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PLASTIC SPIN: NECESSITY OR REDUNDANCY?

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Abstract—The plastic spin concept in large deformation anisotropic elastoplasticity theories with tensorial internal variables, is proved to be a necessary constitutive ingredient. Different inaccurate notions about the plastic spin are dispelled, and its presence in the theory is demystified as something very simple and straightforward. To this extent it is necessary to disassociate the plastic spin concept and the conjugate notion of constitutive spin from the foundation of kinematics, which caused confusion in the past, and define it only in relation to the constitutive equations of evolution of the tensorial internal variables. There, the plastic spin is related to the orientation aspect of such constitutive equations, and the multiplicity of the different internal variables suggests the necessity to have a different spin for each variable. In the process, a straightforward constitutive framework is developed which is based on classical hyperelasticity, yield criteria and invariance requirements of the constitutive functions under superposed rigid body rotation. Ad-hoc assumptions about stress corotational or convected rates and other fuzzy suggestions for different spins are not part of this development. Other topics such as the concept and simplifying effect of the spinless unstressed configuration and its comparison with the isoclinic configuration, some computational aspects, and the effect of small elastic strains are discussed, and all along the significance of plastic spin in the different equations is evaluated.

I. INTRODUCTION

Despite the historically lengthy development of constitutive formulations for large elastoplastic deformations, there are still issues which are being debated at the theoretical level. While many of these debates are justified, given the uncertainty of some basic assumptions particularly in relation to anisotropic plasticity, others have resulted in conflicting propositions affecting the way formulation and computations are made. One of the central points of such debates is whether or not constitutive equations are necessary or redundant for a quantity called the plastic spin in recent publications. The focus of this work is to clarify this particular issue, explain why it is debated, and in the process provide a straightforward constitutive formulation without ambiguities and ad-hoc assumptions. The assumptions which will be made are widely accepted, such as the multiplicative decomposition of the deformation gradient (Lee, 1969) or the notion of a set of scalar and tensor-valued internal variables characterizing the material microstructure in a macroscopic way.

The issue at hand seems to be first a matter of proper definition. The term plastic spin was apparently coined by Dafalias (1985) to express initially the difference of the material and the director vectors spin at an intermediate configuration in the theory developed by Mandel (1971). The same term was also used by others to express different spin entities, such as the antisymmetric part of the velocity gradient at the plastically deforming intermediate configuration. This created a confusion which persists in recent publications, and where the necessity of constitutive relation for what is thought to be the plastic spin is negated, when particular choices of the intermediate configuration are made (Nemat-Nasser, 1990, 1992; Obata *et al.*, 1990; Onat, 1991). This confusion is partly justified because in the original works by Mandel (1971) and Kratochvil (1973), and in other similar works (Dafalias, 1983, 1985, 1987, 1988; Loret, 1983; Paulun and Pecherski, 1987, etc.), the plastic spin appears to be always related to the kinematics of the deformation, a thesis which required some *ad hoc* assumptions about a rotating frame (the director vectors) representing macroscopically the material substructure. While this is not erroneous and in general well-understood in cases where the material substructure was represented by orientation vectors, such as the crystal lattice in polycrystalline materials, it becomes a bit fuzzy when a more direct macroscopic approach in terms of tensorial internal variables is adopted.

In parallel, many research works on the subject focused on how to eliminate stress oscillations appearing in the simulation of fixed-end torsion experiments, and in the process suggesting a plethora of different stress corotational rates associated, one way or the other, with the notion of plastic or similar kind of spin. Such oscillations, however, do actually occur in experiments and need to be simulated (Dafalias, 1985; Ning and Aifantis, 1994; Cho and Dafalias, 1996), but the question is how it is done and at the expense of what kind of *ad hoc* assumptions.

Pausing for a moment to reflect on the foundation of elastoplasticity theory, it is interesting to observe that stress actually appears at the basis of the theory in two ways. First, in the elastic stress–strain relations, which must be in fact hyperelastic. Second, in the definition of a static or dynamic yield criterion in stress space. Stress rate does not constitute a fundamental constitutive ingredient and it appears only in the process of algebraic manipulations associated with the consistency equation (setting the rate of the yield criterion equal to zero). Thus, works which begin with choices of convected or corotational stress rates at the basis of the theory, do actually make an *ad hoc* assumption not necessarily compatible with the notion of hyperelasticity and yield criteria in the classical sense described above.

It appears, therefore, that the subject matter has not yet been appropriately addressed. In the present work a fresh look at the subject is proposed, which in several aspects is related to past works of the author. The correct approach is to disassociate the plastic spin concept from the foundation of the general kinematics of deformation, and simply introduce it as a necessary orientational ingredient (together with the conjugate notion of constitutive spin) of the constitutive rate equations of evolution of the internal variables. In fact, following Dafalias (1993*a*) where still a spin related to the kinematics was unnecessarily (but not erroneously) maintained, a multiplicity of pairs of plastic and constitutive spins is introduced, reflecting the different orientational characteristics of different internal variables.

The formulation then takes a very straightforward path of development during which many issues fall naturally in place. For example, a convenient stress rate naturally arises from the algebraic manipulations rather than assumed, the concepts of spinless and isoclinic (Mandel, 1971) unstressed configurations are presented and used for easy formulation, particularly in connection with some computational aspects, other details such as elastic embedding, the reason for using corotational rather than convected rates with plastic deformation, small elastic strains, etc., are discussed, and all along the presence and influence of the plastic spin concept on the constitutive formulation is shown and evaluated. While the present work is rather abstract because it addresses only the theoretical foundation and formulation of the theory, its practical significance is corroborated by reference to other works where specific applications and comparisons with experiments at large plastic strains have been proved successful.

Direct notation will be used for the algebra of tensor-valued quantities, which will be denoted by bold-face characters if they are vectors or second order tensor, and by italics if they are fourth order tensors. With the summation convention over repeated indices accepted (unless otherwise stated), juxtaposition implies summation over two repeated adjacent indices, i.e. $\mathbf{AB} \rightarrow A_{ij}B_{jk}$. In some cases pre-position of a second order to a fourth order tensor will require special definition of the implied summation. The symbol $\cdot$ between two tensors implies summation over two pairs of repeated adjacent indices in the same order, i.e. $\mathbf{A}\cdot\mathbf{B} \rightarrow A_{ij}B_{ij}$ or $L\cdot\mathbf{A} \rightarrow L_{ijkl}A_{kl}$, while the often used symbol : implies such summation in reverse order, i.e. $\mathbf{A}:\mathbf{B} \rightarrow A_{ij}B_{ji}$ or $L:\mathbf{A} \rightarrow L_{ijkl}A_{lk}$ (trace operation). Notice that $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}:\mathbf{B}^{\mathrm{T}}$ (the T signifies the transpose) and that $\mathbf{A}:\mathbf{BC} = \mathbf{AB}:\mathbf{C} = \mathbf{C}:\mathbf{AB}$, while $\mathbf{A}\cdot\mathbf{BC} \neq \mathbf{AB}\cdot\mathbf{C}$, in general. The symbol $\otimes$ denotes the tensor product, i.e. $\mathbf{A}\otimes\mathbf{B} \rightarrow A_{ij}B_{kl}$. A superposed $\cdot$ denotes the material time derivative or rate. Finally observe that an index used in denoting the multiplicity of tensors may imply repetition of an operation between tensors if it appears twice, unless otherwise stated. For example, if $\mathbf{A}_i$ and $\mathbf{B}_i$ are two families of tensors for $i = 1, 2, \ldots,$ one has that $\mathbf{A}_i\mathbf{B}_i = \mathbf{A}_1\mathbf{B}_1 + \mathbf{A}_2\mathbf{B}_2 + \ldots,$ with each term implying the corresponding indicial summation of juxtaposition as defined above.

II. KINEMATICS

The kinematics associated with the concepts of a current configuration κ, a reference configuration κ_r and an intermediate unstressed and elastically unstrained configuration κ_u (often called relaxed) will be assumed. Consistent with this assumption the multiplicative decomposition of the total deformation gradient $\mathbf{F}$, Lee (1969), yields

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p \tag{1}$$

with $\mathbf{F}^p$ describing the plastic deformation from κ_r to κ_u and $\mathbf{F}^e$ the elastic deformation from κ_u to κ. For given κ and κ_r, the κ_u can be chosen arbitrarily within a rigid body rotation. For example for a κ_u^* which is rotated from κ_u by a rotation described by the orthogonal tensor $\mathbf{Q}$, it follows that $\mathbf{F}^{e^*} = \mathbf{F}^e\mathbf{Q}^{\mathrm{T}}$ and $\mathbf{F}^{p^*} = \mathbf{Q}\mathbf{F}^p$, such that $\mathbf{F}$ remains unchanged since $\mathbf{F}^{e^*}\mathbf{F}^{p^*} = \mathbf{F}^e\mathbf{F}^p = \mathbf{F}$. This arbitrariness of choosing κ_u requires invariance of the constitutive relations under rigid body rotation at κ_u.

The velocity gradient $\dot{\mathbf{F}}\mathbf{F}^{-1}$ at κ follows from eqn (1) as

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = (\dot{\mathbf{F}}\mathbf{F}^{-1})_s + (\dot{\mathbf{F}}\mathbf{F}^{-1})_a = \mathbf{D} + \mathbf{W} = \dot{\mathbf{F}}^e\mathbf{F}^{e^{-1}} + \mathbf{F}^e\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}\mathbf{F}^{e^{-1}} \tag{2}$$

where $\mathbf{D}$ and $\mathbf{W}$ are the rate of deformation and material spin tensors, respectively, at the current configuration κ, and subscripts s and a denote symmetric and antisymmetric parts, correspondingly. No additive decomposition into elastic and plastic parts of $\mathbf{D}$ and $\mathbf{W}$ at κ is proposed, since it is not directly required for the present formulation.

In this work the constitutive framework will be set initially at the unstressed configuration κ_u, where by definition all deformations and rotations are purely plastic in origin, caused by $\mathbf{F}^p$. Hence, the corresponding plastic velocity gradient at κ_u is obtained by

$$\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}} = (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_s + (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \mathbf{D}^p_o + (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a \tag{3}$$

where $\mathbf{D}^p_o = (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_s$ is the plastic rate of deformation at κ_u, while $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ is the material spin at κ_u (corresponds conceptually to $\mathbf{W}$ at κ). Due to its derivation from $\mathbf{F}^p$, the $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ could be called the plastic material spin, but not the plastic spin in the sense defined subsequently. In connection to notation of other relevant works, it should be mentioned that the quantity $\mathbf{D}^p_o$ here was symbolized as $\mathbf{D}^p_u$ in Dafalias (1987).

Under superposed rigid body rotation $\mathbf{Q}(t)$ which brings κ_u to κ_u^*, the new plastic deformation gradient is $\mathbf{F}^{p^*} = \mathbf{Q}\mathbf{F}^p$, which yields

$$(\dot{\mathbf{F}}^{p^*}\mathbf{F}^{p^{*-1}})_s = \mathbf{D}^p_o = \mathbf{Q}\mathbf{D}^p_o\mathbf{Q}^{\mathrm{T}} \tag{4a}$$

$$(\dot{\mathbf{F}}^{p^*}\mathbf{F}^{p^{*-1}})_a = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a\mathbf{Q}^{\mathrm{T}} \tag{4b}$$

for the corresponding plastic rate of deformation and plastic material spin tensors at κ_u^*.

For future use a kinematical relation is derived between the rate of the elastic Green strain tensor $\mathbf{E}^e = (1/2)\ (\mathbf{C}^e - \mathbf{I})$ where $\mathbf{C}^e = \mathbf{F}^{e^{\mathrm{T}}}\mathbf{F}^e$, the total rate of deformation tensor $\mathbf{D}$, and the plastic rate of deformation tensor $\mathbf{D}^p_o$ at κ_u. If one premultiplies by $\mathbf{F}^{e^{\mathrm{T}}}$ and postmultiplies by $\mathbf{F}^e$ the last two members of eqn (2) and takes the symmetric part of the ensuing relation, he obtains the following using eqn (3) and observing that $\dot{\mathbf{E}}^e = (\mathbf{F}^{e^{\mathrm{T}}}\dot{\mathbf{F}}^e)_s$

$$\overset{\nabla}{\mathbf{E}}{}^e = \dot{\mathbf{E}}^e - (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a\mathbf{E}^e + \mathbf{E}^e(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \mathbf{F}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{F}^e - (\mathbf{C}^e\mathbf{D}^p_o)_s \tag{5}$$

The superposed symbol ∇ denotes the corotational rate (also known as Jaumann rate) of $\mathbf{E}^e$ in relation to the plastic material spin $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$, as defined by the second member of eqn (5). This symbol will be used henceforth to denote the corotational rate of a tensor in relation to the spin defined by the antisymmetric part of the velocity gradient pertinent to a configuration where the tensor is defined, e.g. in reference to $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ at κ_u as done above, or in reference to $(\dot{\mathbf{F}}\mathbf{F}^{-1})_a = \mathbf{W}$ at κ, etc. Equation (5) involves no approximations or disputed definitions of elastic and plastic rate deformation measures. It takes the place of the traditional (and often disputed) additive decomposition of $\mathbf{D}$ into elastic and plastic parts.

