Acta Mechanica 73, 121—146 (1988)

ACTA MECHANICA
© by Springer-Verlag 1988

Issues on the Constitutive Formulation at Large Elastoplastic Deformations, Part 2: Kinetics

By

Y. F. Dafalias, Davis, California

(Received August 25, 1986)

Summary

The coupling between kinematics and kinetics and the invariance requirements under superposed rigid body rotations, determine unambiguously the proper general form of the elastoplastic rate constitutive equations, in terms of the values of the state variables and their rates in reference to the current and any choice of the unstressed configuration. Topics such as the effect of changing the stress rate, small elastic deformations with or without large volumetric elastic strains, rate effects and viscoplasticity, an example on single slip, the effect of the plastic spin constitutive relations and the concept of an effectively unstressed configuration, are addressed in detail. Issues and different approaches debated in the past are discussed, compared and clarified.

1. Introduction

In the first part [1] the kinematics were analyzed and the way paved for the development of the kinetics in the constitutive formulation of elastoplasticity at large deformations. The important coupling between kinematics and kinetics had already been implicitly introduced in the first part, by defining the meaning of the different corotational and corodeformational rates of the structure variables. This coupling is of utmost importance for a realistic constitutive development. The notation here will be the same as in the first part.

The constitutive equations can be formulated in reference to any one of the different configurations, the reference $\varkappa_r$ (Lagrangian description), the current $\varkappa$ (Eulerian description) or any one of the unstressed configurations $\varkappa_u$, $\varkappa_0$, $\varkappa_i$ (Intermediate or semi-Eulerian description). It must be stated at the outset that whether Lagrangian, Eulerian or semi-Eulerian, the description is always material in the sense that the constitutive statements refer to the material neighborhood of a specific material point considered in any one of the aforementioned configurations, and not to a control-volume element occupied by different material points at different times (spatial description). Hence, the term Eulerian must be interpreted here as implying a description in terms of the true current values of the related quantities.

One important point is that whichever may be the configuration choice, the corresponding constitutive statements automatically imply, via the kinematics, some equivalent statements in relation to all other configurations, since a physical phenomenon cannot depend on the mode of description. What is at stake, however, is the proper accounting of the physics of the phenomena described by the constitutive relations. Although in principle one can make constitutive statements at the reference configuration $\varkappa_r$, he runs the danger of obscuring the physics of the constitutive reality by the intervention of plastic kinematical variables such as $\boldsymbol{F}^p$, $\boldsymbol{P}$ or $\boldsymbol{F}_i^p$, geometrical entities which are "inert" for the most part in relation to the on-going evolution of the structure variables. The physics of the phenomena involved, can be better understood and realistically describred if one works in terms of the current "true" values of the different quantities, i.e. in a Eulerian description. Then, of course, the Eulerian formulation can be properly transported at $\varkappa_r$ yielding the corresponding Lagrangian description.

In an Eulerian type of description one is faced with the multiple choice of configurations, that is the current $\varkappa$ or one of the intermediate $\varkappa_u$, $\varkappa_0$ and $\varkappa_i$, with the requirement of course that the final result must be the same. Again the physics of the situation dictates that some of the phenomena, such as the elastic relations, lend themselves to an initial constitutive statement at one of the unstressed configurations, while others, such as the yield criterion, are better described initially at the current configuration $\varkappa$. The elastic embedding will always serve as a link between current and unstressed configurations, and will be used in order to have finally everything expressed in terms of current values at $\varkappa$ (purely Eulerian description). Along the lines of development a number of particular topics such as the effect of changing the stress rate, small elastic deformations with or without a large elastic volumetric strain, rate effects and viscoplasticity, an example on single slip, the plastic spin constitutive relations and the concept of an effectively unstressed configuration, are all addressed in detail.

2. Mathematical Preliminaries and General Concepts

It will be expedient to discuss some mathematical generalities pertaining to certain properties of constitutive functions in general, so that repetition can be avoided in the sequel. These functions can represent yield criteria, elastic potentials, rates of deformation and spins, etc. Hence, such a representative function will be denoted by $\boldsymbol{f}(\boldsymbol{\Pi}, \boldsymbol{S})$ being scalar, vector or tensor valued; equivalently the arguments could be $\boldsymbol{\sigma}$ and $\boldsymbol{s}$ in the current configuration (recall that $\boldsymbol{\Pi}$, $\boldsymbol{S}$ or $\boldsymbol{\sigma}$, $\boldsymbol{s}$ are the stress and structure variables in reference to an unstressed or the current configuration, respectively). The tensorial structure variables $\boldsymbol{S}$ can be either purely orientational, such as the unit vectors (and their tensor products thereof) along the axes of orthotropy, or can be evolving tensors with variable eigenvalues and eigenvectors, such as the back-stress or a measure of plastic deformation. It will be seen that no distinction is necessary from the point of view of mathematical representation, and only their law of evolution will distinguish the purely orientational variables from the others. These other structure variables, in addition to their hardening characteristics which distinguish them from the purely orientational ones, have also orientational characteristics (e.g. their eigenvectors).

In the process of investigating the way that $\boldsymbol{f}$ depends on $\boldsymbol{\Pi}$ and $\boldsymbol{S}$, we will be faced with two possible and equivalent cases. In the first case, invariance requirements under superposed rigid body rotations will render $\boldsymbol{f}$ isotropic function of its arguments $\boldsymbol{\Pi}$ and $\boldsymbol{S}$. In the second case, the purely orientational among $\boldsymbol{S}$ (which are non-zero at the initial unstressed configuration) define in fact a group of orthogonal transformations under which they remain invariant. Thus, material symmetries expressed by these orthogonal transformations in reference to an initial relaxed configuration are automatically introduced in $\boldsymbol{f}$ by the mere presence of the corresponding $\boldsymbol{S}$. It follows then, based on the theorems developed in [2], [3], that $\boldsymbol{f}$ must be an isotropic functions of all its arguments, that is including $\boldsymbol{\Pi}$, the "evolving" $\boldsymbol{S}$ and the purely orientational $\boldsymbol{S}$. The important fact is that under any circumstances the $\boldsymbol{f}$ will be an isotropic function of its arguments. A simple example is that of Hill's well known quadratic yield criterion [4] for orthotropic materials. If $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ are unit vectors along two of the axes of orthotogry, the $\boldsymbol{A}_1 = \boldsymbol{N}_1 \otimes \boldsymbol{N}_1$ and $\boldsymbol{A}_2 = \boldsymbol{N}_2 \otimes \boldsymbol{N}_2$ are the purely orientational structure variables [2], [3]. It can be shown that the specific form given by Hill in reference to the orthotropic axes, can be expressed in reference to any set of axes as a polynominal expression of the isotropic invariants $\operatorname{tr} \boldsymbol{\Pi}$, $\operatorname{tr} \boldsymbol{\Pi}^2$, $\operatorname{tr} \boldsymbol{A}_1\boldsymbol{\Pi}$, $\operatorname{tr} \boldsymbol{A}_2\boldsymbol{\Pi}$, $\operatorname{tr} \boldsymbol{A}_1\boldsymbol{\Pi}^2$, $\operatorname{tr} \boldsymbol{A}_2\boldsymbol{\Pi}^2$, [5]. It may be surprising that the isotropy of the constitutive functions arises equivalently either as a result of invariance under superposed rigid body rotations, or as a result of purely mathematical arguments. But the underlying common cause of this equivalence is merely the fact we have assumed that the material substructure can be described in terms of $\boldsymbol{S}$, and in such cases it should be expected that a physical assumption must lead to the same analytical description whichever the chosen methodology may be.

Based on the foregoing, it is obvious that the mathematical isotropy of $\boldsymbol{f}$ does not imply physical material isotropy, due to the tensorial nature of $\boldsymbol{S}$ [6], [7]. In fact almost all possible cases of initial, persisting and/or induced symmetries can be described by the definition of $\boldsymbol{S}$ and their evolution, and there is no reason to make a distinction, from the point of view of mathematical representation, between the so-called "structural" and "induced" anisotropic characteristics [8]. Notably, if all $\boldsymbol{S}$ are scalar valued (or isotropic tensors) the material is and remains isotropic. If $S = 0$ initially (which means they cannot be purely orientational, because the latter are non-zero always) but become non-zero subsequently, the material is isotropic in reference to its initial unstressed configuration, but anisotropic in reference to subsequent relaxed configurations. If all S are purely orientational, the initial symmetries persist in reference to subsequent unstressed configurations. The general case of purely and non-purely orientational or hardening S, the latter being zero or non-zero at the initial unstressed configuration, describes initial and evolving anisotropies. Hence, everything is taken care by the proper choice of S, and the "old-fashioned" way of thinking in terms of groups of orthogonal transformations which define symmetries can be abandoned, since it is already incorporated in the isotropic dependence of f on Π and S according to the modern theorems on representation [2], [3]. These theorems provide the general framework for the dependence of f on its arguments in terms of proper generators and isotropic scalar invariants [9].

As it was shown in the Appendix of [10], the isotropy of f allows us to write the relation

$$\frac{Df}{Dt} = \frac{\partial f}{\partial \Pi} \cdot \frac{D\Pi}{Dt} + \frac{\partial f}{\partial S} \cdot \frac{DS}{Dt} \tag{1}$$

for corotational rates according to Eq. (4) of [1] in relation to any Ω, including $\Omega = 0$, and with the understanding that D/Dt implies the material time derivative for scalar valued quantities. Eq. (1) will be proved very useful in the sequel.

Although it will be assumed in the following that the material substructural characteristics can be described by S, which will imply the isotropy of f, it is instructive to briefly discuss the case where such a description is not possible. In other words, the f can assume a particular analytical form valid only in reference to a particular coordinate system, the director vectors [11], defined by $\hat{x} = \beta_u^T x$ from the fixed cartesial coordinate system x, if one decides to use $\varkappa_u$ as the intermediate configuration (equivalently in what follows, the β or $\beta_i = I$ would substitute for β_u if the configurations $\varkappa_0$ or $\varkappa_i$, respectively, were chosen.) Thus, one has $f(\hat{\Pi}, \hat{S})$, where $\hat{\Pi}$, $\hat{S}$ are the state variables in reference to $\hat{x}$, related to their representation in reference to x by $\hat{\Pi} = \beta_u^T \Pi \beta_u$, $\hat{S} = \beta_u^T[S]$. Notice that the use of S (or part of them) must not be any more such as to define initial symmetries, because then the mathematical theorems in [3] will render f isotropic function of its arguments. The quantity represented by f in $\hat{x}$, will become $\beta_u f(\beta_u^T \Pi \beta_u, \beta_u^T[S]) \beta_u^T$ in x, and it can immediately be seen that if f were isotropic, the β_u would disappear in the x representation, retrieving the previous results [12]. If f is not isotropic, the existence of β_u in the x representation is necessary, and β_u must be calculated by the integration of $\omega_u = \dot{\beta}_u \beta_u^T$. Notice that under a superposed rigid body rotation expressed by Q_u, the fact that $\beta_u^* = Q_u \beta_u$, $\Pi^* = Q_u \Pi Q_u^T$ and $S^* = Q_u[S]$, [1], validates the form invariance of $\boldsymbol{f}(\hat{\boldsymbol{\Pi}}, \hat{\mathbf{S}})$. In other words, invariance requirements under superposed rigid body rotations do not impose any restriction on the form of $\boldsymbol{f}(\hat{\boldsymbol{\Pi}}, \hat{\mathbf{S}})$ because everything is referred to a rotating coordinate system and such invariance is automatically satisfied. This type of description is complicated because of the presence of $\boldsymbol{\beta}_u$, and has been advocated in general by Mandel [11]. Fortunately for almost all materials of interest one can make the assumption that the $\boldsymbol{S}$ can account for initial and subsequent symmetries, in which case $\boldsymbol{f}$ becomes finally an isotropic function of $\boldsymbol{\Pi}$ and $\boldsymbol{S}$ without the presence of β_u, result not forseen by Mandel. The latter approach was proposed by Onat et al. [6], [7].

