} + (1/3)h_\alpha \quad (37b)$$

If the assumptions $\alpha_{11} + \alpha_{22} = 0$ and $\sigma_{33} = 0$ were not made,

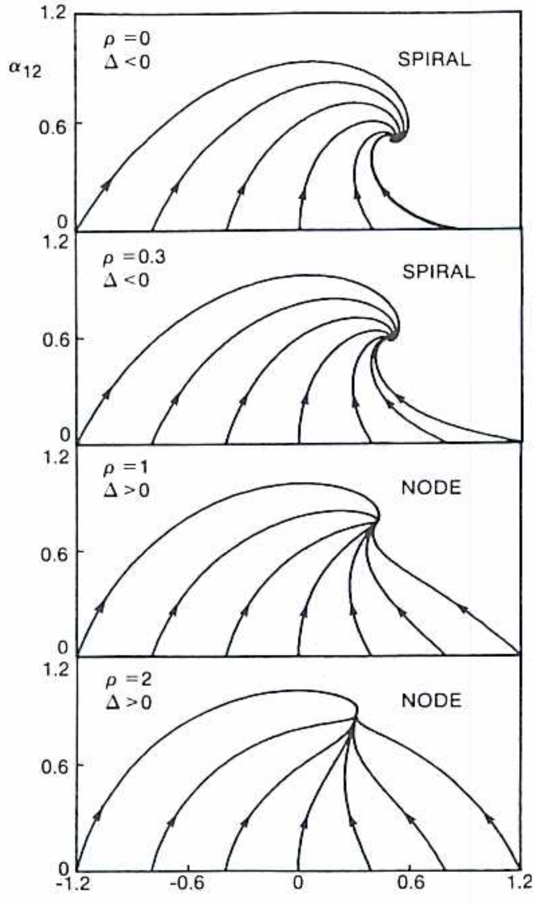

Fig. 1 Back-stress trajectories in simple shear for kinematic hardening with evanescent memory and different values of ρ. Stress quantities are normalized by k.

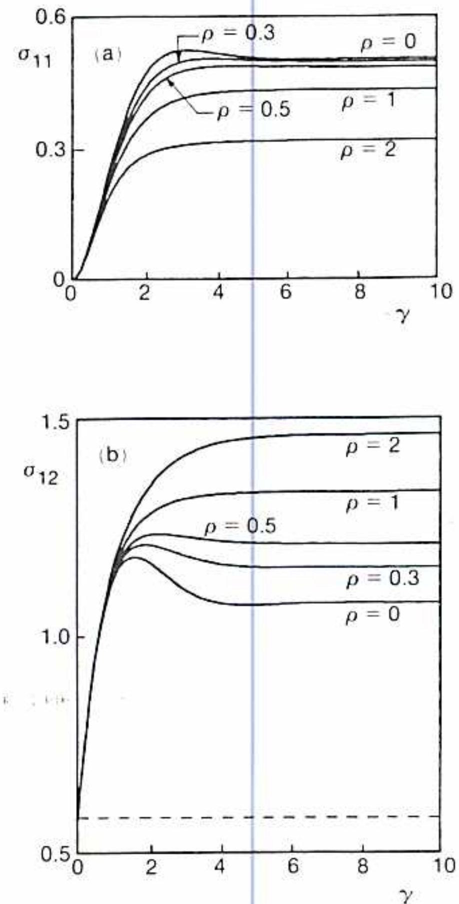

Fig. 2 (a,b) Stress-strain in simple shear for kinematic hardening with evanescent memory and different values of ρ. Stress quantities are normalized by k.