III. STATE VARIABLES AND HYPERELASTICITY

In the present macroscopic theory, the material state at the current configuration κ is defined in terms of the Cauchy stress $\boldsymbol{\sigma}$ and a set of internal variables $\mathbf{a}_i$ (the i indicates the plurality of the variables), encompassing the macroscopic manifestation of the material microstructure. As such they can represent residual stresses, directions of anisotropy, directions of max and min of orientation distribution functions of microstructural aggregates, hardening parameters, etc. Therefore, the $\mathbf{a}_i$ can be scalar or tensor-valued quantities, and it is their orientational properties when they are tensor-valued which yield anisotropic material characteristics. In the sequel the $\mathbf{a}_i$ will be considered second-order symmetric tensors (the most common case) for simplicity of presentation, but it is understood that scalar and vector-valued $\mathbf{a}_i$ can be similarly incorporated. The proposition of using $\mathbf{a}_i$ to define simultaneously the state and orientation of an unstressed material element was first advanced in the works of Onat, e.g. Fardshisheh and Onat (1974) and Onat (1991).

Since the constitutive framework will be set first at the intermediate configuration κ_u, one must transport the $\boldsymbol{\sigma}$ and $\mathbf{a}_i$ from κ to their counterparts $\boldsymbol{\Pi}$ and $\mathbf{A}_i$ at κ_u. The transport is related to $\mathbf{F}^e$, hence, it was called elastic embedding in Dafalias (1985, 1987, 1988). The subsequent use of hyperelastic relations with the elastic Green strain tensor $\mathbf{E}^e$, suggests that it is most convenient to define $\boldsymbol{\Pi}$ at κ_u as the symmetric second Piola–Kirchhoff stress tensor according to

$$\boldsymbol{\Pi} = |\mathbf{F}^e|\mathbf{F}^{e^{-1}}\boldsymbol{\sigma}\mathbf{F}^{e^{-\mathrm{T}}} \tag{6}$$

$\boldsymbol{\Pi}$ is the work conjugate to $\dot{\mathbf{E}}^e$ per unit mass. It is possible to define other stress tensors at κ_u, as for example the non-symmetric $\Sigma = |\mathbf{F}|\mathbf{F}^{e^{-1}}\boldsymbol{\sigma}\mathbf{F}^e$, Aravas (1994) or a similar one by Mandel (1971). Such definitions, including the one in eqn (6), are done for reasons of convenience and not because of a fundamental constitutive requirement.

In contrast, the elastic embedding from $\mathbf{a}_i$ to $\mathbf{A}_i$ reflects a physical attribute of $\mathbf{a}_i$ and becomes a corresponding constitutive assumption, since the $\mathbf{A}_i$ remain attached to the material at κ_u as entities characterizing macroscopically the material state and orientation even after the $\boldsymbol{\sigma}$, and consequently $\boldsymbol{\Pi}$, has been removed. For example, say that $\mathbf{a}_i = \mathbf{m}\otimes\mathbf{m}$, where the vector $\mathbf{m}$ represents a reinforcing thin short fiber at κ which becomes $\mathbf{M}$ at κ_u. The plausible physical assumption of affine embedding with elastic deformation for this fiber is analytically described by $\mathbf{m} = \mathbf{F}^e\mathbf{M}$, which with $\mathbf{A}_i = \mathbf{M}\otimes\mathbf{M}$ yields the embedding relation $\mathbf{A}_i = \mathbf{F}^{e^{-1}}\mathbf{a}_i\mathbf{F}^{e^{-1\mathrm{T}}}$. Other types of embedding can similarly be introduced, Dafalias (1987, 1988), as for example for a residual stress tensor $\mathbf{a}$ at κ which becomes $\mathbf{A}$ at κ_u such that $\mathbf{A} = |\mathbf{F}^e|\mathbf{F}^{e^{-1}}\mathbf{a}\mathbf{F}^{e^{-\mathrm{T}}}$, as done for $\boldsymbol{\Pi}$ and $\boldsymbol{\sigma}$ in eqn (6).

Bearing in mind the foregoing for future reference, an elastic strain energy function Ψ per unit mass is assumed to depend on the elastic Green strain $\mathbf{E}^e$ and the variables $\mathbf{A}_i$, i.e. $\Psi = \Psi(\mathbf{E}^e, \mathbf{A}_i)$. The hyperelastic relations are obtained by

$$\boldsymbol{\Pi} = \rho_o \frac{\partial\Psi(\mathbf{E}^e, \mathbf{A}_i)}{\partial\mathbf{E}^e} \tag{7}$$

where ρ_o is the mass density at κ_u. Invariance requirements under superposed rigid body rotation $\mathbf{Q}(t)$ at κ_u and the ensuing change of $\boldsymbol{\Pi}$, $\mathbf{E}^e$ and $\mathbf{A}_i$ to $\mathbf{Q}\boldsymbol{\Pi}\mathbf{Q}^{\mathrm{T}}$, $\mathbf{Q}\mathbf{E}^e\mathbf{Q}^{\mathrm{T}}$ and $\mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}$, respectively, render Ψ and, therefore, $\boldsymbol{\Pi}$ or $\partial\Psi/\partial\mathbf{E}^e$ isotropic functions of their arguments. However, this does not imply that the material is elastically isotropic due to the tensorial character of $\mathbf{A}_i$ (unless all $\mathbf{A}_i$ are scalar-valued, Mandel, 1971), which allows for the description of different kinds of elastic anisotropy. In fact the possible change of $\mathbf{A}_i$ (scalar or tensor-valued) in the course of plastic deformation describes changing elastic properties in a process which can be called elastoplastic coupling or damage, the latter if $\mathbf{A}_i$ describes a fracture process in a continuous "smeared" sense. Observe also that the Cauchy stress $\boldsymbol{\sigma}$ is fully specified by eqns (6) and (7) given $\mathbf{F}^e$ and $\mathbf{A}_i$ at κ_u, without reference to any specific stress rate.

IV. RATE CONSTITUTIVE EQUATIONS AND THE PLASTIC SPIN

IV.1. *Rate equations and constitutive spins*

At the relaxed configuration κ_u one must now provide constitutive equations for the plastic rate of deformation $\mathbf{D}_o^p$ and the evolution of $\mathbf{A}_i$. The first task is accomplished in the usual way be setting

$$\mathbf{D}_o^p = < \lambda > \mathbf{N}_o^p(\mathbf{\Pi}, \mathbf{A}_i) \tag{8}$$

with λ the scalar-valued plastic loading index or plastic multiplier, to be determined in the sequel, $< >$ the Macauley brackets defining the operation $< \lambda > = \lambda$ if $\lambda > 0$ and $< \lambda > = 0$ if $\lambda \leq 0$, and $\mathbf{N}_o^p$ a symmetric tensor-valued function of the state variables $\mathbf{\Pi}$ and $\mathbf{A}_i$ giving the "direction" of the plastic flow. Clearly plastic deformation occurs only when $\lambda > 0$. Based on eqns (4a) and (8), it follows that under superposed rigid body rotation $\mathbf{Q}(t)$ of κ_u one has $\mathbf{N}_o^p(\mathbf{Q\Pi Q}^{\mathrm{T}}, \mathbf{QA}_i\mathbf{Q}^{\mathrm{T}}) = \mathbf{QN}_o^p(\mathbf{\Pi}, \mathbf{A}_i)\mathbf{Q}^{\mathrm{T}}$, hence, $\mathbf{N}_o^p$ is an isotropic function of its arguments $\mathbf{\Pi}$ and $\mathbf{A}_i$.

In relation now to the rate equations of evolution of $\mathbf{A}_i$, a few observations are pertinent. The physical and, consequently, constitutive meaning of the embedding from $\mathbf{a}_i$ at κ to $\mathbf{A}_i$ at κ_u as far as the elastic deformation gradient $\mathbf{F}^e$ is concerned, discussed in Section III, depended on the way the $\mathbf{a}_i$ was attached, so to speak, to the material in the course of elastic deformation. The corresponding question now arises as to how each $\mathbf{A}_i$ is attached to the material during the plastic deformation $\mathbf{F}^p$ which continuously reshapes the intermediate configuration κ_u where $\mathbf{A}_i$ is referred. Is $\mathbf{A}_i$ again embedded in a definite way in the plastically deforming κ_u, or does it follow its own mode of evolution? A positive answer to the former question would imply the use of convected with the plastically deforming continuum rates of covariant, contravariant or mixed type, in describing the evolution of $\mathbf{A}_i$. Although this cannot be excluded from a general perspective, it is not likely to reflect a physical reality. The reason is that the macroscopic continuous appearance of plastic deformation, is in fact the result of a microscopic intensely discontinuous velocity field accomplished mostly by slipping and rolling processes of grains or part of grains relatively to each other, for both crystalline and granular materials. It seems unlikely then that a short fiber, for example, will be embedded affinely in the bulk of such a discontinuously deforming medium. It will rather evolve in its own way, letting the continuum "flow", so to speak, around it. As another example, the vectors of the crystal lattice of polycrystalline metals follow an orientation quite different from that of the principal plastic stretch directions, for example, while they are still embedded in the material in reference to the elastic part of the deformation according to a physically meaningful rule (Dafalias, 1988). More difficult is to state exactly what a residual stress $\mathbf{A}_i$ will do, but there is nothing which necessitates its convected change with the medium during plastic deformation, in general. In other words the $\mathbf{A}_i$s follow different orientational kinematics from that determined by $\mathbf{F}^p$ for the continuum, in what has been termed the kinematics of the substructure (Dafalias, 1987). Of course some connection must be established between the orientational kinematics of the continuum and its substructure, and that is where the plastic spin concept will play its role. In addition to its orientational change associated with the substructure, each $\mathbf{A}_i$ has also its evolutionary change expressed by a rate equation of evolution.

The simplest way to properly account for these two constitutive aspects, the orientational and the evolutionary, is to consider them separately in the constitutive equations.

Mandel (1971) suggested the existence of a rotating frame of director vectors, as he called them, with respect to which the rate equations of evolution for all $\mathbf{A}_i$ were written. In reference to a fixed frame, the rate form of these equations would be corotational with the director vectors frame. To accept, however, a common rotating frame with respect to which the rate equations of evolution for all $\mathbf{A}_i$ are written, is too restrictive. The difference in nature among the $\mathbf{A}_i$s necessitates the introduction of a different rotating frame for each $\mathbf{A}_i$, in general. Since it may not be possible to always define geometrically this frame given $\mathbf{F}^p$, it is equivalent to introduce instead the notion of its spin, called the constitutive spin and symbolized by $\boldsymbol{\omega}_i$ for each $\mathbf{A}_i$. It follows that the constitutive equation for each $\mathbf{A}_i$ in reference to a fixed frame will be expressed in terms of the corotational rate of $\mathbf{A}_i$ in relation to the constitutive spin $\boldsymbol{\omega}_i$. Denoting this rate by a superposed c (for constitutive) and assuming that $\mathbf{A}_i$ is a second order tensor, one has

$$\overset{\mathrm{c}}{\mathbf{A}}_i = \dot{\mathbf{A}}_i - \boldsymbol{\omega}_i\mathbf{A}_i + \mathbf{A}_i\boldsymbol{\omega}_i = <\lambda> \bar{\mathbf{A}}_i(\boldsymbol{\Pi}, \mathbf{A}_i) \qquad (\text{no sum on } i) \tag{9}$$

while if $\mathbf{A}_i$ is vector-valued the equation becomes

$$\overset{\mathrm{c}}{\mathbf{A}}_i = \dot{\mathbf{A}}_i - \boldsymbol{\omega}_i\mathbf{A}_i = <\lambda> \bar{\mathbf{A}}_i(\boldsymbol{\Pi}, \mathbf{A}_i) \qquad (\text{no sum on } i) \tag{10}$$

Equations (9) and (10) show clearly the two constitutive aspects. The second member involving $\boldsymbol{\omega}_i$ defines the orientational aspect. The last member defines the evolutionary aspect, embodied in the tensor or vector-valued function $\bar{\mathbf{A}}_i$ premultiplied by the plastic multiplier λ to indicate that it occurs only when plastic deformation takes place according to eqn (8). Observe also that when $\mathbf{A}_i$ represents a purely orientational internal variable, e.g. a unit vector $\mathbf{n}$ or its tensor product $\mathbf{n} \otimes \mathbf{n}$, the corresponding $\bar{\mathbf{A}}_i$ in eqns (9) or (10) is identically zero. Thus, $\overset{\mathrm{c}}{\mathbf{A}}_i \equiv \mathbf{0}$ which means that $\mathbf{A}_i$ simply spins by $\boldsymbol{\omega}_i$ in the course of plastic deformation, as expected on physical grounds.