3. Rate Equations for the Plastic Kinematical Variables

With $\boldsymbol{\Pi}$, $\boldsymbol{S}$ defined at $\varkappa_0$ the constitutive equations for the plastic rate of deformation and plastic spin at $\varkappa_0$ are given by [10]

$$\boldsymbol{D}_0{}^p = \langle\lambda\rangle\, \boldsymbol{N}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_0); \qquad \boldsymbol{W}_0{}^p = \langle\lambda\rangle\, \boldsymbol{\Omega}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_0) \qquad (2.1;2)$$

where λ is a scalar loading index associated with a yield criterion, both to be defined subsequently, and $\langle\,\rangle$ are the Macauley brackets such that $\langle\lambda\rangle = \lambda$ if $\lambda > 0$ and $\langle\lambda\rangle = 0$ if $\lambda \leqq 0$. Based on [1, Eq. (24.2)] and the fact that $\boldsymbol{\Pi}^* = \boldsymbol{Q}\boldsymbol{\Pi}\boldsymbol{Q}^T$ and $\boldsymbol{S}^* = \boldsymbol{Q}[S]$ upon a superposed rigid body rotation it follows that necessarily $\boldsymbol{N}_0{}^p$ and $\boldsymbol{\Omega}_0{}^p$ are isotropic functions of their arguments $\boldsymbol{\Pi}$ and $\boldsymbol{S}$. Recall now the discussion following [1, Eqs. (13)] whereby referring to [1, Eqs. (3)] it can be shown that the $\boldsymbol{\Pi}$, $\boldsymbol{S}$ at the relaxed configurations differ only by rotations defined in terms of $\boldsymbol{R}_u{}^e$ and $\boldsymbol{R}_i{}^e$. Hence, based on the isotropy of the $\boldsymbol{N}_0{}^p$ and using [1, Eq. (12.1)] and (2.1), it follows that $\boldsymbol{D}_u{}^p = \langle\lambda\rangle\, \boldsymbol{R}_u{}^{e^T} \boldsymbol{N}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_0)\, \boldsymbol{R}_u{}^e = \langle\lambda\rangle\, \boldsymbol{N}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_u)$, where $\boldsymbol{R}_u{}^{e^T}\boldsymbol{\Pi}$ (at $\varkappa_0$) $\boldsymbol{R}_u{}^e = \boldsymbol{\Pi}$ at $\varkappa_u$, and $\boldsymbol{R}_u{}^{e^T}[\boldsymbol{S} \text{ at } \varkappa_0] = \boldsymbol{S}$ at $\varkappa_u$, were used. Employing similar arguments at both $\varkappa_u$ and $\varkappa_i$ in conjunction with [1, Eqs. (12)] and (2) we have

$$\boldsymbol{D}_u{}^p = \langle\lambda\rangle\, \boldsymbol{N}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_u); \qquad \boldsymbol{W}_u{}^p = \langle\lambda\rangle\, \boldsymbol{\Omega}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_u) \qquad (3)$$

$$\boldsymbol{D}_i{}^p = \langle\lambda\rangle\, \boldsymbol{N}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_i); \qquad \boldsymbol{W}_i{}^p = \langle\lambda\rangle\, \boldsymbol{\Omega}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_i). \qquad (4)$$

Eqs. (2), (3), (4) represent the equivalent constitutive statements about the plastic rate of deformation and spins in reference to $\varkappa_0$, $\varkappa_u$ and $\varkappa_i$, respectively. Notice the important fact that the same functional forms $\boldsymbol{N}_0{}^p$ and $\boldsymbol{\Omega}_0{}^p$ are used in all three configurations, the difference being only that the arguments $\boldsymbol{\Pi}$, $\boldsymbol{S}$ enter by their representation in the corresponding configuration.

In order to transport Eqs. (2), (3), (4) to the current configuration $\varkappa$, one can define first symmetric and antisymmetric tensor $\boldsymbol{N}^p$ and $\boldsymbol{\Omega}^p$, respectively, by

$$\begin{aligned} \boldsymbol{N}^p + \boldsymbol{\Omega}^p &= \boldsymbol{V}[\boldsymbol{N}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_0) + \boldsymbol{\Omega}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_0)]\, \boldsymbol{V}^{-1} \\ &= \boldsymbol{F}^e[\boldsymbol{N}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_u) + \boldsymbol{\Omega}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_u)]\, \boldsymbol{F}^{e^{-1}} \qquad (5) \\ &= \boldsymbol{F}_i{}^e[\boldsymbol{N}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_i) + \boldsymbol{\Omega}_0{}^p(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_i)]\, \boldsymbol{F}_i{}^{e^{-1}} \end{aligned}$$

where the above equivalent definitions are based on the isotropy of $N_0{}^p$, $\Omega_0{}^p$ and the relations among the V, F^e, $F_i{}^e$, [1, Eq. (3.1)], and among the Π, S at the different configurations. Hence, choosing the proper definition from Eq. (5), each one of the Eqs. (2), (3), (4) in conjunction with [1 Eqs. (10.3), (10.4), (11.1), (11.2)] yields

$$D^p = \langle\lambda\rangle\, N^p(\Pi, S \text{ at } \varkappa_0); \qquad W^p = \langle\lambda\rangle\, \Omega^p(\Pi, S \text{ at } \varkappa_0). \tag{6.1, 2}$$

The N^p and Ω^p are isotropic functions of Π, S at $\varkappa_0$ because V (as it will be shown later), $N_0{}^p$ and $\Omega_0{}^p$ are also isotropic functions. Recall now that the elastic embedding relates the Π, S at $\varkappa_0$ to σ, s at $\varkappa$. For example, one has $\sigma = |V|^{-1}\, V\Pi V$ and $a = |V|^{-w}\, VAV$ according to [1, Eq. (13.1)] with V substituting for F^e. The above relations express fully σ, a as isotropic functions of Π, A, since V is isotropic function of Π, A. Thus, in principle, they can be inverted and express Π, A (or more generally Π, S), as isotropic functions of σ, a (or more generally σ, s). Hence, Eq. (6) can be rewritten with N^p and Ω^p being now (different) isotropic functions of the state variables σ, s at the current configuration.

4. Rate Equations for the Structure Variables

The procedure for obtaining the rate equations of evolution for the structure variables at any one of the three intermediate configurations is the same as for the plastic rate of deformation and spin. The important fact is to remember that these equations of evolution must be expressed in terms of rates which are corotational with the substructure. i.e. the corresponding director vectors at the chosen relaxed configuration, Hence, one can equivalently state

$$\mathring{S} = \langle\lambda\rangle\, \bar{S}(\Pi, S \text{ at } \varkappa_0), \quad \text{or} \tag{7.1}$$

$$\overset{\triangle}{S} = \langle\lambda\rangle\, \bar{S}(\Pi, S \text{ at } \varkappa_u), \quad \text{or} \tag{7.2}$$

$$\dot{S} = \langle\lambda\rangle\, \bar{S}(\Pi, S \text{ at } \varkappa_i) \tag{7.3}$$

where $\bar{S}$ is an isotropic function of its arguments; such isotropy follows from the form invariance requirement for $\bar{S}$ under superposed rigid body rotations, and the transformation of S and its corotational rates under such rotation as shown for a second order tensor A in [1, Eqs. (27)].

In order to transport Eqs. (7) from the corresponding intermediate configuration to the current configuration $\varkappa$, one must refer to [1, Eqs. (14)—(17)] and the discussion thereof. From [1, Eq. (14)], exemplifying the situation for a second order relative tensor contravariantly embedded, it is seen that any constitutive statement on the corotational rate of A implies automatically a corresponding statement for the corodeformational rate $\overset{\square}{a}$ of a at $\varkappa$. In addition, [1, Eqs. (16)

and (17)] can be used in conjunction with Eqs. (6) in order to change the constitutive statement made in terms of $\overset{\square}{\boldsymbol{a}}$, to an equivalent constitutive statement made in terms of $\check{\boldsymbol{a}}$. The following express analytically the foregoing equivalence

$$\mathring{\boldsymbol{A}} \text{ or } \overset{\triangle}{\boldsymbol{A}} \text{ or } \dot{\boldsymbol{A}} = \langle\lambda\rangle\, \bar{\boldsymbol{A}}(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_0, \text{ or } \varkappa_u \text{ or } \varkappa_i, \text{ respectively}) \tag{8.1}$$

$$\overset{\square}{\boldsymbol{a}} = \langle\lambda\rangle\, \bar{\boldsymbol{a}}(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_0) \tag{8.2}$$

$$\check{\boldsymbol{a}} = \langle\lambda\rangle \left(\bar{\boldsymbol{a}} - (\boldsymbol{a}\boldsymbol{N}^p + \boldsymbol{N}^p\boldsymbol{a}) + (\boldsymbol{a}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{a}) + w\boldsymbol{a}\,\mathrm{tr}\boldsymbol{N}^p\right) \tag{8.3}$$

$$\begin{aligned}\bar{\boldsymbol{a}} &= |\boldsymbol{V}|^{-w}\, \boldsymbol{V}\bar{\boldsymbol{A}}(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_0)\, \boldsymbol{V} = |\boldsymbol{F}^e|^{-w}\, \boldsymbol{F}^e\bar{\boldsymbol{A}}(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_u)\, \boldsymbol{F}^{e^T} \\ &= |\boldsymbol{F}_i^e|^{-w}\, \boldsymbol{F}_i^e\bar{\boldsymbol{A}}(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_i)\, \boldsymbol{F}_i^{e^T}.\end{aligned} \tag{8.4}$$

In reference to Eq. (8.1) it can be shown, as done in [1, Eq. (12)] for the kinematical variables, that $\mathring{\boldsymbol{A}} = \boldsymbol{R}_u^e \overset{\triangle}{\boldsymbol{A}} \boldsymbol{R}_u^{e^T} = \boldsymbol{R}_i^e \dot{\boldsymbol{A}} \boldsymbol{R}_i^{e^T}$, where of course $\boldsymbol{A}$ is being referred to the corresponding unstressed configuration for each member. It is important to emphasize that whatever is the choice of the intermediate configuration (to which $\boldsymbol{A}$ refers), and the corresponding corotational rate in Eq. (8.1), one single constitutive statement is made at the current configuration $\varkappa$ in terms of $\check{\boldsymbol{a}}$ or $\overset{\square}{\boldsymbol{a}}$. The $\bar{\boldsymbol{a}}$ is an isotropic function of its arguments as it follows from its equivalent three definitions, Eq. (8.4), and the isotropy of $\bar{\boldsymbol{A}}$ and $\boldsymbol{V}$, $\boldsymbol{F}^e$, $\boldsymbol{F}_i^e$ in reference to $\boldsymbol{\Pi}$, $\boldsymbol{S}$. Following similar arguments to the ones applied to $\boldsymbol{N}^p$, $\boldsymbol{\Omega}^p$, the $\bar{\boldsymbol{a}}$ can also be expressed as an isotropic (different) function of $\boldsymbol{\sigma}$ and $\boldsymbol{s}$. For later use, observe that Eq. (8.2) applied to any kind of structure variable can be expressed as $\overset{\square}{\boldsymbol{s}} = \langle\lambda\rangle\, \bar{\boldsymbol{s}}$.

Eq. (8) show clearly the full effect of the elastic embedding and the use of rates corotational with the director vectors. They are by far the most important and influencial contribution of the whole theory, with the most novel feature being the presence of the plastic spin via $\boldsymbol{\Omega}^p$, which explicitly enters Eq. (8.3) and implicitly the rest. Eqs. (8) render irrelevant all past and on-going discussions about the proper choice of corotational and convected rates for the structure variables. For example, Eq. (8.1) is a very clear statement that the constitutive change of $\boldsymbol{A}$ at any intermediate configuration must be corotational with the substructure and not with the continuum aspect of the material. Similarly, the equivalent Eq. (8.2) states that the constitutive change expressed from the viewpoint of the current value $\boldsymbol{a}$, must account not only for the corotation of the transported $\boldsymbol{A}$ at the intermediate configuration, but also for the influence of the elastic embedding occuring from intermediate to current configuration, hence, the use of $\overset{\square}{\boldsymbol{a}}$. Eq. (8.3) is simply the result of mathematical manipulation based on kinematics.

The meaning of these equations can also be understood when one considers the case of unloading, i.e. $\lambda \leqq 0$ and $\langle\lambda\rangle = 0$. Eq. (8.1) will simply state that

$\boldsymbol{A}$ changes only by rotation since its corotational rate becomes zero, which implies that while $\dot{\boldsymbol{A}} \neq \boldsymbol{0}$, the isotropic invariants of $\boldsymbol{A}$ will remain stationary. With $\langle\lambda\rangle = 0$, one can see from [1, Eq. (17)] and (8.2, 3) that $\overset{\square}{\boldsymbol{a}} = \check{\boldsymbol{a}} = \boldsymbol{0}$, which implies not only that $\dot{\boldsymbol{a}} \neq \boldsymbol{0}$, but also that the isotropic invariants of $\boldsymbol{a}$ *are not* stationary. Viewed from a different but equivalent perspective, during unloading the $\boldsymbol{A}$ remains fixed but the $\boldsymbol{a}$ changes due to its elastic embedding, if one refers their components to the director vectors coordinate system.

It is pertinent to discuss here the issue raised by Prager [13] on the appropriateness of using the Jaumann rates (corotational with $\boldsymbol{W}$) in evolution equations, from the point of view of preserving the stationarity of the isotropic invariants under rigid body motion. In fact, Prager referred to stress rates, but within our general framework the requirement for stationarity of invariants can extend his arguments to the choice of rates for the structure variables, as well. To begin with, Prager did not consider the concept of elastic embedding, but it will be shown that Prager's point of view is retrieved anyway. Indeed, under a rigid body motion one has not only $\boldsymbol{D} = \boldsymbol{0}$, but also $\boldsymbol{D}^e = \boldsymbol{D}^p = \boldsymbol{0}$. This implies $\lambda = 0$, hence, according to Eqs. (15) of [1] and (8.2) one has $\overset{\square}{\boldsymbol{a}} = \dot{\boldsymbol{a}} = 0$. But also $\boldsymbol{W}^e = \boldsymbol{W}^p = \boldsymbol{0}$ under rigid body motion, which in combination with [1, Eqs. (9.2) and (10.2)] yields $\boldsymbol{W}^* = \boldsymbol{\omega} = \boldsymbol{W}$. Hence, the $\dot{\boldsymbol{a}} = \boldsymbol{0}$ implies $\overset{\triangledown}{\boldsymbol{a}} = \boldsymbol{0}$ which guarantees the stationarity of the invariants of $\boldsymbol{a}$ (the $\dot{\boldsymbol{a}} = \boldsymbol{0}$ could do that also), and prompted Prager to suggest using the Jaumann rates as the appropriate ones in constitutive equations. We have seen here that this suggestion is not complete if one considers the physical meaning of elastic embedding; the $\overset{\square}{\boldsymbol{a}}$ must be used instead, which simply becomes $\overset{\triangledown}{\boldsymbol{a}}$ under rigid body motion.

Equations similar to Eqs. (8) apply to all other kinds of tensors, vectors and scalars [14]. In particular for scalars one can write

$$K = |\boldsymbol{F}^e|^w k; \quad \dot{K} = \langle\lambda\rangle \overline{K}(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_0, \varkappa_u \text{ or } \varkappa_i) \tag{9.1}$$

$$\overline{K} = |\boldsymbol{F}^e|^w \bar{k}; \quad \check{k} = \dot{k} + wk \operatorname{tr} \boldsymbol{D} = \langle\lambda\rangle (\bar{k} + wk \operatorname{tr} \boldsymbol{N}^p) \tag{9.2}$$

with $\overline{K}$ (or $\bar{k}$) isotropic function of its arguments, and $|\boldsymbol{F}^e| = |\boldsymbol{V}| = |\boldsymbol{F}_i^e|$.