one would have obtained $\sigma'_{11} = \alpha_{11}$, $\sigma'_{22} = \alpha_{22}$, $\sigma'_{33} = \alpha_{33} = -2\alpha_{(+)}$, $\sigma_{11} = 3\alpha_{(+)} + \alpha_{(-)} + \sigma_{33}$, $\sigma_{22} = 3\alpha_{(+)} - \alpha_{(-)} + \sigma_{33}$, with $\alpha_{(+)} = (\alpha_{11} + \alpha_{22})/2 = \alpha^0_{(+)}\exp(-c_r\,|\gamma|/\sqrt{3})$ and $\alpha_{(-)} = (\alpha_{11} - \alpha_{22})/2$ substituting α_{11} in equations (37). Observe that the quantity $(\dot{\gamma}/2)(1 - \rho\alpha_{11})$ represents the ω_{12} component of the spin $\omega = \mathbf{W} - \mathbf{W}^p$, [12], which suggests on physical grounds that $\rho \geq 0$ because otherwise the ω would be "greater" than $\mathbf{W}$ at the first stage of simple shear. Observe also that by setting $c_r = 0$ and $\rho = (2/tr\alpha^2)^{1/2} = (\alpha_{11}^2 + \alpha_{12}^2)^{-1/2}$ the formulation proposed in [22] is obtained, but the continuity requirement $\lim(\rho\alpha) = \mathbf{0}$ as $\boldsymbol{\alpha} \to \mathbf{0}$ is violated. Henceforth, ρ will be considered constant. For monotonic change of γ, such that $\text{sgn}\dot{\gamma} = \text{sgn}\gamma$, and $\rho = 0$ which implies the use of Jaumann rates in equation (34), the solution of equations (37) can be obtained in closed form as

$$\begin{aligned}\alpha_{11} &= [h_\alpha/(3 + c_r^2)][1 - \exp(-c_r\,|\gamma|/\sqrt{3})(\cos\gamma \\ &+ (c_r/\sqrt{3})\sin|\gamma|)] \\ &+ \exp(-c_r\,|\gamma|/\sqrt{3})(\alpha^0_{11}\cos\gamma + \alpha^0_{12}\sin\gamma) \qquad (38a)\end{aligned}$$

$$\begin{aligned}\alpha_{12} &= [(\text{sgn}\gamma)h_\alpha/(3 + c_r^2)][(c_r/\sqrt{3}) \\ &+ \exp(-c_r\,|\gamma|/\sqrt{3})(\sin|\gamma| - (c_r/\sqrt{3})\cos\gamma)] \\ &- \exp(-c_r\,|\gamma|/\sqrt{3})(\alpha^0_{11}\sin\gamma - \alpha^0_{12}\cos\gamma) \qquad (38b)\end{aligned}$$

with α^0_{11}, α^0_{12} the values at $\gamma = 0$. For $\rho > 0$ the stability of the system (37) will be first investigated. By standard methods it can be shown that a unique equilibrium point α^e_{11}, α^e_{12} (in fact two such points exist depending on the $\text{sgn}\gamma$) is given by

$$\alpha^e_{11} = (1/3\rho)[2 + (3\sqrt{q} - p)^{1/3} - (3\sqrt{q} + p)^{1/3}] \qquad (39a)$$

$$\alpha^e_{12} = \text{sgn}\gamma(c_r/\sqrt{3})\alpha^e_{11}/(1 - \rho\alpha^e_{11}) \qquad (39b)$$

$$\begin{aligned}p &= 3[c_r^2 - (1/2)\rho h_\alpha + (1/3)],\ q = (1/9)[p^2 \\ &+ (c_r^2 + \rho h_\alpha - 1)^3] \qquad (39c)\end{aligned}$$

under the condition $q > 0$ which is sufficient and necessary for α^e_{11} to be the unique real root of the cubic equation $(c_r^2/3)\alpha^e_{11}/(1 - \rho\alpha^e_{11}) = \rho(\alpha^e_{11})^2 - \alpha^e_{11} + (1/3)h_\alpha$. A standard investigation of the signs of the left and right-hand sides of this equation, and the fact that the other two roots are complex, yield $0 < \alpha^e_{11} < 1/\rho$. Multiplying now equations (37) by $\text{sgn}\gamma = \text{sgn}\dot{\gamma}$ (monotonic change of γ), observing that $d\,|\gamma| = \text{sgn}\dot{\gamma}d\gamma$ and transferring the origin at α^e_{11}, α^e_{12} without changing notation for the α_{11}, α_{12} for simplicity, equations (37) become

$$\begin{aligned}d\alpha_{11}/d\,|\gamma| &= -[(c_r/\sqrt{3})\alpha_{11}/(1 - \rho\alpha^e_{11})] \\ &+ \text{sgn}\gamma[(1 - \rho\alpha^e_{11})\alpha_{12} - \rho\alpha_{11}\alpha_{12}] \qquad (40a)\end{aligned}$$

$$d\alpha_{12}/d\,|\gamma| = -(c_r/\sqrt{3})\alpha_{12} + \text{sgn}\gamma[(2\rho\alpha^e_{11} - 1)\alpha_{11} + \rho\alpha_{11}^2] \qquad (40b)$$