Constitutive invariance under superposed rigid body rotation $\mathbf{Q}(t)$, requires that $\overset{\mathrm{c}}{\mathbf{A}}_i$ becomes $\overset{\mathrm{c}}{\mathbf{A}}{}_i^* = \mathbf{Q}\,\overset{\mathrm{c}}{\mathbf{A}}_i\mathbf{Q}^{\mathrm{T}}$ if a tensor, and $\overset{\mathrm{c}}{\mathbf{A}}{}_i^* = \mathbf{Q}\,\overset{\mathrm{c}}{\mathbf{A}}_i$ if a vector. This implies that the constitutive spin $\boldsymbol{\omega}_i$ must obey the transformation

$$\boldsymbol{\omega}_i^* = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\boldsymbol{\omega}_i\mathbf{Q}^{\mathrm{T}} \tag{11}$$

as deduced in Lee *et al*. (1983). Equation (11) imposes a restriction on the definition of $\boldsymbol{\omega}_i$, and also yields the relation $\bar{\mathbf{A}}_i(\mathbf{Q}\boldsymbol{\Pi}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\bar{\mathbf{A}}_i(\boldsymbol{\Pi}, \mathbf{A}_i)\mathbf{Q}^{\mathrm{T}}$ for all orthogonal $\mathbf{Q}$ (if $\mathbf{A}_i$ is a second order tensor for simplicity), which renders $\bar{\mathbf{A}}_i$ isotropic function of $\boldsymbol{\Pi}$ and $\mathbf{A}_i$ (Dafalias, 1985).

IV.2. *The plastic spin*

So far, apart from the general invariance requirement expressed by eqn (11), the constitutive spin $\boldsymbol{\omega}_i$ has not been otherwise restricted or related to the kinematics of the unstressed configuration κ_u described by the plastic velocity gradient $\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}}$. It is clear from the reasoning of introducing $\boldsymbol{\omega}_i$ that the latter is not necessarily equal to the antisymmetric part $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$, which was termed the plastic material spin at κ_u. In the absence of plastic rate of deformation, i.e. when $\mathbf{D}_o^p = (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_s = \mathbf{0}$ which implies $<\lambda> = 0$ from eqn (8), it follows that κ_u simply spins by $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$. Simultaneously, it follows from eqns

(9) and (10) that also $\overset{c}{\mathbf{A}}_i = \mathbf{0}$ (because $\langle\lambda\rangle = 0$), i.e. that each $\mathbf{A}_i$ spins by $\boldsymbol{\omega}_i$. Since the $\mathbf{A}_i$ are attached to the material at κ_u, the foregoing necessarily imply that when $\mathbf{D}^p_o = \mathbf{0}$ one has $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \boldsymbol{\omega}_i$, i.e. the continuum and the $\mathbf{A}_i$s spin together as physically expected when no plastic rate of deformation occurs.

The foregoing suggest that one can write for each $\boldsymbol{\omega}_i$ the relation

$$(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \boldsymbol{\omega}_i + \mathbf{W}^p_i = \boldsymbol{\omega}_i + < \lambda > \boldsymbol{\Omega}^p_i(\boldsymbol{\Pi}, \mathbf{A}_i) \tag{12}$$

where $\mathbf{W}^p_i = < \lambda > \boldsymbol{\Omega}^p_i$ is the plastic spin corresponding to the constitutive spin $\boldsymbol{\omega}_i$, such that when $< \lambda > = 0$ one has $\mathbf{W}^p_i = \mathbf{0}$ and $\boldsymbol{\omega}_i = (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$. Thus, the plastic spin for each $\mathbf{A}_i$ is defined as the difference between the plastic material spin $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ and the constitutive spin $\boldsymbol{\omega}_i$, this difference being of constitutive nature and becoming zero when $\mathbf{D}^p_o = \mathbf{0}$. The significance of eqn (12) can be best appreciated if compared with the corresponding equation in crystal plasticity which reads

$$(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \boldsymbol{\omega} + \mathbf{W}^p = \boldsymbol{\omega} + \sum_\alpha \dot{\gamma}_\alpha(\mathbf{m}_\alpha \otimes \mathbf{n}_\alpha)_a \tag{13}$$

where $\boldsymbol{\omega}$ represents the lattice spin, $\mathbf{m}_\alpha$ and $\mathbf{n}_\alpha$ the slip and normal to slip plane unit directions of slip system α at κ_u, and $\dot{\gamma}_\alpha$ the corresponding slip shear strain rate. Notice that researchers in crystal plasticity usually write the right-hand side of eqn (13) without the $\boldsymbol{\omega}$ term, because they tacitly assume lattice co-rotation for $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ in the formulation. The lattice spin is the counterpart of the constitutive spin $\boldsymbol{\omega}_i$, with the lattice itself being the rotating frame with respect to which constitutive rate equations are written. Equations (12) and (13) clearly demonstrate the difference between the antisymmetric part of the plastic velocity gradient $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ (named here the plastic material spin), and the plastic spin $\mathbf{W}^p_i$ in eqn (12) or $\mathbf{W}^p$ in eqn (13). Since the $\mathbf{W}^p$ is related to the plastic slip process, and the $\mathbf{W}^p_i$ is different than zero only when $\mathbf{D}^p_o$ is, the name plastic spin was introduced (Dafalias, 1985). Much confusion in the literature was created because many authors called plastic spin the $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ instead of the $\mathbf{W}^p_i$, as observed in Aravas (1994). This point will be further elaborated in Section V.1.

With the plausible (but not unique) suggestion that $\boldsymbol{\Omega}^p_i$ can be an antisymmetric tensor-valued function of the state variables $\boldsymbol{\Pi}$ and $\mathbf{A}_i$, it follows from eqns (4b), (11) and (12) that $\boldsymbol{\Omega}^p_i$ satisfies $\boldsymbol{\Omega}^p_i(\mathbf{Q}\boldsymbol{\Pi}\mathbf{Q}^{\mathrm{T}}, \mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\boldsymbol{\Omega}^p_i(\boldsymbol{\Pi}, \mathbf{A}_i)\mathbf{Q}^{\mathrm{T}}$, i.e. $\boldsymbol{\Omega}^p_i$ is an isotropic function of its argument. This was the basis for the first suggestion of a definite constitutive equation for $\mathbf{W}^p_i$ or $\boldsymbol{\Omega}^p_i$ in the case of one internal variable representing a back-stress, based on the representation theorems for antisymmetric isotropic functions (Dafalias, 1983, 1985; Loret, 1983). It is possible, however, in addition to the dependence on the state variable to have other kinematical entities (e.g. the Eulerian spin) entering directly the definition of the plastic spin (Dafalias, 1993*a*,*b*). The foregoing clearly demonstrate that no matter what is the value of $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$, constitutive equations for the plastic spin $\mathbf{W}^p_i$ (or equivalently for the constitutive spin $\boldsymbol{\omega}_i$) is a *necessity*, much as it is the provision of the constitutive terms $\dot{\gamma}_\alpha(\mathbf{m}_\alpha \otimes \mathbf{n}_\alpha)_a$ in eqn (13) for crystal plasticity.

Equation (12) can now be used to recast the constitutive rate eqns (9) and (10) for each $\mathbf{A}_i$ in the form

$$\overset{\nabla}{\mathbf{A}}_i = \dot{\mathbf{A}}_i - (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a\mathbf{A}_i + \mathbf{A}_i(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = < \lambda > \left[\bar{\mathbf{A}}_i - \boldsymbol{\Omega}^p_i\mathbf{A}_i + \mathbf{A}_i\boldsymbol{\Omega}^p_i\right] \qquad \text{(no sum on } i\text{)} \tag{14}$$

for a second order tensor, and

$$\overset{\nabla}{\mathbf{A}}_i = \dot{\mathbf{A}}_i - (\dot{\mathbf{F}}^p \mathbf{F}^{p^{-1}})_a \mathbf{A}_i =< \lambda > \left[\bar{\mathbf{A}}_i - \boldsymbol{\Omega}_i^p \mathbf{A}_i \right] \qquad \text{(no sum on } i\text{)} \tag{15}$$

for a vector. Equations (14) and (15) show clearly the role played by the plastic spin term $\boldsymbol{\Omega}_i^p$ if the classical Jaumann rate in relation to the plastic material spin at κ_u is used instead of $\boldsymbol{\omega}_i$ for the evolution of $\mathbf{A}_i$.

Since the original works on equations for the plastic spin were published, many more followed in a number large enough to prohibit comprehensive reference. It would be instructive though to comment on some of them. In a similar way to a work by Dafalias (1993*a*) for a special case of evolving orthotropic symmetries, Aravas (1994) suggested that it is often preferable to specify first the constitutive (or substructural) spin $\boldsymbol{\omega}_i$ and then obtain the plastic spin from eqn (12), given the $(\dot{\mathbf{F}}^p \mathbf{F}^{p^{-1}})_a$, and presented different examples. One interesting example addresses the case whereby an internal variable $\mathbf{A}_i$, namely a back-stress tensor, develops its principal values along the principal plastic stretch directions at κ_u associated with $\mathbf{F}^p$ in a scheme appropriate for anisotropic polymers, Parks *et al.* (1984) and Boyce *et al.* (1988*a*). It follows, therefore, that in this particular case the pertinent choice of the constitutive spin $\boldsymbol{\omega}_i$ for the back-stress is the Eulerian spin $\boldsymbol{\Omega}^E$ at κ_u associated with $\mathbf{F}^p$ (notice that $\boldsymbol{\Omega}^E$ satisfies the required eqn (11)). The corresponding plastic spin is obtained according to eqn (12) by $\mathbf{W}^p = (\dot{\mathbf{F}}^p \mathbf{F}^{p^{-1}})_a - \boldsymbol{\Omega}^E$ and can be expressed component-wise in terms of $\mathbf{D}_o^p = (\dot{\mathbf{F}}^p \mathbf{F}^{p^{-1}})_s$ rather than being defined via an isotropic function $\boldsymbol{\Omega}_i^p$ of $\boldsymbol{\Pi}$ and $\mathbf{A}_i$, since the components of $\boldsymbol{\Omega}^E$ at κ_u can be expressed in terms of the components of $(\dot{\mathbf{F}}^p \mathbf{F}^{p^{-1}})_a$ and $\mathbf{D}_o^p$ (Biot, 1965; Hill, 1970). Based on these expressions it follows that $\mathbf{W}^p = \mathbf{0}$ when $\mathbf{D}_o^p = \mathbf{0}$, while under superposed rigid body rotation one has $\mathbf{W}^{p^*} = \mathbf{Q}\mathbf{W}^p\mathbf{Q}^{\mathrm{T}}$, as it should. Hence, the fact that the orientation of an internal variable is specified directly from the geometry of the kinematics (for example the principal plastic stretch directions in the previous example), does not eliminate the existence of plastic spin constitutive relations, but perhaps renders them impractical for direct use. In many cases, though, the direct geometrical specification of the orientation of an internal variable is not possible and constitutive relations for either the plastic or the constitutive spin must be provided.

Notice though, in reference to the previous example and in contrast to it, that in general the constitutive spin of an internal variable is not necessarily the spin of its eigenvector triad. For example, in crystal plasticity an internal stress tensor may have eigenvectors which spin differently from the underlying lattice frame, the spin of the latter being the natural choice of a constitutive spin for the rate evolution equation of such internal stress tensor. The difference between the spin of eigenvectors and the constitutive spin for a back-stress tensor has been studied analytically in Dafalias (1993*a*).

Plastic and constitutive spin expressions in a macroscopic formulation were obtained analytically in 2-dimensional cases by means of exact averaging procedures and use of orientation distribution functions applied to the assemblage of micro structural elements in van der Giessen and van Houtte (1992) for single slip elements, and in Rashid (1992), Dafalias (1993*a*,*b*) and Prantil *et al.* (1993) for double slip elements. The findings in Dafalias (1993*a*,*b*) were subsequently used in multiple spin applications by Cho and Dafalias (1996), where the importance of using two different $\boldsymbol{\omega}_i$s for the evolution of two internal variables, the back-stress and the direction of orthotropy induced by texture development, was demonstrated by successful comparison of the theory with experiments.