One now can clearly understand the difference between the purely orientational structure variables and the rest. The former are such that their corotational rates in the intermediate configurations always vanish, with plastic loading or unloading. This implies that $\bar{\boldsymbol{S}} \equiv \boldsymbol{0}$ in Eqs. (7) or $\bar{\boldsymbol{A}} \equiv \boldsymbol{0}$ in Eqs. (8). The reader can easily understand this statement imagining that $\boldsymbol{S}$ or $\boldsymbol{A}$ represent for example a preferred vectorial direction $\boldsymbol{M}$ or its tensor product $\boldsymbol{A} = \boldsymbol{M} \otimes \boldsymbol{M}$ in one of the intermediate configurations (e.g. slip plane or orthotropic directions). The $\boldsymbol{M}$ is part of the substructure, it has no constitutive evolution being a purely geometrical (orientational) entity and, therefore, its substructure corotational rate must be zero for any value of λ. Hence, the orientation change of $\boldsymbol{M}$ determines the orientation change of the director vectors (it would not be so if $\boldsymbol{M}$ were a hardening non-purely orientational variable). The $\bar{\boldsymbol{A}} \equiv \boldsymbol{0}$ implies that $\bar{\boldsymbol{a}} = \boldsymbol{0}$ from Eq. (8.4); hence, observe that either $\overset{\square}{\boldsymbol{a}} = \boldsymbol{0}$, Eq. (8.2), or equivalently $\check{\boldsymbol{a}} = \langle\lambda\rangle \big(-(\boldsymbol{a}\boldsymbol{N}^p + \boldsymbol{N}^p\boldsymbol{a}) + (\boldsymbol{a}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{a}) + w\boldsymbol{a}\,\mathrm{tr}\,\boldsymbol{N}^p\big)$, Eq. (8.3). From the last expression it follows that with or without loading the current value of a purely orientational variable at $\varkappa$ changes not only due to rotation (as its transported value to an intermediate configuration does), but also due to its elastic embedding. The isotropic representation of all constitutive functions encountered so far (such as $\boldsymbol{N}_0{}^p, \boldsymbol{\Omega}_0{}^p, \bar{\boldsymbol{S}}$, etc.) is not effected by the fact that some of the $\boldsymbol{S}$ may be purely orientational; this will only effect the corresponding equations of evolution for these $\boldsymbol{S}$, as discussed above.

It is pertinent at this point to discuss another aspect. In all the previous development (and the one to follow), the isotropy of the different functions were derived on the basis of invariance requirements under superposed rigid body rotations in reference to the $\varkappa_0$ (or $\varkappa_u$) intermediate configuration, and the isotropy in reference to $\varkappa_i$ followed. However, if one wanted to work only in reference to the isoclinic configuration $\varkappa_i$ as done in [11], [15], equations such as (24.3) and (27.3) of [1] do not impose explicitly any requirement leading to isotropy. Recalling however, the discussion in a previous section on mathematical preliminaries for a function $\boldsymbol{f}(\boldsymbol{\Pi}, \boldsymbol{S})$, it follows again that such function must be isotropic in relation to its arguments based on the theorems presented in [3]. Also, the case of explicit dependence on β_u (or β) discussed in the same section applies to all constitutive functions presented so far, and those to be presented in the sequel.

5. Elastic Relations with Elastoplastic Coupling and/or Damage

This aspect of the constitutive formulation was developed in reference to the $\varkappa_0$ configuration in [10], [14]. Here, the development in reference to the $\varkappa_u$ configuration will be presented first, and then its equivalence to the $\varkappa_0$ and $\varkappa_i$ configuration will be shown.

Defining by $\boldsymbol{E}^e = (1/2)\,(\boldsymbol{F}^{e^T}\boldsymbol{F}^e - \boldsymbol{I})$ the Green elastic strain measure between $\varkappa_u$ and $\varkappa$, ϱ_0 the mass density at $\varkappa_u$ and $\psi = \bar{\psi}(\boldsymbol{\Pi}, \boldsymbol{S} \text{ at } \varkappa_u)$ the complementary free energy per unit mass, the elastic relation is obtained from $\boldsymbol{E}^e = \varrho_0(\partial\bar{\psi}/\partial\boldsymbol{\Pi})$. Invariance requirements under superposed rigid body rotations render $\bar{\psi}$ and, therefore, $\partial\bar{\psi}/\partial\boldsymbol{\Pi}$ isotropic functions of $\boldsymbol{\Pi}$ and $\boldsymbol{S}$. Notice also that the full invariance requirements, as expressed by [1, Eq. (18.1)] yield $\boldsymbol{E}^{e*} = \boldsymbol{Q}_u\boldsymbol{E}^e\boldsymbol{Q}_u{}^T$, as expected on physical grounds. Based on the definition of $\boldsymbol{E}^e$ in terms of $\boldsymbol{F}^e$, and of $\boldsymbol{D}^e = (\overset{\triangle}{\boldsymbol{F}}{}^e\boldsymbol{F}^{e^{-1}})_s$ from [1, Eq. (10.1)], one can show straightforwardly that $\overset{\triangle}{\boldsymbol{E}}{}^e = \boldsymbol{F}^{e^T}\boldsymbol{D}^e\boldsymbol{F}^e$. A subtle point of the corresponding analytical manipulation is the fact that $\boldsymbol{E}^e$ is isotropic function of $\boldsymbol{F}^e$, hence, one can apply the chain rule in taking the corotational rates according to Eq. (1). The use of the same corotational rate in a chain rule operation applied to $\boldsymbol{E}^e = \varrho_0(\partial\bar{\psi}/\partial\boldsymbol{\Pi})$ in conjunction with Eq. (7.2) and the foregoing relation between $\overset{\triangle}{\boldsymbol{E}}{}^e$ and $\boldsymbol{D}^e$, yields

$$\overset{\triangle}{\boldsymbol{E}}{}^e = \overset{\triangle}{\boldsymbol{E}}{}^r + \overset{\triangle}{\boldsymbol{E}}{}^c = \mathscr{L}^{0^{-1}} : \overset{\triangle}{\boldsymbol{\Pi}} + \langle\lambda\rangle\, \boldsymbol{N}_0^c \tag{10.1}$$

$$\boldsymbol{D}^e = \boldsymbol{D}^r + \boldsymbol{D}^c = \mathscr{L}^{-1} : \overset{\square}{\boldsymbol{\sigma}} + \langle\lambda\rangle\, \boldsymbol{N}^c \tag{10.2}$$

$$\mathscr{L}^0 = (\varrho_0\partial^2\bar{\psi}/\partial\boldsymbol{\Pi} \otimes \partial\boldsymbol{\Pi})^{-1}; \qquad \mathscr{L}_{ijkl} = |\boldsymbol{F}^e|^{-1}\, F^e_{i\alpha}F^e_{j\beta}F^e_{k\gamma}F^e_{l\delta}\, \mathscr{L}^0_{\alpha\beta\gamma\delta} \tag{10.3; 4}$$

$$\boldsymbol{N}^c = \boldsymbol{F}^{e^{-T}}\boldsymbol{N}_0^c\boldsymbol{F}^{e^{-1}} = \varrho_0\boldsymbol{F}^{e^{-T}}[(\partial^2\bar{\psi}/\partial\boldsymbol{\Pi} \otimes \partial\boldsymbol{S}) \cdot \bar{\boldsymbol{S}}]\,\boldsymbol{F}^{e^{-1}} - \boldsymbol{A}^e \operatorname{tr} \boldsymbol{N}^p \tag{10.5}$$

where $\boldsymbol{A}^e = \boldsymbol{F}^{e^{-T}}\boldsymbol{E}^e\boldsymbol{F}^{e^{-1}}$ is the Almansi elastic strain tensor between $\varkappa$ und $\varkappa_u$, the $\overset{\square}{\boldsymbol{\sigma}}$ is related to $\overset{\triangle}{\boldsymbol{\Pi}}$ by $\overset{\triangle}{\boldsymbol{\Pi}} = |\boldsymbol{F}^e|\, \boldsymbol{F}^{e^{-1}}\overset{\square}{\boldsymbol{\sigma}}\boldsymbol{F}^{e^{-T}}$ and is defined from [1, Eq. (15)] with $w = 1$ and $\boldsymbol{\sigma}$ substituting for $\boldsymbol{a}$, the $\mathscr{L}^0$ are the incremental elastic moduli at $\varkappa_u$ and $\mathscr{L}$ their transport at $\varkappa$ (like a relative forth order tensor), according to Eq. (10.4). The $\boldsymbol{D}^r$ and $\boldsymbol{D}^c$ represent the incrementally reversible and elastoplastic coupling or damage induced components of $\boldsymbol{D}^e$ at $\varkappa$, counterparts of $\overset{\triangle}{\boldsymbol{E}}{}^r$ and $\overset{\triangle}{\boldsymbol{E}}{}^c$ at $\varkappa_u$.

While in all the above the $\varkappa_u$ was used as the intermediate unstressed configuration, a similar development can be presented in reference to $\varkappa_0$, as done in [10], or in reference to $\varkappa_i$. It suffices to substitute in the foregoing, the $\boldsymbol{V}$, $\mathring{\boldsymbol{V}}$, $\mathring{\boldsymbol{E}}{}^e$, $\mathring{\boldsymbol{\Pi}}$ or $\boldsymbol{F}_i^e$, $\dot{\boldsymbol{F}}_i^e$, $\dot{\boldsymbol{E}}{}^e$, $\dot{\boldsymbol{\Pi}}$, for $\boldsymbol{F}^e$, $\overset{\triangle}{\boldsymbol{F}}{}^e$, $\overset{\triangle}{\boldsymbol{E}}{}^e$, $\overset{\triangle}{\boldsymbol{\Pi}}$, respectively, and of course refer the $\boldsymbol{\Pi}$ and $\boldsymbol{S}$ to $\varkappa_0$ or $\varkappa_i$ instead of $\varkappa_u$. The isotropy of $\bar{\psi}$ and $\partial\bar{\psi}/\partial\boldsymbol{\Pi}$ follows again either because of invariance requirement under superposed rigid body rotations in reference to $\varkappa_0$, or because of the theorems presented in [3] in reference to $\varkappa_i$, as explained in the mathematical preliminaries section. The values of $\overset{\square}{\boldsymbol{\sigma}}$, $\boldsymbol{N}^c$ and $\mathscr{L}$ at the current configuration $\varkappa$ will always be the same, whichever is our choice of the intermediate configuration; the values of $\mathscr{L}^0$ differ only by rotation.

6. Yield Criterion and Final Form of the Stress-Strain Rate Equations

When a yield criterion is to be considered, one is faced with the dilemma of presenting its analytical expression either in the current configuration $\varkappa$ in terms of $\boldsymbol{\sigma}$ and $\boldsymbol{s}$, (purely Eulerian description), or in any one of the unstressed configurations $\varkappa_0$, $\varkappa_u$ or $\varkappa_i$, in terms of the corresponding transported values $\boldsymbol{\Pi}$ and $\boldsymbol{S}$ (semi-Eulerian description). Although both formulations will be equivalent, it is expedient to present the formulation in terms of the current "true" values $\boldsymbol{\sigma}$, $\boldsymbol{s}$. This is because one has an immediate association with the experimental data used to specify the yield criterion, without the necessity to consider an "elastic unloading" process before such yield criterion is specified. Hence, we can write

$$f(\sigma, \mathbf{s}) = 0; \qquad \boldsymbol{N}^n = \left(\frac{\partial f}{\partial \sigma}\right)_s \tag{11}$$

for the yield criterion $f = 0$ and its symmetric stress gradient $\boldsymbol{N}^n$. The f and $\boldsymbol{N}^n$ are isotropic function of their arguments due to the invariance requirement under superposed rigid body rotations. What is interesting about Eq. (11), is that the $f = 0$ does not define in the stress space σ a fixed "yield surface" for given values of $\mathbf{s}$ in the usual sense, because the $\mathbf{s}$ themselves change when σ does, due to the elastic embedding of $\mathbf{s}$. Of course, if one expresses the $\sigma, \mathbf{s}$ in terms of $\boldsymbol{\Pi}$, $\boldsymbol{S}$ and $\boldsymbol{F}^e$ (or $\boldsymbol{V}$, $\boldsymbol{F}_i^e$), with the elastic deformation gradient being also a function of $\boldsymbol{\Pi}$, $\boldsymbol{S}$, the $f = 0$ will become a different function $\hat{f}(\boldsymbol{\Pi}, \boldsymbol{S}) = 0$ in the corresponding unstressed configuration. In this case the $\hat{f} = 0$ defines a fixed yield surface in the stress space $\boldsymbol{\Pi}$ for given values of $\boldsymbol{S}$. Nevertheless, Eq. (11) defines a yield criterion (rather than a "yield surface"), in the sense that when the value of σ and the ensuing values of $\mathbf{s}$ satisfy $f = 0$, yield is imminent. The result of the foregoing arguments is that the loading direction is diverted from $\boldsymbol{N}^n$, in a way which will be specified subsequently.