Considering the Liapunov function $U = (1/2)(\alpha_{11}^2 + \alpha_{12}^2)$, use of equations (40) yields $dU/d\,|\gamma| = -(c_r/\sqrt{3}(1 - \rho\alpha^e_{11}))\alpha_{11}^2 - (c_r/\sqrt{3})\alpha_{12}^2 + (\text{sgn}\gamma)\rho\alpha^e_{11}\alpha_{11}\alpha_{12}$, quadratic in α_{11}, α_{12}. For asymptotic convergence of the system (40) in the whole it suffices to have $q > 0$ (for uniqueness of α^e_{11}) and $dU/d\,|\gamma| < 0$. On the basis of the sufficient and necessary Sylvester's criterion for negative definiteness [25], the $dU/d\,|\gamma| < 0$ requires that $1 - \rho\alpha^e_{11} > 0$, already satisfied as shown earlier, and $c_r^2 > (3/4)\rho^2(\alpha^e_{11})^2(1 - \rho\alpha^e_{11})$. Observing that the maximum value of the right-hand side of the last inequality is 1/9 for $\alpha^e_{11} = 2/3\rho$, and based on the value of q, equation $(39c)_2$, a stronger but simpler and not unduly restrictive sufficient condition for convergence in the whole is

$$c_r > 1/3 \quad \text{and} \quad c_r^2 + \rho h_\alpha > 1 \tag{41}$$

With ν_1, ν_2 the roots and Δ the discriminant of the characteristic equation of the linear approximation of the system (40), it can be shown that $\nu_1 + \nu_2 < 0$, $\nu_1 \nu_2 > 0$, under the previous conditions, and

$$\Delta = (1/3)c_r^2 \rho^2 (\alpha_{11}^e)^2 + 4(1 - \rho\alpha_{11}^e)^3 (2\rho\alpha_{11}^e - 1) \tag{42}$$

Because the origin is not a center for the system (40) and its linear approximation, their phase portraits are qualitatively equivalent. Hence, provided that $q > 0$ and the aforementioned Sylvester's criterion condition, or the stronger conditions (41), are satisfied, and with α_{11}^e given by equation (39a), the sufficient and necessary condition on h_α, c_r, and ρ for which no stress oscillations occur is $\Delta \geq 0$, in which case the equilibrium point is a stable node (inflected if $\Delta = 0$), while for $\Delta < 0$ the equilibrium point is a stable spiral and stress oscillations occur.

Illustration of the foregoing is shown in Figs. 1 and 2(a,b) obtained by numerical integration of equations (37) together with equations (36) for $\text{sgn}\dot{\gamma} > 0$. The numerical values of all stress quantities are normalized by k. The values $h_\alpha = 3$ and $c_r = \sqrt{3}$ used in the computation, were found to describe sufficiently well the uniaxial response of commercially pure aluminum. For such values the relations (41) are clearly satisfied for any ρ. Figure 1 shows the back-stress trajectories beginning at $\alpha_{12}^0 = 0$ and different α_{11}^0, while Fig. 2(a,b) shows the stress-strain response for $\alpha_{11}^0 = \alpha_{12}^0 = 0$. The α_{12} can be measured from the broken line of Fig. 2(b) corresponding to $\sigma_{12} = (1/\sqrt{3})$. The effect of the different values of ρ (normalized by k^{-1}) is clearly demonstrated, and becomes pronounced for $\gamma \geq 1$, approximately. Using equations (39), (42) and the assumed values of h_α, c_r, it can be computed that $\Delta \geq 0$ for $\rho \geq 0.92$, hence, no oscillations occur for $\rho = 1,2$ in the graphs of Fig. 2.