A two-spin formulation for two superposed back-stress tensors was presented in Zbib and Aifantis (1988) and Ning and Aifantis (1994).

IV.3. *Yield criterion and stress-deformation rate relations*

The plastic multiplier λ which enters all previous constitutive eqns (8)–(10), (12), (14) and (15) and indicates by its sign the event of plastic deformation must now be specified. For the case of rate independent plasticity, the concept of a yield surface in stress space is introduced and together with its stress gradient is defined analytically at κ_u by

$$f(\mathbf{\Pi}, \mathbf{A}_i) = 0; \quad \mathbf{N}_o^n = \frac{\partial f}{\partial \mathbf{\Pi}} \tag{16}$$

Satisfaction of eqn $(16)_1$ by the state variables $\mathbf{\Pi}$ and $\mathbf{A}_i$ is a necessary condition for plastic deformation to occur, given a stress or strain increment of appropriate direction in relation to the normal to $f = 0$ represented by $\mathbf{N}_o^n$. Invariance requirements under superposed rigid body rotation $\mathbf{Q}(t)$ at κ_u, render f isotropic function of its argument.

Since f in eqn (16) and $\mathbf{\Pi} = \rho_o(\partial\Psi/\partial\mathbf{E}^e)$ in eqn (7) are scalar and tensor-valued isotropic functions of their arguments, eqns (A4) and (A8) of the Appendix apply to f and $\mathbf{\Pi}$, respectively, and yield for subsequent use the relations

$$\mathbf{\Pi}\mathbf{N}_o^n - \mathbf{N}_o^n\mathbf{\Pi} + \mathbf{A}_i\frac{\partial f}{\partial \mathbf{A}_i} - \frac{\partial f}{\partial \mathbf{A}_i}\mathbf{A}_i \equiv \mathbf{0} \tag{17}$$

for f, and

$$\left[\mathbf{E}^e\frac{\partial \mathbf{\Pi}}{\partial \mathbf{E}^e} - \frac{\partial \mathbf{\Pi}}{\partial \mathbf{E}^e}\mathbf{E}^e + \mathbf{A}_i\frac{\partial \mathbf{\Pi}}{\partial \mathbf{A}_i} - \frac{\partial \mathbf{\Pi}}{\partial \mathbf{A}_i}\mathbf{A}_i\right\} : \boldsymbol{\omega} + \mathbf{\Pi}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{\Pi} \equiv \mathbf{0} \tag{18}$$

for $\mathbf{\Pi}$, in reference to any spin $\boldsymbol{\omega}$, accounting for the symmetry of $\mathbf{\Pi}$, $\mathbf{A}_i$ and $\mathbf{E}^e$. Recall from the Introduction that summation over i due to the multiplicity of $\mathbf{A}_i$ is implied in eqns (17) and (18), and observe the interpretation given by eqn (A9) of the Appendix, for the fourth and second order tensor multiplication together with the trace operation appearing in eqn (18) and in the following. Equation (A3) of the Appendix from which eqn (17) can be derived as a corolary, was derived in Dafalias (1985), while eqns (17) and (18) were reported in a different setting by Onat (1991). Both equations will be used extensively in the following algebraic operations for simplification reasons.

Returning now to eqn (16), the standard consistency equation of plasticity is obtained by setting the rate of f equal to zero, which yields

$$\dot{f} = \frac{\partial f}{\partial \mathbf{\Pi}} : \ + \frac{\partial f}{\partial \mathbf{A}_i} : \dot{\mathbf{A}}_i = 0 \tag{19}$$

were for simplicity the $\mathbf{A}_i$ are assumed to be tensor-valued and symmetric. For the use of the trace operation symbol : in eqn (19) and in subsequent equations, instead of the symbol $\cdot$ implied by the operation of the rate derivative (see comment in Introduction), the symmetry of $\mathbf{\Pi}$ and $\mathbf{A}_i$ was employed. Using eqn (14) to substitute for $\dot{\mathbf{A}}_i$ in eqn (19), one has

$$\frac{\partial f}{\partial \boldsymbol{\Pi}} : \; + \left(\mathbf{A}_i \frac{\partial f}{\partial \mathbf{A}_i} - \frac{\partial f}{\partial \mathbf{A}_i}\mathbf{A}_i\right) : (\dot{\mathbf{F}}^p \mathbf{F}^{p^{-1}})_a + <\lambda> \left[\frac{\partial f}{\partial \mathbf{A}_i} : \bar{\mathbf{A}}_i - \left(\mathbf{A}_i \frac{\partial f}{\partial \mathbf{A}_i} - \frac{\partial f}{\partial \mathbf{A}_i}\mathbf{A}_i\right) : \boldsymbol{\Omega}_i^p\right] = 0 \tag{20}$$

where summation on i, implying repetition of operation over the indexed quantities, takes place over all terms where i appears two or three times (same holds true in subsequent expressions). Using eqn (17) to substitute $-(\boldsymbol{\Pi}\mathbf{N}_o^n - \mathbf{N}_o^n\boldsymbol{\Pi})$ for the sum $\mathbf{A}_i(\partial f/\partial \mathbf{A}_i) - (\partial f/\partial \mathbf{A}_i)\mathbf{A}_i$ in the second term of eqn (20), and observing that $(\boldsymbol{\Pi}\mathbf{N}_o^n - \mathbf{N}_o^n\boldsymbol{\Pi}) : (\dot{\mathbf{F}}^p - \mathbf{F}^{p^{-1}})_a = \mathbf{N}_o^n : [(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a\boldsymbol{\Pi} - \boldsymbol{\Pi}(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a]$, one can then solve eqn (20) for λ and obtain

$$\lambda = \frac{\mathbf{N}_o^n : \overset{\triangledown}{\boldsymbol{\Pi}}}{H + \left(\mathbf{A}_i \frac{\partial f}{\partial \mathbf{A}_i} - \frac{\partial f}{\partial \mathbf{A}_i}\mathbf{A}_i\right) : \boldsymbol{\Omega}_i^p} \tag{21}$$

with the hardening modulus H given by

$$H = -\frac{\partial f}{\partial \mathbf{A}_i} : \bar{\mathbf{A}}_i \tag{22}$$

Observe the natural emergence of $\overset{\triangledown}{\boldsymbol{\Pi}}$ (corotational rate of $\boldsymbol{\Pi}$ in relation to the plastic material spin $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ in the definition of λ, together with the plastic spin effect portrayed by the $\boldsymbol{\Omega}_i^p$ in eqn (21). Notice that the different $\boldsymbol{\Omega}_i^p$ in the implied sum over i in the denominator of eqn (21), prevents one from using eqn (17) to substitute the $\mathbf{A}_i$ terms by the simpler $\boldsymbol{\Pi}\mathbf{N}_o^n - \mathbf{N}_o^n\boldsymbol{\Pi}$ term. Only if $\boldsymbol{\Omega}_i^p$ were the same for all i, this would have been possible, as it will be discussed in the sequel (case of a single constitutive and plastic spin, as in Mandel, 1971; Dafalias, 1985).

It will be now necessary to express λ in terms of the total rate of deformation tensor $\mathbf{D}$, as a first step towards relating a proper stress rate to $\mathbf{D}$. The rate of both members of eqn (7) yields

$$= \frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{E}^e} : \dot{\mathbf{E}}^e + \frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{A}_i} : \dot{\mathbf{A}}_i + \frac{\dot{\rho}_o}{\rho}\boldsymbol{\Pi} \tag{23}$$

Using the mass conservation equation $\dot{\rho}_o + \rho_o \mathrm{tr}\mathbf{D}_o^p = 0$ at κ_u together with $\mathbf{D}_o^p = <\lambda> \mathbf{N}_o^p$, substituting $\dot{\mathbf{A}}_i$ from eqn (14) in (23) and using eqn (18) with $\boldsymbol{\omega} = (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ to express the sum $\mathbf{A}_i(\partial\boldsymbol{\Pi}/\partial\mathbf{A}_i) - (\partial\boldsymbol{\Pi}/\partial\mathbf{A}_i)\mathbf{A}_i$ over i in terms of the other quantities of eqn (18) for the above choice of $\boldsymbol{\omega}$, one has after some algebra that

$$\overset{\triangledown}{\boldsymbol{\Pi}} = \frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{E}^e} : \overset{\triangledown}{\mathbf{E}} + <\lambda> \left[\frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{A}_i} : \bar{\mathbf{A}}_i - \left(\mathbf{A}_i \frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{A}_i} - \frac{\partial \boldsymbol{\Pi}}{\partial \mathbf{A}_i}\mathbf{A}_i\right) : \boldsymbol{\Omega}_i^p - \boldsymbol{\Pi}\mathrm{tr}\mathbf{N}_o^p\right] \tag{24}$$

After substituting $\overset{\triangledown}{\mathbf{E}}$ from eqn (5) in (24) and using $\mathbf{D}_o^p = <\lambda> \mathbf{N}_o^p$, one can form the trace $\mathbf{N}_o^n : \overset{\triangledown}{\boldsymbol{\Pi}}$ which can be solved for λ accounting for the definition of the latter in eqn (21), and have

$$\lambda = \frac{\mathbf{N}_o^n : L_o : \ddot{\mathbf{F}}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{F}^e}{H + \mathbf{N}_o^n : \left[L_o : \left(\mathbf{C}^e\mathbf{N}_o^p\right)_s + \boldsymbol{\Pi}\mathrm{tr}\mathbf{N}_o^p - C_i : \bar{\mathbf{A}}_i + (\mathbf{A}_iC_i - C_i\mathbf{A}_i) : \boldsymbol{\Omega}_i^p\right] + \left(\mathbf{A}_i \frac{\partial f}{\partial \mathbf{A}_i} - \frac{\partial f}{\partial \mathbf{A}_i}\mathbf{A}_i\right) : \boldsymbol{\Omega}_i^p} \tag{25}$$

where for simplicity the notations

$$L_o = \frac{\partial \mathbf{\Pi}}{\partial \mathbf{E}^e} = \rho_o \frac{\partial^2 \Psi}{\partial \mathbf{E}^e \otimes \partial \mathbf{E}^e} \tag{26a}$$

$$C_i = \frac{\partial \mathbf{\Pi}}{\partial \mathbf{A}_i} = \rho_o \frac{\partial^2 \Psi}{\partial \mathbf{E}^e \otimes \partial \mathbf{A}_i} \tag{26b}$$

were introduced for the fourth order tensors L_o and C_i. The L_o is the usual tensor of tangent elastic moduli, while the C_i represents the elastoplastic coupling due to the presence and evolution of the $\mathbf{A}_i$s entering Ψ. Notice that even if some of the $\mathbf{A}_i$s are purely orientational in nature, i.e. $\bar{\mathbf{A}}_i \equiv \mathbf{0}$, still corresponding C_is, associated with the $\mathbf{\Omega}_i^p$s, exist in eqn (25).

Substitution of eqn (25) in (24) and use again of eqn (5) yields finally the stress-deformation rate relation as

$$\overset{\triangledown}{\mathbf{\Pi}} = \Lambda : \mathbf{F}^{e^{\mathrm{T}}} \mathbf{D} \mathbf{F}^e \tag{27}$$

where the fourth order tensor of elastoplastic moduli Λ is given by

$$\Lambda = L_o - \frac{\left[L_o : \left(\mathbf{C}^e \mathbf{N}_o^p\right)_s + \mathbf{\Pi} \mathrm{tr} \mathbf{N}_o^p - C_i : \bar{\mathbf{A}}_i + (\mathbf{A}_i C_i - C_i \mathbf{A}_i) : \mathbf{\Omega}_i^p\right] \otimes \left[\mathbf{N}_o^n : L_o\right]}{H + \mathbf{N}_o^n : \left[L_o : \left(\mathbf{C}^e \mathbf{N}_o^p\right)_s + \mathbf{\Pi} \mathrm{tr} \mathbf{N}_o^p - C_i : \bar{\mathbf{A}}_i + (\mathbf{A}_i C_i - C_i \mathbf{A}_i) : \mathbf{\Omega}_i^p\right] + \left(\mathbf{A}_i \frac{\partial f}{\partial \mathbf{A}_i} - \frac{\partial f}{\partial \mathbf{A}_i} \mathbf{A}_i\right) : \mathbf{\Omega}_i^p} \tag{28}$$

The reason for the considerably complex form of Λ in eqn (28) is on the one hand the fact that $C_i \neq 0$ because of the inclusion of $\mathbf{A}_i$s in Ψ, and on the other hand the existence of multiple $\mathbf{\Omega}_i^p$. In relation to eqn (28) and all subsequent equations where different forms of Λ appear, it is implied that when elastic unloading occurs and $\lambda \leq 0$ (see definition of λ in eqn (25)), it follows that $\Lambda = L_o$.