Since $f = 0$ is isotropic, Eq. (1) applies for any choice of corotational rates, which here is chosen to be the ones in reference to $\boldsymbol{W}^*$. Hence, the consistency condition $\dot{f} = 0$ yields

$$\dot{f} = \boldsymbol{N}^n : \dot{\sigma} + (\partial f / \partial \mathbf{s}) \cdot \dot{\mathbf{s}} = 0 . \tag{12}$$

The next step is to express $\dot{\mathbf{s}}$ in terms of the corodeformational rate $\overset{\square}{\mathbf{s}}$ (recall Eq. (8.2) for a second order tensor), on order to specify the loading index λ which appears in $\overset{\square}{\mathbf{s}} = \langle \lambda \rangle \, \bar{\mathbf{s}}$. To this extent, an equation similar to [1, Eq. (15)] can relate the $\dot{\mathbf{s}}$ to $\overset{\square}{\mathbf{s}}$ (it is presented only for a second order tensor $\boldsymbol{a}$ in Eq. (15)) depending on the elastic embedding and tensorial nature of $\mathbf{s}$; the $\boldsymbol{D}^e$ which appears in Eq. (15) of [1] can be expressed in terms of $\overset{\square}{\sigma}$ and λ from Eq. (10.2). In addition the $\dot{\sigma}$ and $\overset{\square}{\sigma}$ can be interrelated in a similar manner. While the definition of $\overset{\square}{\sigma}$ follows [1, Eq. (15)] with $w = 1$, one can use a number of other stress rates (including the $\mathring{\sigma}$) by introducing appropriately modified elastic moduli $\mathscr{L}'$ instead of $\mathscr{L}$ (the latter defined by Eq. (10.3; 4)). To this extent, denoting by $\overset{\square}{\sigma}$ any one of these stress rates and with $\mathscr{L}'$ the associated moduli (to be precisely defined in the sequel), we have

$$\boldsymbol{D}^e = \mathscr{L}'^{-1} : \overset{\square}{\sigma} + \langle \lambda \rangle \, \mathscr{L}'^{-1} : \mathscr{L} : \boldsymbol{N}^c \tag{13}$$

instead of Eq. (10.2). Of course, for the choice of $\overset{\square}{\sigma}$ as given by [1, Eq. (15)] with $\boldsymbol{a} = \sigma$ and $w = 1$, one has $\mathscr{L}' = \mathscr{L}$ and Eq. (13) becomes Eq. (10.2). Using the foregoing for changing the $\dot{\mathbf{s}}$ and $\dot{\sigma}$ to $\overset{\square}{\mathbf{s}}$ and $\overset{\square}{\sigma}$, respectively (notice that Eq. (13)

rather than Eq. (10.2) is used now in order to express the $\boldsymbol{D}^e$), Eq. (12) can be rewritten as

$$(\boldsymbol{N}^n - \boldsymbol{Z}^n : \mathcal{L}'^{-1}) : \overset{\square}{\boldsymbol{\sigma}} + \langle\lambda\rangle \left((\partial f/\partial \mathbf{s}) \cdot \bar{\mathbf{s}} - \boldsymbol{Z}^n : \mathcal{L}'^{-1} : \mathcal{L} : \boldsymbol{N}^c\right) = 0 \tag{14}$$

where $\boldsymbol{Z}^n$ is defined in the following equations. Eq. (14) is very important because it specifies the loading index λ in terms of the stress rate $\overset{\square}{\boldsymbol{\sigma}}$ and associated quantities, such as $\mathcal{L}'$ and $\boldsymbol{Z}^n$ which depend on the choice of $\overset{\square}{\boldsymbol{\sigma}}$ (see later how); the $\mathcal{L}$ is always given by Eq. (10.3; 4).

We have reached now the point where combining [1, Eqs. (9.1)] and (6.1), (10.2), and (14), the final form can be expressed by

$$\begin{aligned} \boldsymbol{D} = \boldsymbol{D}^r + \boldsymbol{D}^c + \boldsymbol{D}^p &= \mathcal{L}'^{-1} : \overset{\square}{\boldsymbol{\sigma}} + \langle\lambda\rangle (\boldsymbol{N}^p + \mathcal{L}'^{-1} : \mathcal{L} : \boldsymbol{N}^c) \\ &= \mathcal{L}'^{-1} : \check{\boldsymbol{\sigma}} + \langle\lambda\rangle \boldsymbol{N}' = \Lambda^{-1} : \check{\boldsymbol{\sigma}} \end{aligned} \tag{15}$$

$$\begin{aligned} \lambda &= \frac{\boldsymbol{N} : \overset{\square}{\boldsymbol{\sigma}}}{H + \boldsymbol{Z}^n : \mathcal{L}'^{-1} : \mathcal{L} : \boldsymbol{N}^c} = \frac{\boldsymbol{N} : \check{\boldsymbol{\sigma}}}{H + \boldsymbol{N} : \boldsymbol{Z}^p + \boldsymbol{Z}^n : \mathcal{L}'^{-1} : \mathcal{L} : \boldsymbol{N}^c} \\ &= \frac{\boldsymbol{N} : \mathcal{L} : \boldsymbol{D}}{H + \boldsymbol{N} : \mathcal{L}' : \boldsymbol{N}^p + \boldsymbol{N}^n : \mathcal{L} : \boldsymbol{N}^c} \end{aligned} \tag{16}$$

$$\Lambda = \mathcal{L}' - \bar{h}(\lambda) \frac{(\mathcal{L}' : \boldsymbol{N}') \otimes (\boldsymbol{N} : \mathcal{L}')}{H + \boldsymbol{N} : \mathcal{L}' : \boldsymbol{N}^p + \boldsymbol{N}^n : \mathcal{L} : \boldsymbol{N}^c} \tag{17}$$

$$\boldsymbol{N}' = \boldsymbol{N}^p + \mathcal{L}'^{-1} : \mathcal{L} : \boldsymbol{N}^c - \mathcal{L}'^{-1} : \boldsymbol{Z}^p \tag{18}$$

$$\boldsymbol{N} = \boldsymbol{N}^n - \boldsymbol{Z}^n : \mathcal{L}'^{-1} \tag{19}$$

$$H = -(\partial f/\partial \mathbf{s}) \cdot \bar{\mathbf{s}}\,. \tag{20}$$

Giving the definitions

$$\mathcal{D}^c_{ijkl} = \frac{1}{2} (\delta_{ik}\sigma_{lj} + \delta_{il}\sigma_{kj} + \delta_{jl}\sigma_{ik} + \delta_{jk}\sigma_{il}) \tag{21.1}$$

$$\mathcal{D}^m_{ijkl} = \frac{1}{2} (-\delta_{ik}\sigma_{lj} - \delta_{il}\sigma_{kj} + \delta_{jl}\sigma_{ik} + \delta_{jk}\sigma_{il}) \tag{21.2}$$

$$\begin{aligned} \boldsymbol{\Gamma} = &\pm \varepsilon \left(\boldsymbol{a}^T \frac{\partial f}{\partial \boldsymbol{a}} + \frac{\partial f}{\partial \boldsymbol{a}} \boldsymbol{a}^T\right)_s \pm (1 - \varepsilon) \left(\boldsymbol{a}^T \frac{\partial f}{\partial \boldsymbol{a}} - \frac{\partial f}{\partial \boldsymbol{a}} \boldsymbol{a}^T\right)_s \\ &\pm \left(\frac{\partial f}{\partial \boldsymbol{m}} \otimes \boldsymbol{m}\right)_s + w \left(\frac{\partial f}{\partial \boldsymbol{s}} \cdot \boldsymbol{s}\right) \boldsymbol{I} \end{aligned} \tag{22}$$

with $+$ when the $\boldsymbol{s}$ is transported according to [1, Eqs. (13.2), (13.3.2)]

$-$ when the $\boldsymbol{s}$ is transported according to [1, Eqs. (13.1), (13.3.1)]

$\varepsilon = 1$ when the $\boldsymbol{a}$ is transported according to [1, Eqs. (13.1.1), (13.2.1)]

$\varepsilon = 0$ when the $\boldsymbol{a}$ is transported according to [1, Eqs. (13.3)] the following sets of conjugate quantities enter Eqs. (15)—(20) according to the stress rate used (more sets could be defined).

Set 1 (contravariant; corresponds to $\boldsymbol{\Pi} = |\boldsymbol{F}^e|\,\boldsymbol{F}^{e^{-1}}\boldsymbol{\sigma}\boldsymbol{F}^{e^{-T}}$)

$$\overset{\square}{\boldsymbol{\sigma}} = \overset{*}{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{D}^e - \boldsymbol{D}^e\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D}^e; \quad \check{\boldsymbol{\sigma}} = \overset{\nabla}{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D} \tag{23.1}$$

$$\mathscr{L}' = \mathscr{L} \tag{23.2}$$

$$\boldsymbol{Z}^p = -(\boldsymbol{\sigma}\boldsymbol{N}^p + \boldsymbol{N}^p\boldsymbol{\sigma}) + (\boldsymbol{\sigma}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{\sigma}) + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{N}^p \tag{23.3}$$

$$\boldsymbol{Z}^n = -(\boldsymbol{\sigma}\boldsymbol{N}^n + \boldsymbol{N}^n\boldsymbol{\sigma}) + (\boldsymbol{N}^n : \boldsymbol{\sigma})\,\boldsymbol{I} + \boldsymbol{\Gamma}. \tag{23.4}$$

Set 2 (covariant; corresponds to $\boldsymbol{\Pi} = |\boldsymbol{F}^e|\,\boldsymbol{F}^{e^T}\boldsymbol{\sigma}\boldsymbol{F}^{e}$)

$$\overset{\square}{\boldsymbol{\sigma}} = \overset{*}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{D}^e + \boldsymbol{D}^e\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D}^e; \quad \check{\boldsymbol{\sigma}} = \overset{\nabla}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D} \tag{24.1}$$

$$\mathscr{L}' = \mathscr{L} + 2\mathscr{D}^c \tag{24.2}$$

$$\boldsymbol{Z}^p = (\boldsymbol{\sigma}\boldsymbol{N}^p + \boldsymbol{N}^p\boldsymbol{\sigma}) + (\boldsymbol{\sigma}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{\sigma}) + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{N}^p \tag{24.3}$$

$$\boldsymbol{Z}^n = (\boldsymbol{\sigma}\boldsymbol{N}^n + \boldsymbol{N}^n\boldsymbol{\sigma}) + (\boldsymbol{N}^n : \boldsymbol{\sigma})\,\boldsymbol{I} + \boldsymbol{\Gamma}. \tag{24.4}$$

Set 3 (mixed 1; corresponds to $\boldsymbol{\Pi} = |\boldsymbol{F}^e|\,\boldsymbol{F}^{e^T}\boldsymbol{\sigma}\boldsymbol{F}^{e^{-T}}$)

$$\overset{\square}{\boldsymbol{\sigma}} = \overset{*}{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{D}^e + \boldsymbol{D}^e\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D}^e; \quad \check{\boldsymbol{\sigma}} = \overset{\nabla}{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D} \tag{25.1}$$

$$\mathscr{L}' = \mathscr{L} + \mathscr{D}^c - \mathscr{D}^m \tag{25.2}$$

$$\boldsymbol{Z}^p = -(\boldsymbol{\sigma}\boldsymbol{N}^p - \boldsymbol{N}^p\boldsymbol{\sigma}) + (\boldsymbol{\sigma}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{\sigma}) + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{N}^p \tag{25.3}$$

$$\boldsymbol{Z}^n = (\boldsymbol{N}^n : \boldsymbol{\sigma})\,\boldsymbol{I} + \boldsymbol{\Gamma}. \tag{25.4}$$

Set 4 (mixed 2; corresponds to $\boldsymbol{\Pi} = |\boldsymbol{F}^e|\,\boldsymbol{F}^{e^{-1}}\boldsymbol{\sigma}\boldsymbol{F}^{e}$)

$$\overset{\square}{\boldsymbol{\sigma}} = \overset{*}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{D}^e - \boldsymbol{D}^e\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D}^e; \quad \check{\boldsymbol{\sigma}} = \overset{\nabla}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D} \tag{26.1}$$

$$\mathscr{L}' = \mathscr{L} + \mathscr{D}^c + \mathscr{D}^m \tag{26.2}$$

$$\boldsymbol{Z}^p = (\boldsymbol{\sigma}\boldsymbol{N}^p - \boldsymbol{N}^p\boldsymbol{\sigma}) + (\boldsymbol{\sigma}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{\sigma}) + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{N}^p \tag{26.3}$$

$$\boldsymbol{Z}^n = (\boldsymbol{N}^n : \boldsymbol{\sigma})\,\boldsymbol{I} + \boldsymbol{\Gamma}. \tag{26.4}$$

Set 5 (corotational rates of Cauchy stress $\boldsymbol{\sigma}$)

$$\overset{\square}{\boldsymbol{\sigma}} \to \overset{*}{\boldsymbol{\sigma}}; \quad \check{\boldsymbol{\sigma}} \to \overset{\nabla}{\boldsymbol{\sigma}}; \quad \mathscr{L}' = \mathscr{L} + \mathscr{D}^c - \boldsymbol{\sigma} \otimes \boldsymbol{I} \tag{27.1}$$

$$\boldsymbol{Z}^p = \boldsymbol{\sigma}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{\sigma}; \quad \boldsymbol{Z}^n = \boldsymbol{\Gamma}. \tag{27.2}$$

Set 6 (corotational rates of Kirchhoff stress $\boldsymbol{\tau} = \boldsymbol{\sigma}$ but $\dot{\boldsymbol{\tau}} \neq \dot{\boldsymbol{\sigma}}$)

$$\overset{\square}{\boldsymbol{\sigma}} \to \overset{*}{\boldsymbol{\tau}} = \overset{*}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D}^e; \quad \check{\boldsymbol{\sigma}} \to \overset{\nabla}{\boldsymbol{\tau}} = \overset{\nabla}{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D}; \quad \mathscr{L}' = \mathscr{L} + \mathscr{D}^c \tag{28.1; 2; 3}$$

$$\boldsymbol{Z}^p = \boldsymbol{\sigma}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{\sigma} + \boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{N}^p; \quad \boldsymbol{Z}^n = (\boldsymbol{N}^n : \boldsymbol{\sigma})\,\boldsymbol{I} + \boldsymbol{\Gamma}. \tag{28.4}$$