6 Plane Stress for Orthotropic Symmetries

Assume that the orthotropic axes $\hat{x}_1$, $\hat{x}_2$ form an angle θ with the fixed axes x_1, x_2, measured positive counterclockwise from x_1 to $\hat{x}_1$, and that $\hat{x}_3 = x_3$. Assuming rigid-plastic response, the yield criterion (29), $\sigma_{33} = 0$, the associated flow rule and the simple shear velocity gradient (35), the stress components in reference to $\hat{\mathbf{x}}$ are given by

$$\hat{\sigma}_{11}/B = -\hat{\sigma}_{22}/C = \hat{\sigma}_{12}/(X/F\tan 2\theta) = (k/R)\text{sgn}\dot{\gamma}\,\text{sgnsin}2\theta \tag{43}$$

and $\hat{\sigma}_{13} = \hat{\sigma}_{23} = 0$, with $X = AB + BC + CA$ and $R = [X(B + C + (X/F\tan^2 2\theta))]^{1/2}$. The interesting problem is to find the law of evolution of θ. Based on equation $(30)_1$, one has $\hat{W}_{12}^p = W_{12}^p = (\eta_3'/2F)\hat{D}_{12} = (\eta_3'/4F)\dot{\gamma}\cos 2\theta$, where $\hat{W}_{12}^p = W_{12}^p$ due to the plane transformation. The unit vector $\mathbf{n}_1$ along $\hat{x}_1$ is a purely orientational structure variable and according to equation $(16b)_2$ one has $\overset{\nabla}{\mathbf{n}}_1 = -\mathbf{W}^p\mathbf{n}_1$ (equivalently one may consider the $\overset{\nabla}{\mathbf{a}}_1$ according to equation $(16b)_1$). Hence, based on the values of W_{12}, W_{12}^p and the $\cos\theta$, $\sin\theta$ components of $\mathbf{n}_1$ in reference to x_1, x_2, the expression for $\overset{\nabla}{\mathbf{n}}_1$ yields

$$d\theta/d\gamma = (1/2)[(\eta_3'/2F)\cos 2\theta - 1] \tag{44}$$

As $|\gamma|$ increases either an equilibrium orientation $\theta_e = (1/2)\cos^{-1}(2F/\eta_3')$ is asympotically reached if $|2F/\eta_3'| \leq 1$, or the axes $\hat{x}_1$, $\hat{x}_2$ rotate indefinitely.

As a second example consider a velocity gradient defined by

$$D_{22} = rD_{11}, D_{33} = -(D_{11} + D_{22}), D_{ij} = 0 \quad \text{for } i \neq j, W_{ij} = 0 \tag{45}$$

with r a real number. The same assumptions of the preceding example yield

$$\hat{\sigma}_1/Q_1 = \hat{\sigma}_{22}/Q_2 = \hat{\sigma}_{12}/Q_3 = (k/Q)\text{sgn}D_{11} \tag{46}$$

where $Q_1 = 2(1+r)A + (1 + r + (1-r)\cos 2\theta)B$, $Q_2 = 2(1+r)A + (1 + r - (1-r)\cos 2\theta)C$, $Q_3 = (X/F)(r-1)\sin 2\theta$ and $Q = [A(Q_1 - Q_2)^2 + BQ_2^2 + CQ_1^2 + FQ_3^2]^{1/2}$. Similarly to the previous case one has $\hat{W}_{12}^p = W_{12}^p = (\eta_3'/2F)\hat{D}_{12} = (\eta_3'/4F)(r-1)D_{11}\sin 2\theta$, and since $\overset{\nabla}{\mathbf{n}}_1 = \dot{\mathbf{n}}_1$ because $\mathbf{W} = \mathbf{0}$, the evolution of $\mathbf{n}_1$ according to equation $(16b)_2$ is given by $\dot{\mathbf{n}}_1 = -\mathbf{W}^p\mathbf{n}_1$ which yields

$$d\theta/d\epsilon = (c/2)\sin 2\theta, \quad \tan\theta = \tan\theta_0 e^{c\epsilon}, \quad c = (r-1)\eta_3'/2F \tag{47}$$

where e is the basis of the natural logarithm, ϵ the logarithmic strain along x_1, θ_0 the value of θ at $\epsilon = 0$, and equation $(47)_2$ is obtained from equation $(47)_1$ if c is constant. Depending on the signs of $\tan\theta_0$, c and ϵ, the axis $\hat{x}_1$ tends to align with the axis x_1 or x_2 as $|\epsilon|$ increases (for $c = 0 \Rightarrow \theta = \theta_0$). The results expressed by equations (44) and (47) can be given a plausible physical interpretation when $\eta_3'/2F = \pm 1$ for the case of transverse isotropy and the corresponding visualization in terms of a unidirectionally fiber-reinforced material and a deck of cards, discussed at the end of Section 4. A detailed presentation of this case is given in [26].