Often elastic isotropy is assumed, even if plasticity is anisotropic. In this case all the $\mathbf{A}_i$s which may enter the strain energy density function Ψ must be scalar-valued, portraying a change of elastic properties which does not alter their isotropy (e.g. isotropic damage). Then, the corresponding $\bar{\mathrm{A}}_i$s are also scalar-valued (recall eqns (9) and (10) with $\boldsymbol{\omega}_i = \mathbf{0}$), and the associated C_is become second order tensors (eqn (26b)). As a result, the $C_i : \bar{\mathbf{A}}_i$ terms in eqn (28) yield scalar multiplications of tensors such as $\bar{\mathrm{A}}_i C_i$, while the $(\mathbf{A}_i C_i - C_i \mathbf{A}_i) : \mathbf{\Omega}_i^p$ terms disappear. In fact $\mathbf{\Omega}_i^p$ has no meaning for a scalar-valued A_i. Notice, however, that the $\mathbf{A}_i$s and $\mathbf{\Omega}_i^p$s associated with the $\partial f / \partial \mathbf{A}_i$ terms remain in general in eqn (28), since the $\mathbf{A}_i$s entering eqn (16) of the yield surface are not necessarily the same as the ones entering Ψ, and can be tensor-valued. In the simpler case of unchanged isotropic elastic properties, no $\mathbf{A}_i$ scalar or tensor-valued enters Ψ, hence, all the C_is are zero (eqn (26b)) and eqn (28) simplifies accordingly.

If the material remains elastically and plastically isotropic, it follows that all the $\mathbf{A}_i$s entering Ψ, f and $\mathbf{N}_o^p$ are scalar valued, which implies that $\mathbf{\Omega}_i^p = \mathbf{0}$ for all i, and that $\mathbf{C}^e$, $\mathbf{N}_o^n$ and $\mathbf{N}_o^p$ commute having the same eigenvectors as $\mathbf{\Pi}$ (recall their isotropic dependence on $\mathbf{\Pi}$ and the now scalar-valued $\mathbf{A}_i$). If in addition Ψ depends only on $\mathbf{E}^e$ (no isotropic damage) and the plastic rate of deformation is traceless, one has $C_i = 0$ and $\mathrm{tr} \mathbf{N}_o^p = 0$, respectively. Hence, the elastoplastic tangent moduli Λ of eqn (28) simplifies considerably, with the numerator of the right-hand side of eqn (28) becoming $L_o : \mathbf{N}_o^p\mathbf{C}^e \otimes \mathbf{N}_o^n : L_o$, and the denominator $H + \mathbf{N}_o^n : L_o : \mathbf{N}_o^p\mathbf{C}^e$.

Consider now the particular case where all $\boldsymbol{\omega}_i$ are equal to a common $\boldsymbol{\omega}_o$, hence, $\boldsymbol{\Omega}_i^p = \boldsymbol{\Omega}_o^p$ for all $\mathbf{A}_i$. Recalling the definition of L_o and C_i from eqn (26), using the key eqns (17) and (18) and the relation $(\mathbf{E}^e(\partial\boldsymbol{\Pi}/\partial\mathbf{E}^e) - (\partial\boldsymbol{\Pi}/\partial\mathbf{E}^e)\mathbf{E}^e) : \boldsymbol{\Omega}_o^p = -(\partial\boldsymbol{\Pi}/\partial\mathbf{E}^e) : (\mathbf{C}^e\boldsymbol{\Omega}_o^p)_s$ in the expressions (21), (25) and (28), one finally has

$$\lambda = \frac{\mathbf{N}_o^n : \overset{\nabla}{\boldsymbol{\Pi}}}{H + \mathbf{N}_o^n : (\boldsymbol{\Pi}\boldsymbol{\Omega}_o^p - \boldsymbol{\Omega}_o^p\boldsymbol{\Pi})} = \frac{\mathbf{N}_o^n : L_o : \mathbf{F}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{F}^e}{H + \mathbf{N}_o^n : \left[L_o : \left(\mathbf{C}^e\left(\mathbf{N}_o^p + \boldsymbol{\Omega}_o^p\right)\right)_s + \boldsymbol{\Pi}\mathrm{tr}\mathbf{N}_o^p - C_i : \bar{\mathbf{A}}_i\right]} \tag{29}$$

$$\Lambda = L_o - \frac{\left[L_o : \left(\mathbf{C}^e\left(\mathbf{N}_o^p + \boldsymbol{\Omega}_o^p\right)\right)_s + \boldsymbol{\Pi}\mathrm{tr}\mathbf{N}_o^p - C_i : \bar{\mathbf{A}}_i + 2\left(\boldsymbol{\Omega}_o^p\boldsymbol{\Pi}\right)_s\right] \otimes \left[\mathbf{N}_o^n : L_o\right]}{H + \mathbf{N}_o^n : \left[L_o : \left(\mathbf{C}^e\left(\mathbf{N}_o^p + \boldsymbol{\Omega}_o^p\right)\right)_s + \boldsymbol{\Pi}\mathrm{tr}\mathbf{N}_o^p - C_i : \bar{\mathbf{A}}_i\right]} \tag{30}$$

with eqn (27) still being valid. Observe that now the C_is appear in conjunction with hardening $\mathbf{A}_i$s only, i.e. when $\bar{\mathbf{A}}_i \neq \mathbf{0}$. If one symbolizes by a superposed o the corotational rate in relation to $\boldsymbol{\omega}_o$, it easily follows based on $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \boldsymbol{\omega}_o + < \lambda > \boldsymbol{\Omega}_o^p$ that $\overset{\nabla}{\boldsymbol{\Pi}} - \overset{\circ}{\boldsymbol{\Pi}} = < \lambda > (\boldsymbol{\Pi}\boldsymbol{\Omega}_o^p - \boldsymbol{\Omega}_o^p\boldsymbol{\Pi})$. Hence, instead of eqn (27) one can have the relation

$$\overset{\circ}{\boldsymbol{\Pi}} = \Lambda_o : \mathbf{F}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{F}^e \tag{31}$$

where Λ_o is given by eqn (30) without the term $2(\boldsymbol{\Omega}_o^p\boldsymbol{\Pi})_s$ in the first bracketed quantity of the numerator. Also in this case the expression for λ in eqn (29) in terms of the new stress rate becomes $\lambda = (\mathbf{N}_o^n : \overset{\circ}{\boldsymbol{\Pi}})/H$, while the second expression in terms of $\mathbf{D}$ remains as is.

The rate independence of eqns (27) and (28) is based on the derivation of λ in eqn (21) or eqn (25) in association with a yield surface, eqn (16). In rate dependent theories, the plastic multiplier λ represents simply a scalar-valued overstress isotropic function of $\boldsymbol{\Pi}$ and $\mathbf{A}_i$, related to a so-called "dynamic" yield criterion in stress space without the need of satisfying the consistency eqn (19) (Dafalias, 1990). Denoting then by z the $< \lambda >$, the relation between stress rate and rate of deformation is obtained from eqn (24) with the substitution of $\overset{\nabla}{\mathbf{E}}{}^e$ from eqn (5) to yield

$$\overset{\nabla}{\boldsymbol{\Pi}} = L_o : \mathbf{F}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{F}^e - z\left[L_o : (\mathbf{C}^e\mathbf{N}_o^p)_s + \boldsymbol{\Pi}\mathrm{tr}\mathbf{N}_o^p - C_i : \bar{\mathbf{A}}_i + (\mathbf{A}_iC_i - C_i\mathbf{A}_i) : \boldsymbol{\Omega}_i^p\right] \tag{32}$$

It is important to emphasize that z substitutes $< \lambda >$ in all other pertinent equations, such as eqns (8)–(10), (12), (14) and (15). A properly chosen form of z, whereby the overstress concept uses a zero stress as reference, can accommodate smoothened out yield conditions, e.g. the power law which is useful in practical large strain crystal and polycrystal plasticity calculations.

Notice finally that the $\mathbf{F}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{F}^e$ is work-conjugate to $\boldsymbol{\Pi}$ at the unstressed configuration κ_u since with $|\mathbf{F}^e| = \rho_o/\rho$, where ρ_o and ρ are the mass densities at κ_u and κ, respectively, and the definition of $\boldsymbol{\Pi}$ from eqn (6), it follows that

$$\frac{\boldsymbol{\Pi} : (\mathbf{F}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{F}^e)}{\rho_o} = \frac{\boldsymbol{\sigma} : \mathbf{D}}{\rho} \tag{33}$$

Observe that throughout the preceding development it was not necessary to decompose a total strain measure or the total rate of deformation $\mathbf{D}$ into elastic and plastic parts. Elasticity was defined directly from eqn (7) without the need to introduce any specific stress rate, while the plastic rate of deformation was unambiguously defined at κ_u by eqn (8), and its presence is expressed by $\mathbf{N}_o^p$ in the final equations.

V. ISSUES RELATED TO THE CHOICE OF CONFIGURATIONS

V.1. *Spinless and isoclinic configurations*

So far the choice of the intermediate unstressed configuration κ_u was left indeterminate within a rigid body rotation. Equivalent to this indeterminacy is the fact that the plastic material spin $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ at κ_u was unspecified, while recall that $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_s = \mathbf{D}_o^p$ was given by the constitutive eqn (8). The arbitrariness in choosing κ_u or equivalently specifying $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$, a fact also recognized in Lubarda and Shih (1994), was accounted for in the formulation by the natural appearance of the corotational rates $\overset{\nabla}{\mathbf{\Pi}}$ and $\overset{\nabla}{\mathbf{A}}_i$ in relation to $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ in all relevant constitutive equations. In other words, one can choose *any* value he desires for $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ as long as the key eqn (12) reflects this choice onto $\boldsymbol{\omega}_i$ and $\mathbf{W}_i^p$ for each $\mathbf{A}_i$.

The forms of the development for different prescribed values of $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ are entirely equivalent. The only effect the different choices for $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ will have is a rotational difference in $\mathbf{F}^p$ and $\mathbf{F}^e$ for the same $\mathbf{F}$, e.g. $\mathbf{F}^{p^*} = \mathbf{Q}\mathbf{F}^p$ and $\mathbf{F}^{e^*} = \mathbf{F}^e\mathbf{Q}^{\mathrm{T}}$. As already noticed in relation to invariance under superposed rigid body rotation at κ_u, the foregoing will simply induce a difference by rotation compatible with the $\mathbf{F}^{p^*} = \mathbf{Q}\mathbf{F}^p$, i.e. $\mathbf{\Pi}^* = \mathbf{Q}\mathbf{\Pi}\mathbf{Q}^{\mathrm{T}}$, $\mathbf{A}_i^* = \mathbf{Q}\mathbf{A}_i\mathbf{Q}^{\mathrm{T}}$, $\mathbf{E}^{e^*} = \mathbf{Q}\mathbf{E}^e\mathbf{Q}^{\mathrm{T}}$, etc. The functional form of Ψ, f, $\mathbf{N}_o^p$ and $\mathbf{\Omega}_i^p$ will remain the same since they are constitutive functions. But it is of cardinal importance to observe that it is their isotropic dependence on $\mathbf{\Pi}$, $\mathbf{A}_i$ and $\mathbf{E}^e$ which makes inconsequential (within a rotation) which is the choice of the relaxed configuration or of $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ (Dafalias, 1988; Aravas, 1994).

Different authors (Nemat-Nasser, 1990, 1992; Obata *et al*., 1990; Onat, 1991) have considered special choices of κ_u which were associated with either a symmetric elastic deformation gradient $\mathbf{F}^e = \mathbf{F}^{e^{\mathrm{T}}}$, or a symmetric plastic deformation gradient $\mathbf{F}^p = \mathbf{F}^{p^{\mathrm{T}}}$. Due to these choices they were able to express the $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ in terms of $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_s = \mathbf{D}_o^p$, and identifying the $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ with the plastic spin (i.e. calling by the name of ''plastic spin'' a totally different quantity than what is defined as plastic spin here), they concluded that no constitutive relation is necessary for the latter since it can be expressed in terms of $\mathbf{D}_o^p$. They totally ignored, in reference to eqn (12), the distinction between the antisymmetric part of the plastic velocity gradient $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ (here also called the plastic material spin) and the correct concept of plastic spin $\mathbf{W}_i^p$. Their extensive effort to express the former in terms of $\mathbf{D}_o^p$ is but a particular case of choosing any value one desires for $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$, as mentioned before. Clearly, no matter what the choice for $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$, one is forced to additionally specify either $\boldsymbol{\omega}_i$ or $\mathbf{W}_i^p$ so that eqn (12) is satisfied, and the proper corotational rate is used for the evolution of $\mathbf{A}_i$ in relation to $\boldsymbol{\omega}_i$.