The foregoing relations generalize the ones presented in [10], where only the sets 1 and 6 were used (for set 1 observe that $\mathscr{L}' = \mathscr{L}$, hence $\mathscr{L}'^{-1} : \mathscr{L} : \boldsymbol{N}^c = \boldsymbol{N}^c$ in all previous equations), and the term of $\boldsymbol{\Gamma}$ which is premultiplied by $1 - \varepsilon$, Eq. (22), was not included. The $\boldsymbol{Z}^p$ is introduced as a result of changing the stress rate from $\overset{\square}{\sigma}$ to $\check{\sigma}$. The $\boldsymbol{Z}^n$ consists of two groups of terms; the group defining $\boldsymbol{\Gamma}$ which reflects the effect of the elastic embedding of $\mathbf{s}$ and is the same for any set, and the group of the remaining terms which reflect the choice of the stress rate and are different for each set. The $\boldsymbol{Z}^n$ is responsible for diverting the loading direction from $\boldsymbol{N}^n$ to $\boldsymbol{N}$, Eq. (19); this is the analytical statement of the discussion in the paragraph preceding Eq. (12). In reference to the expression for $\boldsymbol{\Gamma}$, Eq. (22), it is implied that each term is repeated so many times as many different $\boldsymbol{a}, \boldsymbol{m}$ or $\mathbf{s}$ enter $f = 0$; the w is the weight of their elastic embedding. The elastoplastic moduli $\boldsymbol{\Lambda}$, Eq. (17), possess the summetries $ij \leftrightarrow kl$ (the so-called "normality structure") if the corresponding $\mathscr{L}'$ do, and $\boldsymbol{N}'$ is proportional to $\boldsymbol{N}$. The former occurs for the $\mathscr{L}'$ of the sets 1, 2 and 6, based on the definition of $\mathscr{L}$, Eq. (10.3; 4), and of $\mathscr{D}^c$, $\mathscr{D}^m$, Eqs. (21). The latter occurs for the set 1 (for which recall that $\mathscr{L}' = \mathscr{L}$) if $\boldsymbol{N}^p + \boldsymbol{N}^c$ and $\boldsymbol{Z}^p$ are similarly proportional to $\boldsymbol{N}^n$ and $\boldsymbol{Z}^n$, respectively, or for any set with $\boldsymbol{N}^c = 0$ if $\boldsymbol{N}^p$ and $\boldsymbol{Z}^p$ are similarly proportional to $\boldsymbol{N}^n$ and $\boldsymbol{Z}^n$. The novel aspects here are the generalization of the final form of the rate equations in comparison to the one presented in [10], and the rigorous proof of obtaining the same result no matter what is the choice of the intermediate unstressed configuration.

7. Particular Topics

7.1 The Single Slip

One of the simplest and best understood mechanisms of large elastoplastic deformations is that of the single slip of a monocrystal. It will be used as an example on the effect of elastic embedding and other aspects of the theory developed so far. If $\boldsymbol{m}$ and $\boldsymbol{n}$ are two orthogonal (but not necessarily unit) vectors at $\varkappa$ along the slip and normal to the slip plane directions, respectively, let the orthonormal vectors $\boldsymbol{M} = \boldsymbol{V}^{-1}\boldsymbol{m}$ and $\boldsymbol{N} = |\boldsymbol{V}|^{-1}\,\boldsymbol{n}\boldsymbol{V}$ be their transported counterparts at $\varkappa_0$, according to the elastic embedding first proposed by Rice [16]. Observe that while $\boldsymbol{M}$ transforms elastically exactly as a material line segment does at $\varkappa_0$, $\boldsymbol{N}$ does not. This is related to the statement made before Eq. (1) of part 1 [1], that the elastic transformations of the continuum and its substructure are determined by the elastic part of $\boldsymbol{F}$ (here the $\boldsymbol{V}$), but are not necessarily identical. The structure variables can be defined at $\varkappa_0$ as the asymmetric orientation tensor $\boldsymbol{A} = \boldsymbol{M} \otimes \boldsymbol{N}$, and a scalar measure τ_0 of the resolved shear stress yield threshold. With $\dot{\gamma}_0$ denoting the slip rate at $\varkappa_0$, the hardening law $\dot{\tau}_0 = h_0\dot{\gamma}_0$ is assumed, h_0 being the slip hardening modulus. Then, the following relations can be established on the basis of the general formulation, at an order which is indicative of the way calculations were made.

$$\boldsymbol{A} = \boldsymbol{M} \otimes \boldsymbol{N}; \qquad \boldsymbol{a} = \boldsymbol{m} \otimes \boldsymbol{n}; \qquad \boldsymbol{A} = |\boldsymbol{V}|^{-1}\, \boldsymbol{V}^{-1}\boldsymbol{a}\boldsymbol{V} \tag{29.1; 2; 3}$$

$$f = \boldsymbol{a}^T : \boldsymbol{\sigma} - \tau_0 = \boldsymbol{A}^T : \boldsymbol{\Pi} - \tau_0 = 0; \qquad \boldsymbol{\Pi} = |\boldsymbol{V}|\, \boldsymbol{V}\boldsymbol{\sigma}\boldsymbol{V}^{-1} \tag{30}$$

$$\boldsymbol{N}^n = \boldsymbol{N}^p = \boldsymbol{a}_s; \qquad \boldsymbol{\Omega}^p = \boldsymbol{a}_a; \qquad \mathrm{tr}\, \boldsymbol{N}^p = 0 \tag{31}$$

$$\boldsymbol{N}_0{}^p + \boldsymbol{\Omega}_0{}^p = \boldsymbol{V}^{-1}(\boldsymbol{N}^p + \boldsymbol{\Omega}^p)\, \boldsymbol{V} = |\boldsymbol{V}|\, \boldsymbol{A}; \qquad \mathrm{tr}\, \boldsymbol{N}_0{}^p = 0 \tag{32}$$

$$\boldsymbol{N}^p : \boldsymbol{\sigma} = \tau_0; \qquad (\partial f/\partial \boldsymbol{a}) \cdot \boldsymbol{a} = \tau_0; \qquad (\partial f/\partial \tau_0)\, \tau_0 = -\tau_0 \tag{33}$$

$$\boldsymbol{\Gamma} = -\left(\boldsymbol{a}^T \frac{\partial f}{\partial \boldsymbol{a}} - \frac{\partial f}{\partial \boldsymbol{a}}\, \boldsymbol{a}^T\right)_s - \left(\frac{\partial f}{\partial \boldsymbol{a}} \cdot \boldsymbol{a}\right) \boldsymbol{I} = \boldsymbol{\sigma}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{\sigma} - \tau_0 \boldsymbol{I} \tag{34}$$

$$\boldsymbol{Z}^n = (\boldsymbol{N}^n : \boldsymbol{\sigma})\, \boldsymbol{I} + \boldsymbol{\Gamma} = \boldsymbol{\sigma}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{\sigma} = \boldsymbol{Z}^p \tag{35}$$

$$\mathring{\boldsymbol{A}} = |\boldsymbol{V}|^{-1}\, \boldsymbol{V}^{-1}\overset{\square}{\boldsymbol{a}}\boldsymbol{V} = 0 \Rightarrow \overset{\square}{\boldsymbol{a}} = \dot{\boldsymbol{a}} + \boldsymbol{a}\boldsymbol{D}^e + \boldsymbol{D}^e\boldsymbol{a} - \boldsymbol{a}\, \mathrm{tr}\, \boldsymbol{D}^e = 0 \tag{36.1}$$

$$\overset{\nabla}{\boldsymbol{a}} = \dot{\boldsymbol{a}} - \boldsymbol{W}^p\boldsymbol{a} + \boldsymbol{a}\boldsymbol{W}^p = -\boldsymbol{a}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{a} + \boldsymbol{a}\, \mathrm{tr}\, \boldsymbol{D} \tag{36.2}$$

$$\hat{\boldsymbol{P}}\boldsymbol{P}^{-1} = \langle\lambda\rangle\, (\boldsymbol{N}_0{}^p + \boldsymbol{\Omega}_0{}^p) = \langle\lambda\rangle\, |\boldsymbol{V}|\, \boldsymbol{A} = \dot{\gamma}_0\boldsymbol{A} \Rightarrow \dot{\gamma}_0 = |\boldsymbol{V}|\, \langle\lambda\rangle \tag{37}$$

$$\dot{\tau}_0 = \langle\lambda\rangle\, \bar{\tau}_0 = h_0\dot{\gamma}_0 \Rightarrow H = -(\partial f/\partial \tau_0)\, \bar{\tau}_0 = \bar{\tau}_0 = |\boldsymbol{V}|\, h_0 \tag{38}$$

$$\begin{aligned} H\lambda = \dot{\tau}_0 = \boldsymbol{A}^T : \mathring{\boldsymbol{\Pi}} &= \boldsymbol{A}^T : |\boldsymbol{V}|\, \boldsymbol{V}\overset{\square}{\boldsymbol{\sigma}}\boldsymbol{V}^{-1} = \boldsymbol{a}^T : (\dot{\boldsymbol{\sigma}} - \boldsymbol{\sigma}\boldsymbol{D}^e + \boldsymbol{D}^e\boldsymbol{\sigma} + \boldsymbol{\sigma}\, \mathrm{tr}\, \boldsymbol{D}^e) \\ &= \boldsymbol{a} : (\dot{\boldsymbol{\tau}} + \boldsymbol{\sigma}\boldsymbol{D}^e - \boldsymbol{D}^e\boldsymbol{\sigma}) = \boldsymbol{a} : (\overset{\nabla}{\boldsymbol{\tau}} + \boldsymbol{\sigma}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{\sigma}) \end{aligned} \tag{39}$$

where the set 6, Eqs. (28), was chosen for the stress rates and the expressions of $\boldsymbol{Z}^p$ and $\boldsymbol{Z}^n$. In deriving some of the foregoing results, intermediate steps were often not shown for simplicity; for example, in going from $\dot{\boldsymbol{\tau}}$ and $\boldsymbol{D}^e$ to $\overset{\nabla}{\boldsymbol{\tau}}$ and $\boldsymbol{D}$ at the end of Eq. (39), use of $\boldsymbol{D}^e = \boldsymbol{D} - \boldsymbol{D}^p$, $\boldsymbol{W}^* = \boldsymbol{W} - \boldsymbol{W}^p$ and Eq. (31) was made. Perhaps the most interesting calculation is that of $\boldsymbol{\Gamma}$, Eq. (34); the role of the mixed embedding of $\boldsymbol{a}$ with weight $w = -1$ according to Eq. (29.3), played an instrumental role when Eq. (22) for the derivation of $\boldsymbol{\Gamma}$ was applied. The resulting value of $\boldsymbol{\Gamma}$ rendered $\boldsymbol{Z}^n = \boldsymbol{Z}^p$, Eq. (35); hence, with $\boldsymbol{N}^n = \boldsymbol{N}^p$ and the choice of $\mathscr{L}'$ as given in Eq. (28.3), the "normality structure" of $\boldsymbol{A}$ is achieved (recall discussion at the end of previous section). Eqs. (36) yield the evolution of the purely orientational structure variable $\boldsymbol{a}$ in different but equivalent ways. In Eq. (39) the loading index λ is expressed in terms of $\dot{\boldsymbol{\tau}}$ or $\overset{\nabla}{\boldsymbol{\tau}}$ in combination with $\boldsymbol{D}^e$ or $\boldsymbol{D}$, respectively, in order to show the similarity of the present formulation with that in [17] were the single slip was analyzed. In fact the results of [17] can be obtained exactly if a few more steps are taken, as shown in [18] where the $\boldsymbol{m}$ and $\boldsymbol{n}$, instead of $\boldsymbol{a}$, were equivalently used as structure variables. But of course the λ could be expressed in terms of $\dot{\boldsymbol{\tau}}$ or $\overset{\nabla}{\boldsymbol{\tau}}$ only, which would involve a diversion of the loading direction from $\boldsymbol{N}^n = \boldsymbol{a}_s$ to $\boldsymbol{N} = \boldsymbol{N}^n - \boldsymbol{Z}^n : \mathscr{L}'^{-1}$, according to the general theory, Eqs. (16) and (19). Finally notice the important fact that it is the transport of $\boldsymbol{N}_0{}^p + \boldsymbol{\Omega}_0{}^p$ which defines $\boldsymbol{N}^p + \boldsymbol{\Omega}^p$, Eq. (32), and not the separate transport of $\boldsymbol{N}_0{}^p$ and $\boldsymbol{\Omega}_0{}^p$ which define $\boldsymbol{N}^p$ and $\boldsymbol{\Omega}^p$, respectively. This point has been discussed in the general development presented in the first part of this work [1].