A remarkable difference exists between the anisotropy induced by kinematic hardening and the initial orthotropic one. In the former case the effect of the plastic spin is pronounced after large strains occurred in order to induce sufficiently "intense" anisotropy, while in the latter case the plastic spin affects the response at the early stage of small to moderate strains due to the pre-existing orthotropy.

7 Discussion and Conclusion

The generality of the development is partially limited because of two restrictions. The first is the omission of tensorial structural variables of higher order than two for reasons of simplicity. Even under this restriction, a sufficiently realistic set of constitutive models can be obtained. Including higher-order tensors can certainly be done along the lines presented in [4, 21]; one should expect then greater difficulties in their experimental determination and more complex expressions for the plastic spin and elastic embedding. The second restriction is the assumption of a smooth yield criterion. While for single crystals there is experimental and theoretical evidence that corners are induced by the superposed action of smooth yield criteria, in materials such as polycrystals for which a macroscopic approach can be applied, no conclusive experimental evidence of macroscopically detectable corners exists at present. Should such evidence become available the present formulation must be extended appropriately.

Within the aforementioned limitations the present work achieved its stated objectives, namely to demonstrate the effect of elastically embedding the structure variables and, mainly, of the constitutive relations for the plastic spin on a macroscopic formulation of anisotropic continua. We would like to emphasize here three points. First, the importance of the forms of equations (15) and (16) for the evolution laws of the structure variables, where the purely constitutive part associated with $\bar{\mathbf{a}}$, $\bar{\mathbf{m}}$, $\bar{\mathbf{s}}$ is clearly distinguished from the kinematical terms associated with $\mathbf{N}^p$ and $\mathbf{\Omega}^p$. Second, the form of the constitutive relations for the plastic spin via $\mathbf{\Omega}^p$, as in equation (27), together with the subsequently discussed continuity requirement as $\mathbf{a}' \to \mathbf{0}$. And third, the fact that Mandel's director vectors are not necessary in general for the definition of the spin ω which simply can be obtained from the kinematics, equation (6), given the constitutive relation for $\mathbf{W}^p$. This has not been recognized in the earlier works of the author [12–14], but has been pointed out in [26]. A number of additional topics including extension to rate dependent viscoplastic response and examples on the effect of elastic embedding, such as in the single slip theory [7, 9], can be found in [27] and [28]; reference [28] provides also more details on the calculations performed in the present work.

Much can be gained by parallel microscopic studies but also, vice-versa, the present formulation can serve as a guideline for such studies when the bridging of the gap from micro to macrolevel is attempted by means of proper averaging procedures, through which macroscopic tensorial structure variables emerge. At present one can attempt to calibrate macroscopically constitutive parameters such as ρ and η_3' related to the plastic spin, and whose effect was clearly demonstrated by the theoretical analysis of the cases presented in Sections 5 and 6. For example, the h_α and c_r for the kinematic hardening can be first calibrated from the experimental data of radial deviatoric stress paths (particular case being the uniaxial) where the plastic spin is zero because $\alpha\sigma' = \sigma'\alpha$, equations (32)–(34); then the value of ρ can be determined by fitting experimental data where the plastic spin is active and affects the stress-strain response, as shown eloquently in Fig. 2(a,b) for simple shear. The fact that the aforementioned cases refer to partially idealized situations cannot be overlooked, but at the same time they should not be discarded from a practical point of view before actual experimental data become available for comparison. Even if such data show the weakness of the idealization, the analysis of these examples is of value because it illustrates the way of approach for more complex considerations (e.g., more than one evolving structure variables) within the general formulation.