In fact a natural corollary of this free choice is to opt for the simplest case of setting $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a \equiv \mathbf{0}$, thus, choosing what one can call a ''spinless'' unstressed configuration. An allusion to this kind of free choice and its inconsequence for the final formulation was already made in Boyce *et al*. (1988*b*) for a specific constitutive model related to glossy polymers and a description using principal stretch directions. Moreover, here it is proved that such arbitrary choice, including the spinless configuration, can be made inconsequentially for any model whose state is defined in terms of $\boldsymbol{\Pi}$ and $\mathbf{A}_i$, and which due to invariance under rotation results in having all constitutive functions isotropic in their arguments, a case which includes the foregoing suggestion by Boyce *et al.* (1988*b*). The spinless configuration choice simplifies eqns (14), (15), (21), (27) and (29) by changing the corotational rates $\overset{\triangledown}{\boldsymbol{\Pi}}$ and $\overset{\triangledown}{\mathbf{A}}_i$ to regular rates and $\dot{\mathbf{A}}_i$. Then, together with eqns (3) and (8) it follows that

$$\dot{\mathbf{F}}\mathbf{F}^{p^{-1}} =< \lambda > \mathbf{N}_o^p \tag{34}$$

$$\dot{\mathbf{A}}_i =< \lambda > \left[\bar{\mathbf{A}}_i - \boldsymbol{\Omega}_i^p\mathbf{A}_i + \mathbf{A}_i\boldsymbol{\Omega}_i^p\right] \qquad \text{(no sum on } i) \tag{35}$$

$$= \Lambda : \mathbf{F}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{F}^e \tag{36}$$

Equations (34)–(36) are in fact all that is needed for the implementation of the theory. The Λ is still given by eqn (28) while eqns (34) and (35) are used for the updating of $\mathbf{F}^p$ and $\mathbf{A}_i$, the former because $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a \equiv \mathbf{0}$.

Mandel (1971) introduced another unstressed configuration, the isoclinic, by setting the constitutive spin $\boldsymbol{\omega}_i \equiv \mathbf{0}$ in eqn (12), with two consequences. First, the $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \mathbf{W}_i^p =< \lambda > \boldsymbol{\Omega}_i^p$ as a result of the choice $\boldsymbol{\omega}_i \equiv \mathbf{0}$. Second, eqns (9) and (10) yield $\dot{\mathbf{A}}_i =< \lambda > \bar{\mathbf{A}}_i$, i.e. the frame of reference for $\mathbf{A}_i$ which spins by $\boldsymbol{\omega}_i$ remains fixed, hence, the name isoclinic which based on its greek root means "of equal inclination" (with time). In the case of multiple spins, though, presented in this work, one is faced with the necessity to introduce as many isoclinic configurations as there are $\mathbf{A}_i$s and associated $\boldsymbol{\omega}_i$s, hence, to use different values for $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ for each $\boldsymbol{\omega}_i \equiv \mathbf{0}$. This is of course impractical and, in fact, Mandel introduced the notion of the isoclinic configuration because in his theory only one common constitutive spin, the spin of the director vectors $\boldsymbol{\omega}_o = \boldsymbol{\omega}_i$ for all i was considered and set equal to zero. Then, indeed, the notion of the isoclinic configuration is a useful one, while for multiple spins the spinless configuration is the simplest choice, appropriate also for a single spin theory.

The analytical description of the isoclinic configuration can easily be retrieved from eqns (3), (8)–(10), (12) and (31). Since all $\boldsymbol{\omega}_i = \boldsymbol{\omega}_o = \mathbf{0}$, it follows that $\overset{c}{\mathbf{A}}_i$ in eqns (9) and (10) becomes $\dot{\mathbf{A}}_i$ and that all $\mathbf{W}_i^p = \mathbf{W}_o^p =< \lambda > \boldsymbol{\Omega}_o^p = (\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$. In addition, eqn (31) which was derived on the basis of $\boldsymbol{\omega}_i = \boldsymbol{\omega}_o$ for all i, is also valid for the particular case $\boldsymbol{\omega}_o = \mathbf{0}$ which changes $\overset{\circ}{\boldsymbol{\Pi}}$ to . Hence, for the isoclinic configuration one has

$$\dot{\mathbf{F}}\mathbf{F}^{p^{-1}} =< \lambda > \left(\mathbf{N}_o^p + \boldsymbol{\Omega}_o^p\right) \tag{37}$$

$$\dot{\mathbf{A}}_i =< \lambda > \bar{\mathbf{A}}_i \tag{38}$$

$$= \Lambda_o : \mathbf{F}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{F}^e \tag{39}$$

A comparison of the sets of eqns (34)–(36) and (37)–(39) shows the similarities and differences between the choice of the spinless and the isoclinic unstressed configuration (notice difference in Λ and Λ_o). Remember, however, that the isoclinic configuration choice requires a common constitutive and plastic spin quantity (Mandel, 1971), while the spinless choice is valid in the presence of different constitutive and plastic spins for different $\mathbf{A}_i$s. Finally, it is instructive to show with an example the invariance of the plastic spin expression while the $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ can be chosen in different ways. If $\mathbf{n}$ is a unit vector which remains along a material line element at κ_u during plastic deformation, its rate is given by a standard continuum mechanics derivation as

$$\dot{\mathbf{n}} = \left[(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a + \mathbf{D}^p_o(\mathbf{n}\otimes\mathbf{n}) - (\mathbf{n}\otimes\mathbf{n})\mathbf{D}^p_o\right]\mathbf{n} = \boldsymbol{\omega}_i\mathbf{n} \tag{40}$$

where $\boldsymbol{\omega}_i$ equals the bracketed quantity of the second member of eqn (40) and is the constitutive spin for $\mathbf{n}$. For the spinless configuration choice, $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \mathbf{0}$, and it follows from eqns (12) and (40) that $\mathbf{W}^p_i = -\boldsymbol{\omega}_i = -\left[\mathbf{D}^p_o(\mathbf{n}\otimes\mathbf{n}) - (\mathbf{n}\otimes\mathbf{n})\mathbf{D}^p_o\right]$. For the isoclinic configuration choice, one has $\boldsymbol{\omega}_i = \boldsymbol{\omega}_o = \mathbf{0}$, and from eqn (12) it follows that $\mathbf{W}^p_i = \mathbf{W}^p_o = (\dot{\mathbf{F}}^p\mathbf{F}^p)_a = -\left[\mathbf{D}^p_o(\mathbf{n}\otimes\mathbf{n}) - (\mathbf{n}\otimes\mathbf{n})\mathbf{D}^p_o\right]$, same as before. This is because any choice of $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$, or equivalently of the relaxed configuration, reflects into $\boldsymbol{\omega}_i$ and vice versa, letting the plastic spin $\mathbf{W}^p_i$ maintain the same functional form. The significance of eqn (40) in applications can be seen in relation to a finite plastic deformation theory for fibrous metal matric composites by Fares and Dvorak (1991), where $\mathbf{n}$ is along a fiber and no shear can occur in directions normal to it.

V.2. *Computational aspects*

A tangent stiffness matrix numerical approach can be obtained as a straightforward application of the set of eqns (34)–(36) or eqns (37)–(39) in the particular cases of spinless and isoclinic configurations, respectively. Alternatively one can consider λ as an unknown to be specified by the satisfaction of the yield criterion with the updated variables, and is closely related to the previous subsection on the choice of the relaxed configuration. Following the method of Aravas (1992, 1994) which was applied to the isoclinic configuration, the statement of the problem and its solution in relation to eqns (7), (34) and (35) for the spinless, instead of the isoclinic, relaxed configuration is as follows.

At a material point the solution $\boldsymbol{\Pi}^{(n)}$, $\mathbf{A}_i^{(n)}$, $\mathbf{F}^p_n$ and $\mathbf{F}_n$ at time t_n is known. For given $\mathbf{F}_{n+1}$ at time $t_{n+1} = t_n + \Delta t$, one must determine $\boldsymbol{\Pi}^{(n+1)}$, $\mathbf{A}_i^{(n+1)}$ and $\mathbf{F}^p_{n+1}$. The starting point is eqn (34) for the spinless choice which yields

$$\dot{\mathbf{F}}^p = < \lambda > \mathbf{N}^p_o\mathbf{F}^p \tag{41}$$

Assuming the plastic flow direction $\mathbf{N}^p_o$ constant over the increment and equal to $(\mathbf{N}^p_o)_n \equiv \mathbf{X}_n$, eqn (41) can be integrated to yield:

$$\mathbf{F}^{p^{-1}}_{n+1} = \mathbf{F}^{p^{-1}}_n \exp(-\delta\lambda\mathbf{X}_n) = \mathbf{F}^{p^{-1}}_n\left[\mathbf{I} - \delta\lambda\mathbf{X}_n + \frac{1}{2}(\delta\lambda)^2\mathbf{X}^2_n + 0(\delta\lambda)^3\right] \tag{42}$$

which truncated yields

$$\mathbf{F}^{p^{-1}}_{n+1} = \mathbf{F}^{p^{-1}}_n\left(\mathbf{I} - \delta\lambda\mathbf{X}_n + \frac{1}{2}(\delta\lambda)^2\mathbf{X}^2_n\right) \tag{43}$$

The algorithm then becomes

$$\mathbf{F}^e_{n+1} = \mathbf{F}_{n+1}\mathbf{F}^{p^{-1}}_n\left(\mathbf{I} - \delta\lambda\mathbf{X}_n + \frac{1}{2}(\delta\lambda)^2\mathbf{X}^2_n\right) \tag{44a}$$

$$\mathbf{F}^p_{n+1} = \mathbf{F}_{n+1}\mathbf{F}^{e^{-1}}_{n+1} \tag{44b}$$

$$\mathbf{E}^e_{n+1} = \frac{1}{2}\left(\mathbf{F}^{e^{\mathrm{T}}}_{n+1}\mathbf{F}^e_{n+1} - \mathbf{I}\right) \tag{44c}$$

$$\mathbf{\Pi}^{(n+1)} = \rho_0(\partial\Psi/\partial\mathbf{E}^e)_{n+1} \tag{44d}$$

$$\mathbf{A}^{(n+1)}_i = \mathbf{A}^{(n)}_i + \delta\lambda\left[\bar{\mathbf{A}}^{(n)}_i - \mathbf{\Omega}^{p(n)}_i\mathbf{A}^{(n)}_i + \mathbf{A}^n_i\mathbf{\Omega}^{p(n)}_i\right] \tag{44e}$$

$$f\left(\mathbf{\Pi}^{(n+1)}, \mathbf{A}^{(n+1)}\right) = 0 \tag{44f}$$

where $\bar{\mathbf{A}}^{(n)}_i$ and $\mathbf{\Omega}^{p(n)}_i$ are functions of $\mathbf{\Pi}^{(n)}$ and $\mathbf{A}^{(n)}_i$ of the nth step. The last eqn (44f) is in fact the one that provides the value of $\delta\lambda$, necessary for the construction of the algorithm. If the same approach were applied in the case of the isoclinic configuration for a single spin in relation to eqns (7), (37) and (38), the difference would be that now $\mathbf{X}_n = \left(\mathbf{N}^p_o + \mathbf{\Omega}^p_o\right)_n$ and the $\mathbf{\Omega}^{p(n)}_i = \mathbf{\Omega}^{p(n)}_o$ terms would be absent from eqn (44e), according to eqn (38) (Aravas, 1994).

It is clear from the foregoing that no stress rate was used in the process, and elasticity was introduced straightforwardly via the hyperplastic relation, eqn (44d). Notice also that incremental objectivity requirements were not necessary due to the choice of the spinless configuration and absence of corotational rates from eqns (34) and (35). However, a more careful consideration of eqn (35) and its numerical counterpart eqn (44e) together with the observation that $\mathbf{W}^{p(n)}_i = \delta\lambda\mathbf{\Omega}^{p(n)}_i$, shows that eqn (44e) incorporates in the updating process a corotational rate of $\mathbf{A}_i$ in relation to the corresponding plastic spin, and implies that the $\mathbf{\Omega}^{p(n)}_i$ is considered constant over the increment. But also one must observe that the order of magnitude of $\mathbf{W}^p_i$ is the same as $\mathbf{D}^p_o$, hence the same applies for $\mathbf{\Omega}^p_i$ and $\mathbf{N}^p_o$ (recall eqns (8) and (12)). If one then considers $\mathbf{N}^p_o$ constant over the increment, it must allow the same approximation for $\mathbf{\Omega}^p_i$. It would have been a different matter if the spin in eqn (44e) was not the plastic spin, but the plastic material spin $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a$ which can be of much larger order than $\mathbf{D}^p_o$ hence, not accurate to be considered constant over the increment, in which case one must seek the satisfaction of incremental objectivity. It is exactly this point that one avoids by choosing the spinless configuration.