7.2 Small Elastic Deformations

Much of the complexity of the foregoing equations is due to the large elastic deformations. For many materials the assumption of small elastic strains is a realistic one, and considerably simplifies the formulation. Analytically it implies $\boldsymbol{V} \simeq \boldsymbol{I}$ and the omission of terms of the order stress/elastic moduli compared to unity. As a result the basic kinematical decomposition, Eqs. (9) of [1], becomes

$$\boldsymbol{D} = \boldsymbol{D}^e + \boldsymbol{D}^p; \qquad \boldsymbol{W} = \boldsymbol{\omega} + \boldsymbol{W}^p. \qquad (40.1;2)$$

The elastic embedding looses its meaning, and in referenced to $\varkappa_0$ one can set $\boldsymbol{s} \simeq \boldsymbol{S}$ and $\boldsymbol{\sigma} \simeq \boldsymbol{\Pi}$, while in reference to $\varkappa_u$ and $\varkappa_i$ the $\boldsymbol{s}$ and $\boldsymbol{\sigma}$ differ only by rotation from $\boldsymbol{S}$ and $\boldsymbol{\Pi}$, respectively, because $\boldsymbol{F}^e \simeq \boldsymbol{R}_u{}^e$ and $\boldsymbol{F}_i{}^e \simeq \boldsymbol{R}_i{}^e$. Since $\boldsymbol{D}_0{}^p \simeq \boldsymbol{D}^p$ and $\boldsymbol{W}_0{}^p \simeq \boldsymbol{W}^p$ at $\varkappa_0$, the kinematical constitutive Eqs. (2) and (6) are identical, while the $\boldsymbol{D}_u{}^p$ and $\boldsymbol{W}_u{}^p$, or $\boldsymbol{D}_i{}^p$ and $\boldsymbol{W}_i{}^p$, Eqs. (3) and (4), differ from $\boldsymbol{D}^p$ and $\boldsymbol{W}^p$ only by the rotation $\boldsymbol{R}_u{}^e$ or $\boldsymbol{R}_i{}^e$, respectively. Eqs. (7) for the structure variables can be rewritten as:

$$\dot{\boldsymbol{s}} = \langle \lambda \rangle \, \bar{\boldsymbol{s}}(\sigma, \boldsymbol{s} \text{ at } \varkappa_0) = \boldsymbol{R}_u{}^e[\overset{\triangle}{\boldsymbol{s}}] = \boldsymbol{R}_i{}^e[\dot{\boldsymbol{s}}] \qquad (41)$$

with $\bar{\boldsymbol{s}}$ an isotropic function of its arguments. In fact, restricting attention to the $\varkappa_0$ configuration and using Eq. (40.2) to express $\boldsymbol{\omega} = \boldsymbol{W} - \boldsymbol{W}^p$, the first two members of Eq. (41) can be written for a tensor $\boldsymbol{a}$ and a vector $\boldsymbol{m}$ in terms of Jaumann corotational rates as

$$\overset{\triangledown}{\boldsymbol{a}} = \langle \lambda \rangle \, (\bar{\boldsymbol{a}} + \boldsymbol{a}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{a}); \qquad \overset{\triangledown}{\boldsymbol{m}} = \langle \lambda \rangle \, (\bar{\boldsymbol{m}} - \boldsymbol{\Omega}^p\boldsymbol{m}). \qquad (42.1;2)$$

Finally, the stress-strain rate equations can be written as

$$\overset{\triangledown}{\boldsymbol{\sigma}} = \left[\mathscr{L} - \bar{h}(\lambda) \, \frac{[\mathscr{L} : (\boldsymbol{N}^p + \boldsymbol{N}^c)] \otimes [\boldsymbol{N}^n : \mathscr{L}]}{H + \boldsymbol{N}^n : \mathscr{L} : (\boldsymbol{N}^p + \boldsymbol{N}^c)} \right] : \boldsymbol{D}. \qquad (43)$$

In deriving Eq. (43) from the general expression (17) in association with set 1, Eqs. (23), notice that due to the absence of elastic embedding ($\boldsymbol{V} \simeq \boldsymbol{I}$ and $\boldsymbol{Z}^n = \boldsymbol{0}$) the elastic moduli are the same at $\varkappa$ and $\varkappa_0$, i.e. $\mathscr{L} \simeq \mathscr{L}_0$. Also, the $\boldsymbol{N}' \simeq \boldsymbol{N}^p + \boldsymbol{N}^c$ if the term $\mathscr{L}^{-1} : \boldsymbol{Z}^p$ is neglected compared to unity, being of the order of stress/elastic moduli (which occurs if $\boldsymbol{\Omega}^p$ is of the same order in σ as $\boldsymbol{N}^p$ is, according to the definition of $\boldsymbol{Z}^p$). The λ is obtained from Eqs. (16) incorporating the above modifications, with $\dot{\boldsymbol{\sigma}}$ and $\overset{\triangledown}{\boldsymbol{\sigma}}$ substituting for $\overset{\square}{\sigma}$ and $\check{\boldsymbol{\sigma}}$, respectively. The H is always given by Eq. (20).

Since the term $\boldsymbol{Z}^p$, which includes the $\boldsymbol{\Omega}^p$, has been neglected, the elastoplastic moduli relating $\overset{\triangledown}{\boldsymbol{\sigma}}$ and $\boldsymbol{D}$ in Eq. (43) become independent of an explicit expression for the plastic spin; in fact, Eq. (43) is the classical form used in many theories and their numerical implementation. This apparent absence of $\boldsymbol{\Omega}^p$ from the stress-strain rate relation is misleading. The reason is that all the entities which enter the elastoplastic moduli in Eq. (43) (such as the $\boldsymbol{N}^p$ and $\boldsymbol{N}^n$) depend on the structure variables $\boldsymbol{s}$ (or $\boldsymbol{a}$, $\boldsymbol{m}$ and k), whose law of evolution, Eqs. (41) or (42), strongly depend on $\boldsymbol{\Omega}^p$. In fact, it can be said that by far the most important practical aspect of the present theoretical development, is the use of Eqs. (42) for the evolution of $\boldsymbol{a}$ and $\boldsymbol{m}$, with $\boldsymbol{\Omega}^p$ determined by constitutive relations.

7.3 Small Deviatoric but Large Volumetric Elastic Deformation

Often the volumetric part of the elastic deformation is finite while the deviatoric is small, as it would occur under high pressures on metals. In this case the $\boldsymbol{V}$ becomes approximately a spherical tensor, but not the unit one as in the previous subsection. If v is the unique eigenvalue of $\boldsymbol{V}$, the effect on the kinematics is analytically described by

$$\boldsymbol{V} = |\boldsymbol{V}|^{1/3}\,\boldsymbol{I} = v\boldsymbol{I}; \quad \mathring{\boldsymbol{V}}\boldsymbol{V}^{-1} = (\dot{v}/v)\,\boldsymbol{I} = \boldsymbol{D}^e; \quad \operatorname{tr}\boldsymbol{D}^e = 3(\dot{v}/v) \qquad (44.1;2;3)$$

$$\boldsymbol{D}^p = \boldsymbol{D}_0{}^p; \quad \boldsymbol{W}^p = \boldsymbol{W}_0{}^p; \quad \boldsymbol{W}^e = (\mathring{\boldsymbol{V}}\boldsymbol{V}^{-1})_a = \boldsymbol{0} \qquad (44.4;5;6)$$

$$\boldsymbol{D} = \boldsymbol{D}^e + \boldsymbol{D}^p; \quad \boldsymbol{W} = \boldsymbol{\omega} + \boldsymbol{W}^p; \quad \boldsymbol{W}^* = \boldsymbol{\omega}. \qquad (44.7;8;9)$$

In relation to the elastic embedding and the associated rates, [1, Eqs. (13) and (14)], one can state the following in reference to $\varkappa_0$

$$\boldsymbol{S} = v^{\overline{w}}\boldsymbol{s}; \quad \mathring{\boldsymbol{S}} = v^{\overline{w}}\overset{\square}{\boldsymbol{s}}; \quad \overset{\square}{\boldsymbol{s}} = \mathring{\boldsymbol{s}} + \frac{1}{3}\,\overline{w}\boldsymbol{s}\operatorname{tr}\boldsymbol{D}^e \qquad (45.1;2;3)$$

where the $\overline{w}$ takes the values $\overline{w} = 3w - 2$, $3w + 2$, $3w$, $3w$, $3w - 1$, $3w + 1$, $3w$, for the embedding expressed by [1, Eqs. (13.1.1), (13.2.1), (13.3.1), (13.3.2), (13.1.2), (13.2.2), (13.4)], respectively, with $\boldsymbol{V}$ substituting for $\boldsymbol{F}^e$. A direct consequence of the kinematical relations (45), is that the constitutive rate equation for the structure variables take the equivalent forms

$$\mathring{\boldsymbol{S}} = \langle\lambda\rangle\,\overline{\boldsymbol{S}}; \quad \overset{\square}{\boldsymbol{s}} = \langle\lambda\rangle\,\overline{\boldsymbol{s}} \quad \text{with} \quad \overline{\boldsymbol{S}} = v^{\overline{w}}\overline{\boldsymbol{s}} \qquad (46.1;2)$$

and $\overline{w}$ defined as before. Expressions in terms of convected rates, such as in Eq. (8.3), do not change, or based on Eqs. (44.8), (45.3) and (46.2) they can be recasted into the form (for the $\boldsymbol{a}$ defined in Eqs. (8))

$$\overset{\triangledown}{\boldsymbol{a}} + \frac{1}{3}\,\boldsymbol{a}\operatorname{tr}\boldsymbol{D} = \langle\lambda\rangle\left(\overline{\boldsymbol{a}} + \boldsymbol{a}\boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p\boldsymbol{a} + \frac{1}{3}\,\boldsymbol{a}\operatorname{tr}\boldsymbol{N}^p\right). \qquad (47)$$

Observe the difference of Eq. (47) from Eq. (42.1), where the elastic volumetric strain is small. Based on the foregoing assumptions, the elastic relations in reference to $\varkappa_0$ are given from Eqs. (10) with $\boldsymbol{V}$ substituting for $\boldsymbol{F}^e$, yielding

$$\boldsymbol{\Pi} = v\boldsymbol{\sigma}; \quad \boldsymbol{E}^e \approx \frac{1}{2}(v^2 - 1)\,\boldsymbol{I}; \quad \mathscr{L} = v\mathscr{L}^0; \quad \boldsymbol{N}^c = \frac{1}{v^2}\boldsymbol{N}_0^{\ c}. \tag{48}$$

The elastoplastic rate equations are given from Eqs. (15)—(28) with the following differences:

$$\boldsymbol{\Gamma} = \overline{w}\left(\frac{\partial f}{\partial \boldsymbol{s}} \cdot \boldsymbol{S}\right)\boldsymbol{I}; \quad \boldsymbol{Z}^n = \varepsilon(\boldsymbol{N}^n : \boldsymbol{\sigma})\,\boldsymbol{I} + \boldsymbol{\Gamma}; \quad \overset{\square}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} + \varepsilon\boldsymbol{\sigma}\,\mathrm{tr}\,\boldsymbol{D}^e \quad (49.1;2;3)$$

where $\overline{w}$ is defined as after Eq. (45) and $\varepsilon = 1/3,\ 5/3,\ 1,\ 1,\ 0,\ 1$ for the sets 1—6, respectively.

As an example of the foregoing consider an isotropic elastic relation given by

$$\boldsymbol{E}^e = \mathscr{L}_0^{-1} : \boldsymbol{\Pi} = \frac{1}{2G}\boldsymbol{\Pi}' + \frac{1}{9K}(\mathrm{tr}\,\boldsymbol{\Pi})\,\boldsymbol{I} \tag{50}$$

with $\boldsymbol{\Pi}'$ the deviator of $\boldsymbol{\Pi} = v\boldsymbol{\sigma}$, and G, K constant shear and bulk moduli (from Eq. (50) it is straightforward to obtain the $\bar{\psi}$, quadratic in $\boldsymbol{\Pi}$). Using Eqs. (44), (48) and (49.2) one can establish the relations

$$v = \left[1 + \frac{2}{9K}\,\mathrm{tr}\,\boldsymbol{\Pi}\right]^{1/2} = \frac{1}{9K}\,\mathrm{tr}\,\boldsymbol{\sigma} + \left[1 + \left(\frac{\mathrm{tr}\,\boldsymbol{\sigma}}{9K}\right)^2\right]^{1/2} \tag{51.1}$$

$$\mathrm{tr}\,\boldsymbol{D}^e = \frac{\mathrm{tr}\,\dot{\boldsymbol{\sigma}}}{3B}; \quad B = Kv - \frac{1}{9}\,\mathrm{tr}\,\boldsymbol{\sigma} = \left[K^2 + \left(\frac{\mathrm{tr}\,\boldsymbol{\sigma}}{9}\right)^2\right]^{1/2}. \tag{51.2}$$

The $\mathrm{tr}\,\boldsymbol{D}^e$ which enters many of the foregoing relations is expressed in terms of $\mathrm{tr}\,\dot{\boldsymbol{\sigma}}$ and a variable bulk modulus B, Eq. (51.2).

7.4 *Elastoviscoplasticity*

Rate dependent response can be described macroscopically within the framework of elastoviscoplasticity. All the kinematics developed in [1] apply again, and the only, but important difference from plasticity, is that what causes the plastic rate of deformation and spin, as well as the evolution of the structure variables, is not any more the loading index λ, but a positive scalar valued function ϕ of an overstress measure [19]. Hence, it suffices to substitute $\langle\phi\rangle$ for $\langle\lambda\rangle$ in Eqs. (2), (3), (4), (6)—(8.3), (10.1, 2), (13) and (15), in order to obtain the corresponding elastoviscoplastic general formulation. Similarly for Eqs. (37), (41), (42), (46) and (47). The overstress measure is usually defined as the norm of the distance in stress space between the current stress point and a properly defined reference stress point. Such reference stress point is located on a so-called static yield surface by a proper mapping rule. The "center" $\boldsymbol{a}$ of the static yield surface may be fixed (e.g. the stress origin) or variable (kinematic hardening). If the static yield surface "shrinks" towards its center and degenerates into one point, the latter becomes the reference stress; this scheme has been used by Anand [20] in defining the overstress (although not explicitly stated as such). The important element again becomes the evolution law for the internal variables. For example, the "center" $\boldsymbol{a}$ of the static yield surface evolves according to Eq. (42.1), with $\langle\phi\rangle$ substituting for $\langle\lambda\rangle$ at small elastic deformations. One must therefore define not only the $\bar{\boldsymbol{a}}$ (usually along the deviatoric part of $\boldsymbol{\sigma} - \boldsymbol{a}$), but also the $\boldsymbol{\Omega}^p$ as isotropic functions of the state variables $\boldsymbol{\sigma}$ and $\mathbf{s}$ ($\mathbf{s} = \boldsymbol{a}$ in the simplest case). The form of $\boldsymbol{\Omega}^p = \eta(\boldsymbol{a\sigma} - \boldsymbol{\sigma a})$ according to the original suggestion in [21], was followed in [20] for the viscoplastic formulation. The original viscoplasticity formulation along with elastoplasticity (in fact in combination with it) for large deformations in the spirit of the present development, was presented by Mandel [11] and followed by Halphen [22].