Finally, observe that the general formulation was presented in terms of the values of the state variables at the current and relaxed configurations as defined by equation (1). Equivalent descriptions in reference to other configurations, such as the isoclinic [1], are also possible; but the important fact is that a change of the mode of description cannot eliminate a physical necessity such as the requirement of constitutive relations for the plastic spin.

8 References

1 Mandel, J., "Plasticité Classique et Viscoplasticité," *Courses and Lectures, No. 97*, International Center for Mechanical Sciences, Udine, Springer, New York, 1971.

2 Kratochvil, J., "Finite-Strain Theory of Crystalline Elastic-Inelastic Materials," *J. Appl. Phys.*, Vol. 42, 1971, pp. 1104–1108.

3 Wang, C. C., "A New Representation Theorem for Isotropic Functions: An Answer to Professor G. F. Smith's Criticism of My Paper on Representations for Isotropic Functions," *Arch. Rat. Mech. Anal.*, Vol. 36, 1970, pp. 198–223.

4 Fardshisheh, F., and Onat, E. T., "Representation of Elastoplastic Behavior by Means of State Variables," *Problems of Plasticity*, Sawczuk, A., ed., Noordhoff, Leyden, 1974, pp. 89–115.

5 Lee, E. H., "Elastic-Plastic Deformations at Finite Strains," ASME Journal of Applied Mechanics, Vol. 36, 1969, pp. 1–6.

6 Rice, J. R., "Inelastic Constitutive Relations for Solids: An Internal-Variable Theory and its Application to Metal Plasticity," *J. Mech. Phys. Solids*, Vol. 19, 1971, pp. 433–455.

7 Asaro, R. J., and Rice, J. R., "Strain Localization in Ductile Single Crystals," *J. Mech. Phys. Solids*, Vol. 25, 1977, pp. 309–338.

8 Weng, G. J., "The Yield Surface of Single Crystals at Arbitrary Strain," *Acta Mechanica*, Vol. 37, 1980, pp. 231–245.

9 Peirce, D., Asaro, R. J., and Needleman, A., "An Analysis of Nonuniform and Localized Deformation in Ductile Single Crystals," *Acta Metal.*, Vol. 30, 1982, pp. 1087–1119.

10 Nemat-Nasser, S., "On Finite Deformation Elasto-Plasticity," *Int. J. Solids Structures*, Vol. 18, 1982, pp. 857–872.

11 Havner, K. S. and Shalaby, A. H., "Further Investigation of a New Hardening Law in Crystal Plasticity," ASME Journal of Applied Mechanics, Vol. 45, 1978, pp. 500–506.

12 Dafalias, Y. F., "A Missing Link in the Macroscopic Constitutive Formulation of Large Plastic Deformations," Chapter 8 in *Plasticity Today*, Sawczuk, A., and Bianchi, G., eds., Elsevier Applied Science, U.K., 1985, pp. 135–151. From the International Symposium on Current Trends and Results in Plasticity, CISM, Udine, Italy, June 1983.

13 Dafalias, Y. F., "On the Evolution of Structure Variables in Anisotropic Yield Criteria at Large Plastic Transformations," Volume from the Colloque International du C.N.R.S. No. 351 sur "Critères de Rupture des Materiaux à Structure Interne Orientée", Villard-de-Lans, France, June 1983, Boehler, J. P., ed., Éditions du C.N.R.S., Paris, in press.

14 Dafalias, Y. F., "Corotational Rates for Kinematic Hardening at Large Plastic Deformations," ASME Journal of Applied Mechanics, Vol. 50, 1983, pp. 561–565.

15 Loret, B., "On the Effects of Plastic Rotation in the Finite Deformation of Anisotropic Elastoplastic Materials," *Mechanics of Materials*, Vol. 2, 1983, pp. 287–304.

16 Boehler, J. P., "A Simple Derivation of Representations for Non-Polynomial Constitutive Equations in Some Cases of Anisotropy," *ZAMM*, Vol. 59, 1979, pp. 157–167.