V.3. *Yield surface from current to unstressed configuration*

One aspect of the formulation which simplified the formulation is the consideration of the analytical expression, eqn (16), for the yield surface at the unstressed configuration in terms of $\mathbf{\Pi}$ and $\mathbf{A}_i$. In Dafalias (1985, 1987, 1988) the yield surface was first defined in the current configuration by $f(\boldsymbol{\sigma}, \mathbf{a}_i) = 0$ and the formulation accounted for the elastic embedding of the $\mathbf{a}_i$, e.g. $\mathbf{a}_i = |\mathbf{F}^e|^{-1}\mathbf{F}^e\mathbf{A}_i\mathbf{F}^{e^{\mathrm{T}}}$, hence, became much more complex. If the question was just one of convenience, then certainly eqn (16) is the most convenient. What is at stake, though, is the physical significance of stating for example an equation like eqn (16) for a Mises type yield condition with kinematic and isotropic hardening, directly at the unstressed configuration as

$$f = \frac{3}{2}\mathrm{tr}(\mathbf{\Pi}' - \mathbf{A}')^2 - k^2 = 0 \tag{45}$$

where tr means the trace, in terms of the deviatoric part $\mathbf{\Pi}'$ of $\mathbf{\Pi}$ and the deviatoric part $\mathbf{A}'$ of a back-stress tensor $\mathbf{A}$. If eqn (45) is accepted, the rest follows in a very straightforward manner. If, on the other hand, one must state the Mises type yield condition in terms of the deviatoric Cauchy stress $\boldsymbol{\sigma}'$ and the deviatoric part $\boldsymbol{\alpha}$ of a back-stress $\mathbf{a}$ defined at the current configuration, it takes some algebra and the way $\boldsymbol{\sigma}'$ and $\boldsymbol{\alpha}$ are related to $\mathbf{\Pi}$ and $\mathbf{A}$, e.g. $\boldsymbol{\sigma}' = |\mathbf{F}^e|^{-1}\mathbf{F}^e[\mathbf{\Pi} - (1/3)\mathbf{C}^{e^{-1}}\mathrm{tr}(\mathbf{C}^e\mathbf{\Pi})]\mathbf{F}^{e^{\mathrm{T}}}$ and similarly for $\boldsymbol{\alpha}$ and $\mathbf{A}$, to have instead of eqn (45) the expression

$$f = \frac{3}{2}\mathrm{tr}(\boldsymbol{\sigma}' - \boldsymbol{\alpha})^2 - k^2 = \frac{3}{2}|\mathbf{C}^e|^{-1}\left[\mathrm{tr}(\mathbf{C}^e(\mathbf{\Pi} - \mathbf{A}))^2 - \frac{1}{3}\mathrm{tr}^2(\mathbf{C}^e(\mathbf{\Pi} - \mathbf{A}))\right] - k^2 = 0 \quad (46)$$

The main complexity arising from eqn (46) is not just the appearance of $\mathbf{C}^e$, but the fact that if one wants to use eqn (46) as a particular case of eqn (16) together with the mechanism of formulation that follows, he must express f exclusively in terms of $\mathbf{\Pi}$ and $\mathbf{A}$. It means that $\mathbf{C}^e$ must also be considered a function of $\mathbf{\Pi}$ and $\mathbf{A}$, in general, found by inverting the dependence of $\mathbf{\Pi}$ on $\mathbf{E}^e = (1/2)(\mathbf{C}^e - \mathbf{I})$ and possibly $\mathbf{A}$. Then, the expressions for $\partial f/\partial\mathbf{\Pi}$ and $\partial f/\partial\mathbf{A}_i$ must include also the $\partial\mathbf{C}^e/\partial\mathbf{\Pi}$ and $\partial\mathbf{C}^e/\partial\mathbf{A}_i$ terms, which indeed complicates the actual application. It is complex but not impossible. As an approximation, particularly when elastic deformations are small, one can either use directly eqn (45) or eqn (46) with $\mathbf{C}^e$ a constant quantity when considering the $\partial f/\partial\mathbf{\Pi}$ and $\partial f/\partial\mathbf{A}_i$. In the latter case it is straightforward to show that $\mathbf{N}^n_o = \partial f/\partial\mathbf{\Pi} = |\mathbf{F}^e|^{-1}\mathbf{F}^{e^{\mathrm{T}}}\mathbf{N}^n\mathbf{F}^e$, with $\mathbf{N}^n = \partial f/\partial\boldsymbol{\sigma} = 3(\boldsymbol{\sigma}' - \boldsymbol{\alpha})$.

V.4. *Small elastic strains*

In the case of small elastic strains of order ε, one has that $\mathbf{F}^e = \mathbf{V}^e\mathbf{R}^e = (\mathbf{I} + \varepsilon\mathbf{V}')\mathbf{R}^e \simeq \mathbf{R}^e$ with $\mathbf{V}'$ being of order 1, if terms of order ε are neglected in comparison to 1 (a more detailed consideration of small elastic strain linearization can be found in Aravas (1992)). It then follows that $\mathbf{\Pi} = \mathbf{R}^{e^{\mathrm{T}}}\boldsymbol{\sigma}\mathbf{R}^e$ and from eqn (27) that

$$\overset{\triangledown}{\mathbf{\Pi}} = \Lambda : \mathbf{R}^{e^{\mathrm{T}}}\mathbf{D}\mathbf{R}^e \quad (47)$$

with Λ obtained from eqn (28) as

$$\Lambda = L_o - \frac{(L_o : \mathbf{N}^p_o) \otimes (\mathbf{N}^n_o : L_o)}{H + \mathbf{N}^n_o : L_o : \mathbf{N}^p_o} \quad (48)$$

The simplified form of Λ in eqn (48) is obtained by neglecting in eqn (28) all terms of order ε in comparison with 1 (recall that L_{o} is of the order $\boldsymbol{\sigma}/\varepsilon$), and using $\mathbf{C}^e = \mathbf{I}$. Equations (14) and (15) remain, since they are not affected by the elastic part of the deformation.

The choice of spinless relaxed configuration changes $\overset{\triangledown}{\mathbf{\Pi}}$ to in eqn (47), while eqn (35) substitutes for eqn (14). Also, one can choose the current configuration as relaxed within order ε. In this case $\mathbf{R}^e = \mathbf{I}, \mathbf{\Pi} = \boldsymbol{\sigma}$, and since $\mathbf{F} \simeq \mathbf{F}^p$ within order ε it follows that $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \mathbf{W}$, hence, $\overset{\triangledown}{\mathbf{\Pi}} = \overset{\triangledown}{\boldsymbol{\sigma}}$ the $\overset{\triangledown}{\boldsymbol{\sigma}}$ being the classical Jaumann corotational rate of $\boldsymbol{\sigma}$ in reference to $\mathbf{W}$. Then, eqn (47) becomes $\overset{\triangledown}{\boldsymbol{\sigma}} = \Lambda : \mathbf{D}$ with Λ given by eqn (48), which is the classical result obtained in Dafalias (1988, 1993*a* with a change of notation).

Along the same line of reasoning, and with $\mathbf{A}_i \simeq \mathbf{a}_i$ and $\bar{\mathbf{A}}_i \simeq \bar{\mathbf{a}}_i$ for the choice of $\mathbf{F}^e \simeq \mathbf{R}^e = \mathbf{I}$, one has in lieu of eqn (14)

$$\overset{\triangledown}{\mathbf{a}}_i = <\lambda> (\bar{\mathbf{a}}_i - \boldsymbol{\Omega}_i^p \mathbf{a}_i + \mathbf{a}_i \boldsymbol{\Omega}_i^p) \text{ (no sum on } i) \tag{49}$$

where again the corotational rate $\overset{\triangledown}{\mathbf{a}}_i$ is understood in relation to $\mathbf{W}$, and the $\boldsymbol{\Omega}_i^p$ depends on $\boldsymbol{\sigma}$ and $\mathbf{a}_i$ at the current configuration ($\mathbf{F}^e \simeq \mathbf{I}$), as it did on $\boldsymbol{\Pi} = \mathbf{R}^{e^{\mathrm{T}}} \boldsymbol{\sigma} \mathrm{R}^e$ and $\mathbf{A}_i = \mathbf{R}^{e^{\mathrm{T}}} \mathbf{a}_i \mathbf{R}^e$ at the unstressed configuration for $\mathbf{F}^e = \mathbf{R}^e$ (recall isotropy of $\boldsymbol{\Omega}_i^p$). It is interesting to observe that no trace of plastic spin appears in the elastoplastic moduli Λ of eqn (48) for small elastic strains, while the plastic spin via $\boldsymbol{\Omega}_i^p$ is part of the constitutive rate equations of evolution (49) (and (14), (15)) for the internal variables. This is where finally the plastic spin plays its significant role, since the updating of $\mathbf{a}_i$ through eqn (49) (and (14), (15) and (35) for large elastic strains) depends on it, and $\mathbf{a}_i$ in turn determines $\mathbf{N}_o^p$, $\mathbf{N}_o^n$, H, etc. in eqn (48).

VI. CONCLUDING REMARKS

The underlying motivation for this work was to show the necessity of considering constitutive equations for the plastic spin in a large deformation elastoplasticity theory. To achieve this, it was necessary to dispel with different inaccurate notions about the plastic spin and, in fact, to demystify its presence as something rather simple. It was clarified that a plastic spin term is introduced not at the foundation of the kinematics of the deformation, but simply as a term associated with the orientation part of a rate constitutive equation of evolution for tensorial internal variables. In this respect the plastic spin appears as a conjugate notion to the constitutive spin, the latter used in the corotational rate equation of evolution of an internal variable, such that their sum equals the antisymmetric part of the velocity gradient at the configuration where they are defined. Since each tensorial internal variable evolves in a different way, it was theoretically necessary to introduce as many pairs of constitutive and plastic spins as the number of internal variables. This multiple spin formulation proved to be very useful in the practical sense of matching theory and experiments (Cho and Dafalias, 1996) where the orientation of kinematic hardening and orthotropic symmetries tensors evolved according to different and appropriately chosen constitutive spins.

Once the significance and proper place in the theory for the plastic spin was established, it was straightforward to present a comprehensive formulation of rate-independent, primarily, and rate-dependent constitutive theory for large elastoplastic deformations accounting for possible change of elastic properties with plastic deformation. The formulation is set first at the plastically deformed relaxed configuration, where the plastic rate of deformation is unambiguously defined. No additive decomposition of the total rate of deformation at the current configuration in elastic and plastic parts was constitutively necessary (an issue of debate in the past). One important aspect of the development was that the emerging stress rate was not a-priori decided to be one or the other kind of corotational or convected time derivatives, whose arbitrary choice characterizes a plethora of recent works. Rather a straightforward and classical approach was followed, whereby stress is introduced from the hyperelastic relations without reference to any stress rate. As a result of the isotropic functional dependence of the yield criterion and the elastic strain energy on the state variables due to invariance requirements at the relaxed configuration

κ_u, eqns (17) and (18) apply. Use of these equations in the consistency equation provides a formulation where a stress rate corotational with the antisymmetric part of the plastic velocity gradient, called the plastic material spin (not the plastic spin) emerges. It is important to emphasize that such stress rate appears necessarily in the course of algebraic manipulations, and is not a-priori chosen. While one can always subsequently change to other stress rates with proper accommodation for the tangent moduli (Dafalias, 1988), the foregoing stress rate offers a simple computational environment if combined with the choice of the so-called spinless unstressed configuration in reference to which it becomes a simple rate. Thus, the need to maintain incremental objectivity in a corresponding numerical scheme is eliminated. During the formulation the plastic spin terms associated with different internal variables appear in the tangent moduli relating stress rate and rate of deformation, in addition to their presence in the equations of evolution of the corresponding internal variables. For small elastic strains it is the latter equations only which depend directly on the corresponding plastic spin, and it is there where one can find its significance.