In fact the results obtained in [10] by analyzing specific examples of homogeneous deformations for rate-independent rigid-plastic material response, would be exactly the same for rate-dependent rigid-viscoplastic response. Indeed, the expressions for the stress components (e.g. [10, Eqs. (43) and (46)]) are normalized by k which measures the "size" of the yield surface. In a viscoplastic consideration the only difference is in the interpretation of k, which measures now the "size" of the dynamic yield surface passing through the current stress point, and which is a function of the strain rate according to well known relations based on the overstress concept [19].

8. Residual Stresses and the Effectively Unstressed Configuration

The concept of an intermediate "unstressed" or "relaxed" configuration was criticized in the case where residual stresses within the macroscopically homogeneous (but microscopically inhomogeneous) material element, for which the constitutive formulation is being developed, bring the stress origin outside the current yield surface, as shown so often by Phillips et al. [23], [24], for either small or large deformations. Then, it is impossible to elastically unload (i.e. to set $\boldsymbol{\sigma} = \boldsymbol{\Pi} = \boldsymbol{0}$) without causing additional plastic deformation [25], [26], [27], thereby altering the "unstressed" configuration from which elastic strains are measured in relation to the current one. Against this criticism the concept of a "virtual" elastic unloading was proposed [28], [29], where the $\boldsymbol{\sigma}$ is brought to zero not actually but virtually, i. e. by "freezing" all the mechanisms of plastic deformation during unloading. In this case, however, the material macroelement at the intermediate "virtual" configuration will still be subjected to its own locked-in residual stresses, hence, raising the question of how much "unstressed" is the intermediate configuration.

We will propose here a different answer to this problem, which may have some advantages, and study its consequence on the overall formulation. The constitutive essence of an intermediate unstressed configuration is that the elastic strain $\boldsymbol{E}^e = (1/2)\,(\boldsymbol{F}^{e^T}\boldsymbol{F}^e - \boldsymbol{I})$ is related to the stress $\boldsymbol{\Pi}$ in such a way that $\boldsymbol{E}^e = \boldsymbol{0}$ when $\boldsymbol{\Pi} = \boldsymbol{0}$ and vice-versa. The elastic strain, however, is only a relative geometrical measure between the current and what is termed "unstressed" configuration. It can very well be argued that a configuration which is subjected to a macroscopic stress $\boldsymbol{\sigma}$ which is exactly the opposite of the residual locked-in stress $\boldsymbol{\sigma}_r$, is more "unstressed" than the one for which $\boldsymbol{\sigma} = \boldsymbol{0}$. The rationale behind this argument is that the corresponding relative measure of elastic strain is closer to a physical situation of zero elastic deformation, if the macroscopic and residual stresses cancel each other on the average over the volume of the macroelement. Denoting by $\boldsymbol{a}$ the opposite of the residual stress, i.e. $\boldsymbol{a} = -\boldsymbol{\sigma}_r$, it is the effective stress $\boldsymbol{\sigma} - \boldsymbol{a}$ which controls the "elasticity" rather than $\boldsymbol{\sigma}$ itself. The intermediate configuration for which $\boldsymbol{\sigma} - \boldsymbol{a} = \boldsymbol{0}$ can be called the *effectively unstressed configuration* from which $\boldsymbol{E}^e$ is measured. In such case no problem of additional plastic deformation arises upon unloading to $\boldsymbol{\sigma} = \boldsymbol{a}$.

Considering the same elastic embedding for $\boldsymbol{a}$ as for $\boldsymbol{\sigma}$, the foregoing have the following analytical description:

$$\boldsymbol{A} = |\boldsymbol{F}^e|\,\boldsymbol{F}^{e^{-1}}\boldsymbol{a}\boldsymbol{F}^{e^{-T}}; \qquad \boldsymbol{\Pi} - \boldsymbol{A} = |\boldsymbol{F}^e|\,\boldsymbol{F}^{e^{-1}}(\boldsymbol{\sigma} - \boldsymbol{a})\,\boldsymbol{F}^{e^{-T}} \tag{52.1; 2}$$

$$\psi = \bar{\psi}(\boldsymbol{\Pi} - \boldsymbol{A}, \boldsymbol{S}); \qquad \boldsymbol{E}^e = \varrho_0\,\frac{\partial\bar{\psi}}{\partial\boldsymbol{\Pi}} = -\varrho_0\,\frac{\partial\bar{\psi}}{\partial\boldsymbol{A}} \tag{52.3; 4}$$

$$f(\boldsymbol{\sigma} - \boldsymbol{a}, \boldsymbol{s}) = 0; \qquad \boldsymbol{N}^n = \left(\frac{\partial f}{\partial\boldsymbol{\sigma}}\right)_s = -\left(\frac{\partial f}{\partial\boldsymbol{a}}\right)_s. \tag{52.5; 6}$$

Clearly $\boldsymbol{E}^e = 0$ when $\boldsymbol{\Pi} = \boldsymbol{\sigma} = \boldsymbol{a} = \boldsymbol{A}$, i.e. $\boldsymbol{\sigma}' = \boldsymbol{a}'$ and $\operatorname{tr}\boldsymbol{\sigma} = \operatorname{tr}\boldsymbol{a}$, where a prime denotes the deviatoric part. Eq. (52.5; 6) represents the classical description of kinematic hardening due to $\boldsymbol{a}$, in addition to other possible hardening due to the remaining $\boldsymbol{s}$. It is worth mentioning, that if all other $\boldsymbol{s}$ and $\boldsymbol{S}$ in Eqs. (52.3; 4) and (52.5; 6) are scalar valued, the material is elastically and plastically isotropic in reference to the effectively unstressed configuration $\boldsymbol{\sigma} = \boldsymbol{a}$, and orthotropic in reference to the unstressed configuration $\boldsymbol{\sigma} = \boldsymbol{0}$ with the axes of orthotogry along the eigenvectors of $\boldsymbol{a}$ [30]. These conclusions are based on the analytical isotropy of f and $\bar{\psi}$ due to invariance requirements.

As to the evolution law for $\boldsymbol{A}$ or $\boldsymbol{a}$, it is given by Eqs. (8) setting $w = 1$ due to Eq. (52.1). It is usually assumed that $\operatorname{tr}\bar{\boldsymbol{a}} = 0$ (notice that then $\operatorname{tr}\bar{\boldsymbol{A}} \neq 0$), as for example when $\bar{\boldsymbol{a}} = (2/3)\,h_a\boldsymbol{n} - (2/3)^{1/2}\,c_r\alpha$, where $\alpha = \boldsymbol{a}'$, $\boldsymbol{n}$ is the unit "vector" along $\boldsymbol{N}^n$, and $f = (3/2)\,(\boldsymbol{\sigma}' - \alpha) : (\boldsymbol{\sigma}' - \alpha) - k^2 = 0$ [10] (for $c_r = 0$ this corresponds to Prager's kinematic hardening). In such a case, from $\overset{\square}{\boldsymbol{a}} = \langle\lambda\rangle\,\bar{\boldsymbol{a}}$, Eq. (8.2), one has $\operatorname{tr}\overset{\square}{\boldsymbol{a}} = 0$ whether $\lambda \gtrless 0$, which on the basis of Eq. (15) of [1] yields

$$\operatorname{tr}\dot{\boldsymbol{a}} = 2\operatorname{tr}(\boldsymbol{a}\boldsymbol{D}^e) - \operatorname{tr}\boldsymbol{a}\operatorname{tr}\boldsymbol{D}^e. \tag{53}$$

Eq. (53) determines the evolution of $\operatorname{tr} \boldsymbol{a}$ which is always linked to the elastic rate of deformation only, while the $\overset{\square}{\boldsymbol{a}} = \langle \lambda \rangle \overline{\boldsymbol{a}}$ yields the evolution of $(\overset{\square}{\boldsymbol{a}})'$. Observe, that it is only the $\boldsymbol{a}' = \alpha$ which enters the $f = 0$ as defined before, but in general it is the $\boldsymbol{a}$ (and its transport $\boldsymbol{A}$) which enters the elastic relations, Eqs. (52.3; 4). For small elastic deformations recall that $\mathring{\boldsymbol{a}} = \langle \lambda \rangle \overline{\boldsymbol{a}}$, Eq. (41), which means that $\boldsymbol{a}$ remains always deviatoric when $\operatorname{tr} \overline{\boldsymbol{a}} = 0$, since $\operatorname{tr} \mathring{\boldsymbol{a}} = \operatorname{tr} \dot{\boldsymbol{a}} = 0$. Notice that all the foregoing observations on the evolution of $\boldsymbol{a}$, apply to the concept of kinematic hardening with or without the concept of an effectively unstressed configuration.

The effect of the residual stress on the general form of Eqs. (15)—(22) can be presented in two stages. First, the classical definition of the unstressed configuration at $\boldsymbol{\sigma} = \boldsymbol{\Pi} = \boldsymbol{0}$ renders $\bar{\psi}$ a function of $\boldsymbol{\Pi}$ but not of $\boldsymbol{\Pi} - \boldsymbol{A}$ (although $\boldsymbol{A}$ could be one of the $\boldsymbol{S}$). Eqs. (52.5; 6) holds true, where $\boldsymbol{a}$ could be one of the $\boldsymbol{s}$ in addition to its presence via $\boldsymbol{\sigma} - \boldsymbol{a}$. Under these conditions the presence of $\boldsymbol{a}$ results in changing the H and $\boldsymbol{\Gamma}$, Eqs. (20) and (22), which are due to the presence of $\boldsymbol{s}$, to $H + H_a$ and $\boldsymbol{\Gamma} + \boldsymbol{\Gamma}_a$, respectively, where

$$H_a = \boldsymbol{N}^n : \overline{\boldsymbol{a}}; \qquad \boldsymbol{\Gamma}_a = \boldsymbol{a}\boldsymbol{N}^n + \boldsymbol{N}^n\boldsymbol{a} - (\boldsymbol{N}^n : \boldsymbol{a})\,\boldsymbol{I}. \tag{54.1; 2}$$

Eq. (54) are plausible, since $\boldsymbol{a}$ becomes one additional structure variable effecting H, via its hardening, and $\boldsymbol{\Gamma}$ via its elastic embedding according to Eq. (52.1). If now the notion of the effectively unstressed configuration is introduced, i.e. Eqs. (52.3; 4) is assumed, a necessary modification in addition to Eqs. (54) is to change $\boldsymbol{N}^c$ to $\boldsymbol{N}^c + \boldsymbol{N}_a^c$, where

$$\boldsymbol{N}_a^c = -\mathscr{L}^{-1} : \overline{\boldsymbol{a}}. \tag{55}$$

Again Eq. (55) is plausible, since $\boldsymbol{a}$ contributes to the elastoplastic coupling term $\boldsymbol{N}^c$, being introduced in $\bar{\psi}$ in addition to $\boldsymbol{S}$, Eqs. (52.3; 4).

Based on Eqs. (54) and (55) it is straightforward to identify the changes which occur in the elastoplastic moduli, Eq. (17). The change of $\boldsymbol{\Gamma}$ by $\boldsymbol{\Gamma}_a$, Eq. (54.2), will always effect the $\boldsymbol{N}$ via $\boldsymbol{Z}^n$, according to Eq. (19) and the subsequent definitions of $\boldsymbol{Z}^n$ in the six sets. Observe, however, that when both Eqs. (54.1) and (55) are used (i.e. the effectively unstressed configuration is introduced). the quantity $H + \boldsymbol{N}^n : \mathscr{L} : \boldsymbol{N}^c$ in the denominator of Eq. (17) remains unchanged since the modification of H by H_a, cancels out the modification of $\boldsymbol{N}^c$ by $\boldsymbol{N}_a^c$. The effect of the notion of the effectively unstressed configuration can be seen clearer in Eq. (43) for small elastic deformations (no sense of elastic embedding). Eqs. (54.1) and (55) result in one change only; the first bracked in the numerator of the right-hand side of Eq. (43) becomes $\mathscr{L} : (\boldsymbol{N}^p + \boldsymbol{N}^c) - \overline{\boldsymbol{a}}$. The unpleasant consequence is that the $\boldsymbol{A}$ in $\overset{\triangledown}{\boldsymbol{\sigma}} = \boldsymbol{A} : \boldsymbol{D}$, Eq. (43), looses in general the normality structure even if $\boldsymbol{N}^p + \boldsymbol{N}^c$ is proportional to $\boldsymbol{N}^n$. An exception would be the case where $(1/c)\,\overline{\boldsymbol{a}} = \boldsymbol{N}^n = \boldsymbol{N}^p$ (Prager's kinematic hardening and associated flow rule), $N^c = \mathbf{0}$, $\operatorname{tr} \bar{\boldsymbol{a}} = 0$ and $\mathscr{L}$ are the isotropic elastic tangent moduli; then it can be shown that $\mathscr{L} : (N^p + N^c) - \bar{\boldsymbol{a}} = \big(1 - (c/2G)\big) \mathscr{L} : N^n$ with G the shear elastic modulus, hence, the normality structure is preserved. In general, based on Eq. (42.1) for $\overset{\triangledown}{\boldsymbol{a}}$ and the definition of λ in terms of $\boldsymbol{D}$, the use of Eqs. (54.1) and (55) in Eq. (43) yields

$$\overset{\triangledown}{\boldsymbol{\sigma}} - \overset{\triangledown}{\boldsymbol{a}} = \boldsymbol{\Lambda} : \boldsymbol{D} \tag{56}$$

with $\boldsymbol{\Lambda}$ being exactly the one defined in the right-hand side of Eq. (43), if terms of the order of $\boldsymbol{a}$ (which is of the order of $\boldsymbol{\sigma}$) over elastic moduli are neglected compared to 1. In Eq. (56) the normality structure is obtained now if $N^p + N^c$ is proportional to N^n.