17 Liu, I. S., "On Representations of Anisotropic Invariants," *Int. J. Eng. Sci.*, Vol. 20, 1982, pp. 1099–1109.

18 Hill, R., *The Mathematical Theory of Plasticity*, Oxford University Press, London, 1950.

19 Armstrong, P. J., and Frederick, C. O., "A Mathematical Representation of the Multiaxial Bauschinger Effect," C.E.G.B. Report RD/B/N 731, 1966.

20 Nagtegaal, J. C., and de Jong, J. E., "Some Aspects of Nonisotropic Work Hardening in Finite Strain Plasticity," in: *Plasticity of Metals at Finite Strain: Theory, Experiment and Computation*, Lee, E. H., and Mallett, R. L., eds., Div. Appl. Mech., Stanford University and Dept. Mech. Eng., R.P.I., 1982, pp. 65–102.

21 Onat, E. T., "Representation of Inelastic Behavior in the Presence of Anisotropy and of Finite Deformations," in: *Recent Advances in Creep and Fracture of Engineering Materials and Structures*, Wilshire, B., and Owen, D. R. J., eds., Pineridge Press, Swansea, U.K., 1982, pp. 231–264.

22 Lee, E. H., Mallet, R. L., and Wertheimer, T. B., "Stress Analysis for Anisotropic Hardening in Finite-Deformation Plasticity," ASME Journal of Applied Mechanics, Vol. 50, 1983, pp. 554–560.

23 Fressengeas, C., and Molinari, A., "Modèles d'Ecrouissage Cinématique en Grande Déformation," *C.R.A.S.*, Vol. 297, Série II, 1983, pp. 93–96.

24 Johnson, G. C., and Bammann, D. J., "A Discussion of Stress Rates in Finite Deformation Problems," Sandia Report SAND82-8821, Aug. 1982.

25 Ogata, K., *State Space Analysis of Control Systems*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

26 Dafalias, Y. F., "The Plastic Spin Concept and a Simple Illustration of its Role in Finite Plastic Transformations," *Mechanics of Materials*, Vol. 3, 1984, pp. 223–233.

27 Dafalias, Y. F., "A Missing Link in the Formulation and Numerical Implementation of Finite Transformation Elastoplasticity," in: *Constitutive Equations: Macro and Computational Aspects*, William, K. J., ed., ASME, 1984, pp. 25–40.

28 Dafalias, Y. F., "The Plastic Spin," A three-part Report No. 83-1-2-3, Department of Civil Engineering, University of California at Davis, Oct. 1983.

APPENDIX

A scalar, vector, or tensor-valued isotropic function $\mathbf{f}$ of the scalar, vector, and tensor-valued variables $\mathbf{s}$, is defined by $\mathbf{f}(\mathbf{s}) = \mathbf{Q}[\mathbf{f}(\mathbf{Q}^T[\mathbf{s}])]$ for any orthogonal transformation $\mathbf{Q}$. For such a function it can be shown that

$$D\mathbf{f}/Dt = (\partial\mathbf{f}/\partial\mathbf{s})\cdot(D\mathbf{s}/Dt) \tag{A1}$$

for corotational rates according to equation (2) with respect to any $\mathbf{\Omega}$ including $\mathbf{\Omega} = \mathbf{0}$, and with the understanding that D/Dt implies the material time derivative for scalar valued quantities. The proof is based on the fact that one can always write $\mathbf{\Omega} = \dot{\mathbf{Q}}\mathbf{Q}^T$ since $\mathbf{Q}(t)$ is arbitrary. As an example, for a scalar-valued isotropic function $f(\mathbf{s}) = f(\mathbf{Q}^T[\mathbf{s}])$, one has

$$\dot{f} = \frac{\partial f}{\partial \mathbf{Q}^T[\mathbf{s}]}\cdot\dot{\overline{\mathbf{Q}^T[\mathbf{s}]}} = \mathbf{Q}^T\left[\frac{\partial f}{\partial \mathbf{s}}\right]\cdot\mathbf{Q}^T\left[\frac{\partial \mathbf{s}}{Dt}\right] = \frac{\partial f}{\partial \mathbf{s}}\cdot\frac{D\mathbf{s}}{Dt}$$

$$= \frac{\partial f}{\partial \mathbf{s}}\cdot\overset{\circ}{\mathbf{s}} \tag{A2}$$

The proof for tensor-valued $\mathbf{f}$ is longer but similar.