Additional issues are also briefly addressed in an effort to clarify further aspects of large elastoplastic deformation theories, such as the elastic embedding of internal variables, the reason for using corotational rather than convected rates at the plastically deformed relaxed configuration, and the simplifying effect of choices such as the spinless and isoclinic configurations. All along the importance of the isotropic dependence of the different constitutive functions Ψ, f, $\mathbf{N}_o^p$, $\mathbf{\Omega}_i^p$ on the state variables $\mathbf{\Pi}$ and $\mathbf{A}_i$, due to invariance requirements at κ_u, cannot be overstated. This is the reason why one can state the formulation at any arbitrarily chosen relaxed configuration κ_u, and still maintain the same form of the constitutive functions, since the state variables' orientation will account for such choice automatically. For example, with $\mathbf{N}_o^p$ isotropic in $\mathbf{\Pi}$ and $\mathbf{A}_i$ one has $\mathbf{N}_o^p(\mathbf{Q\Pi Q}^{\mathrm{T}}, \mathbf{QA}_i\mathbf{Q}^{\mathrm{T}}) = \mathbf{QN}_o^p(\mathbf{\Pi}, \mathbf{A}_i)\mathbf{Q}^{\mathrm{T}}$ for two configurations which differ by a rotation $\mathbf{Q}$.

The freedom of choosing a relaxed configuration κ_u at any orientation, should not be confused with the additional freedom of choosing a reference observer's frame which may spin differently from κ_u and with respect to which one can write the equations and express corotational rates. In this paper such frame was a fixed cartesian one, but could as well be one that spins by any chosen spin $\boldsymbol{\omega}$ and associated plastic spin $\mathbf{W}^p = <\lambda> \mathbf{\Omega}^p$, not necessarily associated with a constitutive requirement for the evolution of a specific internal variable. In this case one has that $(\dot{\mathbf{F}}^p\mathbf{F}^{p^{-1}})_a = \boldsymbol{\omega} + <\lambda> \mathbf{\Omega}^p$, in a parallel notion to eqn (12). The theory then can be developed in a way that the relative constitutive and plastic spin terms $\boldsymbol{\omega}_i - \boldsymbol{\omega}$ and $\mathbf{\Omega}_i^p - \mathbf{\Omega}^p$ appear instead of $\boldsymbol{\omega}_i$ and $\mathbf{\Omega}_i^p$ in the equations of evolution of internal variables $\mathbf{A}_i$ and all that follows (Dafalias, 1993*a*). The corotational rates with respect to $\boldsymbol{\omega}$, reflect what the spinning observer sees in the process. In fact, this last observation is relevant to Mandel's (1971) original work and Dafalias' subsequent development (1985, 1987, 1988), where the substructural or director vector's spin $\boldsymbol{\omega}_o$ was used also as an observer's frame spin in an effort to show the independence of the final formulation from such a choice. Recall that in this case $\boldsymbol{\omega}_o$ is also the common constitutive spin $\boldsymbol{\omega}_i$ of all $\mathbf{A}_i$s, hence, the relative spins $\boldsymbol{\omega}_i - \boldsymbol{\omega}_o = \mathbf{0}$. In the process, the rates of $\mathbf{F}^e$ and $\mathbf{F}^p$ were expressed corotationally with $\boldsymbol{\omega}_o$ defined at different relaxed configurations (for the isoclinic $\boldsymbol{\omega}_o = \mathbf{0}$), and this perhaps may have been an additional reason for the confusion related to the plastic spin and its incorporation into kinematics. No compelling reason, however, exists, to introduce a spinning observer's frame, since any constitutive requirement can be accounted for by the constitutive and plastic spins $\boldsymbol{\omega}_i$ and $\mathbf{W}_i^p$ for each variable $\mathbf{A}_i$, and can be expressed in reference to a fixed observer's frame in association with an arbitrarily chosen relaxed configuration (the simplest choice of which is the spinless).

In conclusion one may reflect on the significance of the plastic spin issue vis-a-vis other elements of a constitutive formulation. It is certain that such significance is not necessarily superior to, say, the hardening aspects under cyclic loading or the characterization of initial anisotropy, but constitutes an integral part of the formulation. The importance of the plastic and constitutive spins in the formulation will increase with the magnitude of plastic deformation which induces new or alters existing anisotropic properties due to microscopic texture formation. The sensitivity of the material response to the orientational evolution aspects portrayed by the plastic spin has been proved to be an important feature in the analysis of localization phenomena (Tvergaard and van der Giessen, 1991; Zbib, 1993; Lee *et al.*, 1995; Kuroda, 1996), in addition to the important role of the plastic spin in the realistic simulation of the response of anisotropic materials (orthotropic) under large plastic deformations for monotonic and reversed loading conditions (Cho and Dafalias, 1996; Kuroda, 1997). From this perspective the present work offers a theoretical clarification of the plastic spin concept and an associated convenient constitutive framework which may be proved useful in analyses of the foregoing nature.

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APPENDIX

Relations for isotropic functions

If $\mathbf{Q}(t)$ is any orthogonal tensor ($\mathbf{QQ}^{\mathrm{T}} = \mathbf{I}$) function of time t, one can associate a spin $\boldsymbol{\omega} = \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$ (observe $\boldsymbol{\omega}^{\mathrm{T}} = -\boldsymbol{\omega}$) and define the corotational rate $\overset{\circ}{\mathbf{a}}_i$ of a second order tensor $\mathbf{a}_i$ by

$$\overset{\circ}{\mathbf{a}}_i = \dot{\mathbf{a}}_i - \boldsymbol{\omega}\mathbf{a}_i + \mathbf{a}_i\boldsymbol{\omega} \tag{A1}$$

If f is a scalar-valued isotropic function of $\mathbf{a}_i$ it follows by definition

$$f(\mathbf{a}_i) = f(\mathbf{Q}^{\mathrm{T}}\mathbf{a}_i\mathbf{Q}) \tag{A2}$$

From the rate of both members of eqn (A2) and use of eqn (A1) together with the observation of $(\partial f/\partial \mathbf{a}_i)\cdot\dot{\mathbf{a}}_i = (\partial f/\partial \mathbf{a}_i) : \dot{\mathbf{a}}_i^{\mathrm{T}}$, it follows

$$\dot{f}(\mathbf{a}_i) = \frac{\partial f}{\partial \mathbf{a}_i} : \dot{\mathbf{a}}_i^{\mathrm{T}} = \frac{\partial f}{\partial (\mathbf{Q}^{\mathrm{T}}\mathbf{a}_i\mathbf{Q})} : \overline{(\mathbf{Q}^{\mathrm{T}}\mathbf{a}_i^{\mathrm{T}}\mathbf{Q})}^{\cdot} = \mathbf{Q}^{\mathrm{T}} \frac{\partial f}{\partial \mathbf{a}_i}\mathbf{Q} : \mathbf{Q}^{\mathrm{T}}\ \overset{\circ}{\mathbf{a}}{}_i^{\mathrm{T}}\mathbf{Q}$$
$$= \frac{\partial f}{\partial \mathbf{a}_i} :\ \overset{\circ}{\mathbf{a}}{}_i^{\mathrm{T}} = \frac{\partial f}{\partial \mathbf{a}_i} : (\dot{\mathbf{a}}_i^{\mathrm{T}} + \mathbf{a}_i^{\mathrm{T}}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{a}_i^{\mathrm{T}}) = \frac{\partial f}{\partial \mathbf{a}_i} : \mathbf{a}_i^{\mathrm{T}} - (\mathbf{a}_i^{\mathrm{T}}\frac{\partial f}{\partial \mathbf{a}_i} - \frac{\partial f}{\partial \mathbf{a}_i}\mathbf{a}_i^{\mathrm{T}}) : \boldsymbol{\omega} \tag{A3}$$

Since $\boldsymbol{\omega}$ is arbitrary, it follows from the second and last member of eqn (A3) that:

$$\mathbf{a}_i^{\mathrm{T}} : \frac{\partial f}{\partial \mathbf{a}_i} - \frac{\partial f}{\partial \mathbf{a}_i}\mathbf{a}_i^{\mathrm{T}} \equiv \mathbf{0} \tag{A4}$$

If now $\mathbf{f}$ is a tensor-valued isotropic function of $\mathbf{a}_i$, by definition one has

$$\mathbf{Q}^{\mathrm{T}}\mathbf{f}(\mathbf{a}_i)\mathbf{Q} = \mathbf{f}(\mathbf{Q}^{\mathrm{T}}\mathbf{a}_i\mathbf{Q}) \tag{A5}$$

Taking again the rate of both members of eqn (A5) one has

$$\overline{(\mathbf{Q}^{\mathrm{T}}\mathbf{f}(\mathbf{a}_i), \mathbf{Q})}^{\cdot} = \mathbf{Q}^{\mathrm{T}}\ \overset{\circ}{\mathbf{f}}\mathbf{Q} = \frac{\partial \mathbf{f}}{\partial (\mathbf{Q}^{\mathrm{T}}\mathbf{a}_i\mathbf{Q})} : \overline{(\mathbf{Q}^{\mathrm{T}}\mathbf{a}_i^{\mathrm{T}}\mathbf{Q})}^{\cdot} = \mathbf{Q}^{\mathrm{T}}\frac{\partial \mathbf{f}}{\partial \mathbf{a}_i}\mathbf{Q} : \mathbf{Q}^{\mathrm{T}}\ \overset{\circ}{\mathbf{a}}{}_i^{\mathrm{T}}\mathbf{Q} = \frac{\partial \mathbf{f}}{\partial \mathbf{a}_i} :\ \overset{\circ}{\mathbf{a}}{}_i^{\mathrm{T}} \tag{A6}$$

Choosing $\mathbf{Q} = \mathbf{I}$ it follows from the second and last member of eqn (A6) and from eqn (A1) with $\mathbf{f}$ substituting for $\mathbf{a}_i$, that

$$\overset{\circ}{\mathbf{f}} = \frac{\partial \mathbf{f}}{\partial \mathbf{a}_i} : \dot{\mathbf{a}}_i^{\mathrm{T}} - \boldsymbol{\omega}\mathbf{f} + \mathbf{f}\boldsymbol{\omega} = \frac{\partial \mathbf{f}}{\partial \mathbf{a}_i} : \dot{\mathbf{a}}_i^{\mathrm{T}} = \frac{\partial \mathbf{f}}{\partial \mathbf{a}_i} : (\dot{\mathbf{a}}_i^{\mathrm{T}} + \dot{\mathbf{a}}_i^{\mathrm{T}}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{a}_i^{\mathrm{T}}) = \frac{\partial \mathbf{f}}{\partial \mathbf{a}_i} : \dot{\mathbf{a}}_i^{\mathrm{T}} - (\mathbf{a}_i^{\mathrm{T}}\frac{\partial \mathbf{f}}{\partial \mathbf{a}_i} - \frac{\partial \mathbf{f}}{\partial \mathbf{a}_i}\mathbf{a}_i^{\mathrm{T}}) : \boldsymbol{\omega} \tag{A7}$$

From the second and last member of eqn (A7) follows then

$$(\mathbf{a}_i^{\mathrm{T}}\frac{\partial \mathbf{f}}{\partial \mathbf{a}_i} - \frac{\partial \mathbf{f}}{\partial \mathbf{a}_i}\mathbf{a}_i^{\mathrm{T}}) : \boldsymbol{\omega} + \mathbf{f}\boldsymbol{\omega} - \boldsymbol{\omega}\mathbf{f} \equiv \mathbf{0} \tag{A8}$$

for any $\boldsymbol{\omega}$. The multiplication of $\mathbf{a}_i^{\mathrm{T}}$ and $\partial\mathbf{f}/\partial\mathbf{a}_i$ as well as the trace operation with $\boldsymbol{\omega}$ takes place over the indices of $\mathbf{a}_i^{\mathrm{T}}$ and $\partial/\partial\mathbf{a}_i$, without involving the indices of $\mathbf{f}$. To make it clear, eqn (A8) is written in indicial notation as (observe sum over i for multiplicity of $\mathbf{a}_i$, in addition to standard indicial sum)

$$\left[(\mathbf{a}_i^{\mathrm{T}})_{mn}\frac{\partial f_{rs}}{\partial (\mathbf{a}_i)_{nk}} - \frac{\partial f_{rs}}{\partial (\mathbf{a}_i)_{mn}}(\mathbf{a}_i^{\mathrm{T}})_{nk}\right]\omega_{km} + f_{ra}\omega_{as} - \omega_{ra}f_{as} \equiv 0 \tag{A9}$$

The arbitrariness of $\boldsymbol{\omega}$ allows one to write eqn (A8) as follows

$$\mathbf{a}_i^{\mathrm{T}}\frac{\partial \mathbf{f}}{\partial \mathbf{a}_i} - \frac{\partial \mathbf{f}}{\partial \mathbf{a}_i}\mathbf{a}_i^{\mathrm{T}} = \Delta; \quad \boldsymbol{\omega}\mathbf{f} - \mathbf{f}\boldsymbol{\omega} = \Delta : \boldsymbol{\omega} \tag{A10a}$$

$$\Delta_{ijkl} = \frac{1}{2}(f_{ik}\delta_{jl} + f_{jk}\delta_{il} - f_{jl}\delta_{ik} - f_{il}\delta_{jk}) \tag{A10b}$$