9. The Plastic Spin Effect: Intuition or Definition?

If one would like to single out the most important novel aspect of the theoretical macroscopic constitutive framework proposed by Mandel [11], [28] and Kratochvil [31], it would be the requirement to obtain the plastic spin by constitutive relations, as done for the plastic rate of deformation. A brief review of recent achievements in this direction is, therefore, pertinent, and will allow a clear comparison of Mandel's and Onat's approach on the kinetics of the subject, in addition to the comparison made in the first part [1] on the kinematical aspects.

Kratochvil [31] appears to be the first suggesting the use of representation theorem for isotropic functions in order to conclude that the plastic spin is identically zero for isotropic materials, but did not provide and/or investigate any plastic spin form for anisotropic materials. Hahn [32] actually used the representation theorems to obtain an explicit plastic spin constitutive relation in terms of an asymmetric stress tensor, but has not introduced and used the concept of tensorial structure variables to describe the material substructure. It was in the work of Dafalias [21] that for the first time a specific form of a plastic spin constitutive relation appeared in print, and its effect on the simple shear response discussed in relation to a kinematically hardening constitutive model. Recalling the notation of the previous sections, Eq. (23) of [21] can be written as

$$\begin{aligned}(\dot{\boldsymbol{P}}\boldsymbol{P}^{-1})_a = \boldsymbol{W}^p &= \langle \lambda \rangle\, \boldsymbol{\Omega}^p(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \langle \lambda \rangle\, \eta(\boldsymbol{\alpha}\boldsymbol{\sigma}' - \boldsymbol{\sigma}'\boldsymbol{\alpha}) \\ &= \langle \lambda \rangle\, \eta\big(\boldsymbol{\alpha}(\boldsymbol{\sigma}' - \boldsymbol{\alpha}) - (\boldsymbol{\sigma}' - \boldsymbol{\alpha})\,\boldsymbol{\alpha}\big) = \frac{1}{2}\,\varrho(\boldsymbol{\alpha}\boldsymbol{D}^p - \boldsymbol{D}^p\boldsymbol{\alpha})\end{aligned} \tag{57}$$

where η and ϱ are isotopic scalar valued isotopic functions of $\boldsymbol{\sigma}'$, $\boldsymbol{\alpha}$ and other scalar variables. The transition to the last member of Eq. (57) is obtained by making use of the associated flow rule according to which $\boldsymbol{D}^p$ is along $\boldsymbol{\sigma}' - \boldsymbol{\alpha}$. The key step in Eq. (57) is the expression $\boldsymbol{\Omega}^p = \eta(\boldsymbol{\alpha}\boldsymbol{\sigma}' - \boldsymbol{\sigma}'\boldsymbol{\alpha})$, as the simplest possible case obtained from the representation theorems. During the month of publication of [21], Loret [15] submitted his own independent work on the subject which appeared in print three months later; the suggestion for the plastic spin in the case of kinematic hardening was identical to Eq. (57). In addition, both Dafalias [10], [14], [33], [34] and Loret [15] pursued the matter of anisotropies different than the one induced by kinematic hardening (such as orthotopy), suggesting corresponding equations for the plastic spin and investigating the results; the reader can refer to the aforementioned works for further details.

Use of Eqs. (42.1) and (57) can now provide a very clear comparison of Onat's and Mandel's approach on the general subject of evolution equations for the structure variables. Onat et al. [6], [7] have reached the conclusion that the evolution of α is given by $\overset{\triangledown}{\alpha} = \langle \lambda \rangle \bar{\boldsymbol{A}}(\boldsymbol{\sigma}', \alpha)$, with $\bar{\boldsymbol{A}}$ an isotropic second order symmetric tensor-valued function of $\boldsymbol{\sigma}'$ and α (the interpretation of λ is slightly different from the one given here). Considering Eq. (42.1) applied to the back-stress, in which case one has $\boldsymbol{a} = \alpha$, it can be seen that in Onat's approach all the terms of the expression $\bar{\boldsymbol{a}} + \alpha \boldsymbol{\Omega}^p - \boldsymbol{\Omega}^p \alpha$ have been "lumped" into just $\bar{\boldsymbol{A}}$. Since $\bar{\boldsymbol{a}}$ and $\boldsymbol{\Omega}^p$ are isotropic functions of $\boldsymbol{\sigma}'$ α (and so is $\bar{\boldsymbol{A}}$), there is no contradiction. However, Eq. (42), rseulting from Mandel's proposition [11], provides a clear distinction between the constitutive part $\bar{\boldsymbol{a}}$ for the structure variable, and the remaining terms which involve the constitutive part $\boldsymbol{\Omega}^p$ for the plastic spin; in Onat's approach such a distinction has been lost. Nevertheless, Onat [7] porposed a form for $\bar{\boldsymbol{A}}(\boldsymbol{\sigma}', \alpha)$ which yielded exactly the same result for $\overset{\triangledown}{\alpha}$ obtained by substituting the expression (57) for the plastic spin in Eq. (42.1). Hence, Onat obtained the evolution equation for α earlier than Dafalias and Loret, but one can observe the following difference. While in Onat's approach a kind of intuition is necessary in order to obtain the form of $\bar{\boldsymbol{A}}$, in Dafalias's and Loret's approach it is the determination of the plastic spin by a well defined constitutive relation which yields the final result for $\overset{\triangledown}{\alpha}$. Such difference can become even more important in other cases with different anisotropics. The plastic spin will always be analytically defined by rigorously derived constitutive relations, and no need for intuitive suggestions on the form of rate equations of evolution of the structure variables is necessary.

Conclusion

It is believed that satisfactory answers were provided to the issues raised in the introduction of [1, part 1]. It will be a redundancy to repeat the conclusions here, especially because sometimes it was necessary to provide the answer partially in each part, being related to both kinematics and kinetics. Reference to these issues should not overshadow, however, other important aspects of the general theoretical development, some of them quite novel as for example the concept of the effectively unstressed configuration.

We would like to close this work with one thought. What characterized the effort in this development, was the desire to account as realistically as possible for the physics of the situation from a continuum mechanics point of view. Abstract formulations and/or numerical convenience came secondary in this effort. Still, it was possible to have some problems solved in closed or semi-closed form [10], [12], [34]. There are other theories in the area of large elastoplastic deformations, which do not take this position. For the followers of these works, we would like to present an open challenge; considering the example of single slip from the continuum point of view, one must be able to obtain the corresponding basic equations as a particular case of the developed theory, (as done in this paper). If this is impossible, the adopted theory is defective in its basis, because it cannot describe one of the most fundamental mechanisms of elastoplastic deformation.

References

[1] Dafalias, Y. F.: Issues on the constitutive formulation at large elastoplastic deformations, part 1: Kinematics. Acta Mechanica **69**, 119—138 (1987).

[2] Boehler, J. P.: A simple derivation of representations for non-polynomial constitutive equations in some cases of anisotropy. ZAMM **59**, 157—167 (1979).

[3] Liu, I. S.: On representations of anisotropic invariants. International Journal of Solids and Structures **20**, 1099—1109 (1982).

[4] Hill, R.: The mathematical theory of plasticity, p. 318. London: Oxford University Press 1950.

[5] Dafalias, Y. F.: Anisotropy, reference configuration and residual stresses, in: Desai, C. S., at al, eds.; Constitutive laws for engineering materials: Theory and application, from Second Int. Conference and Short Course, Tucson, Arizona, January 1987, Elsevier, in press.

[6] Fardshisheh, F., Onat, E. T.: Representation of elastoplastic behavior by means of state variables, in: Sawczuk, A., ed.; Problems of plasticity, pp. 89—115, Leyden: Noordhoff 1974.

[7] Onat, E. T.: Representation of inelastic, behavior in the presence of anisotropy and of finite deformations, in: Wilshire, B., Owen, D. R. J., eds.; Recent advances in creep and fracture of engineering materials and structures, pp. 231—264. Swansea: Pineridge Press 1982.

[8] Dashner, P. A.: Invariance considerations in large strain elastoplasticity. ASME Journal of Applied Mechanics **53**, 55—60 (1986).

[9] Wang, C. C.: A new representation theorem for isotropic functions: an answer to Professor G. F. Smith's criticism of my paper on representation for isotropic functions. Arch. Rat. Mech. Anal. **36**, 198—223 (1970).

[10] Dafalias, Y. F.: The plastic spin. ASME Journal of Applied Mechanics **52**, 865—871 (1985).

[11] Mandel, J.: Plasticité classique et viscoplasticité, Courses and Lectures, No. 97, Internationald Center for Mechanical Sciences, Udine, Wien—New York: Springer 1971.

[12] Dafalias, Y. F.: A missing link in the macroscopic constitutive formulation of large plastic deformations, in: Sawczuk, A., Bianchi, G., eds.; Plasticity today, pp. 135–15. Barking, Essex: Elsevier 1985, from the International Symposium on Recent Trends and Results in Plasticity, Udine, Italy, June 1983.

[13] Prager, W.: An elementary discussion of definitions of stress rate. Quarterly of Applied Mathematics **18**, 403–407 (1961).

[14] Dafalias, Y. F.: The plastic spin. A three-part Report No. 83-1-2-3, Department of Civil Engineering, University of California at Davis, October 1983.

[15] Loret, B.: On the effects of plastic rotation in the finite deformation of anisotropic elastoplastic materials. Mechanics of Materials **2**, 287–304 (1983).

[16] Rice, J. R.: Inelastic constitutive relations for solids: an internalvariable theory and its application to metal plasticity. Journal of the Mechanics and Physics of Solids **19**, 433–455 (1971).

[17] Asaro, R. J., Rice, J. R.: Strain localization in ductile single crystals. Journal of the Mechanics and Physics of Solids **25**, 309–338 (1977).

[18] Dafalias, Y. F.: A missing link in the formulation and numeral implementation of finite-transformation elastoplasticity, in: Willam, K. J., ed.; Constitutive equations: Macro and computational aspects, pp. 25–40, ASME 1984.

[19] Yao, O., Krempl, E.: Viscoplasticity theory based on overstress. The prediction of monotonic and cyclic proportional and unproportional loading paths of an aluminum alloy. International Journal of Plasticity **1**, 259–274 (1985).

[20] Anand, L.: Constitutive equations for hot-working of metals. International Journal of Plasticity **1**, 213–231 (1985).

[21] Dafalias, Y. F.: Corotational rates for kinematic hardening at large plastic deformations. ASME Journal of Applied Mechanics **50**, 561–565 (1983).

[22] Halphen, B.: Sur le champ des vitesses en thermoplasticité finie. International Journal of Solids and Structure **11**, 947–960 (1975).

[23] Phillips, A., Weng, G. J.: An analytical study of an experimentally verified hardening law. ASME Journal of Applied Mechanics **42**, 375–378 (1975).

[24] Phillips, A., Lee, C. W.: Yield surfaces and loading surfaces. Experiments and recommendations. International Journal of Solids and Structures **15**, 715–729 (1979).

[25] Green, A. E., Naghdi, P. M.: Some remarks on elastic-plastic deformation at finite strain. International Journal of Engineering Science **9**, 1219–1229 (1971).

[26] Casey, J., Naghdi, P. M.: A remark on the use of the decomposition $F = F^eF^p$ in plasticity. ASME Journal of Applied Mechanics **47**, 672–675 (1980).

[27] Nemat-Nasser, S.: Decomposition of strain measures and their rates in finite deformation elastoplasticity. International Journal of Solids and Structures **15**, 155–166 (1979).

[28] Mandel, J.: Equations constitutives et directeurs dans les milieux plastiques et viscoplastiques. International Journal of Solids and Structures **9**, 725–740 (1973).

[29] Lee, E. H.: Some comments on elastic-plastic analysis. International Journal of Solids and Structures **17**, 859–872 (1981).

[30] Dafalias, Y. F.: Lagrangian and Eulerian description of plastic anisotropy at large strains. Case study: orthotropy and isotropy, in: Boehler, J. P., ed.; Plastic behavior of anisotropic solids, pp. 357–374, Paris: Éditions du C. N. R. S. 1985, from the International Colloquium No. 319, Villard-de-Lans, June 1981.

[21] Kratochvil, J.: On a finite-strain theory of elastic-inelastic materials. Acta Mechanica **16**, 127–142 (1973).

[32] Hahn, H. T.: A finite-deformation theory of plasticity. International Journal of Solids and Structures **10**, 111–121 (1974).

[33] Dafalias, Y. F.: On the evolution of structure variables in anisotropic yield criteria

at large plastic transformations, in: Boehler, J. P., ed.; Failure criteria of structured media, Éditions du C. N. R. S., in press, from the International Colloquium, No. 351, Villard-de-Lans, June 1983.

[34] Dafalias, Y. F.: The plastic spin concept and a simple illustration of its role in finite plastic transformations. Mechanics of Materials **3**, 223—233 (1984).

Professor Y. F. Dafalias
Department of Civil Engineering
University of California at Davis
Davis, CA 95616
U.S.A.