V. Uncertainty Sensitivity Index Method

PROBLEM V.1: Developing a Computer Program

This example presents a sales model of a computer program which is based on two variables, technological complexity and market price. It is assumed that the product is affected by two choices: i) the program's level of complexity, and ii) its market price, which is an exogenous parameter of the company's simple profit model.

DESCRIPTION

The producer's dilemma is further described as follows. As the program to be sold becomes more complex, it becomes more powerful and more and more non-professional consumers want to purchase it. However, after a certain point (such as the global maximum in Figure V.1.1), the program can become so complex that sales to the general public will drop drastically. Past this point, however, the program becomes complex/powerful enough for academic professionals to use. Therefore, with more technological complexity, sales begin to rise again. Nevertheless, the new local maximum (at high levels of complexity) is not as high as the global maximum because there are fewer academic professionals than there are general consumers.

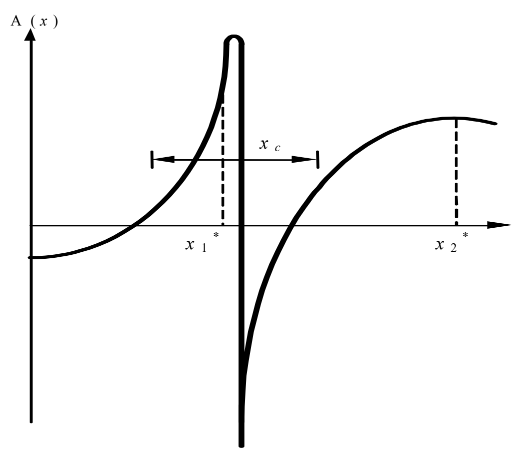

Figure V.1.1. Sensitivity band, adapted from the companion textbook [Haimes 2009]

Note that due to manufacturing constraints, the company can make only one version, with technological complexity (x) as the decision variable. Also, the objective of maximizing sales can be alternatively represented by minimizing lost sales (defined here as the objective function $f_1(x, \alpha)$, where α is the market price parameter).

METHODOLOGY

Use the Uncertainty Sensitivity Index Method (USIM) to solve this problem.

The mathematical formulation of the objective function is as follows:

$$f_1(x, \alpha) = \text{lost sales} = 4x^2 - 2.25\alpha(x-2) + 1.5\alpha^2 \tag{V.1.1}$$

where:

x = technological complexity
α = market price

We look at the problem from two angles: 1) the "business-as-usual" case (minimize objective function to get x^*) and 2) the most conservative case (minimize sensitivity function to get $\hat{x}$).

SOLUTION

To determine x^*, we have to take the derivative function with respect to x, and let $\alpha = \hat{\alpha} = \$20$:

Based on market prices, and the company's long history in this industry, they have set the nominal value α as equal to \$20.

$$f_1(x, \hat{\alpha}) = 4x^2 - 45(x-2) + 600 = 4x^2 - 45x + 690 \tag{V.1.2}$$

$$\frac{\partial f_1(x, \hat{\alpha})}{\partial x} = 8x - 45 = 0 \tag{V.1.3}$$

$$\therefore x^* = \frac{45}{8}$$

Now we need to determine $\hat{x}$ to derive a sensitivity function from (1):

$$f_1(x, \hat{\alpha}) = 4x^2 - 2.25\alpha(x-2) + 1.5\alpha^2$$

$$f_2(x, \hat{\alpha}) = \left(\frac{\partial f_1(x, \alpha)}{\partial \alpha} \Big|_{\alpha=\hat{\alpha}} \right)^2 = \left(-2.25(x-2) + 3\hat{\alpha} \right)^2 \tag{V.1.4}$$

$$= 5.0625x^2 + 20.25 + 9\hat{\alpha}^2 - 20.25x - 13.5x\hat{\alpha} + 27\hat{\alpha} \tag{V.1.5}$$

$$= 5.0625x^2 + 20.25 + 3600 - 20.25x - 270x + 540 \tag{V.1.6}$$

$$= 5.0625x^2 - 290.25x + 4160.25 \tag{V.1.7}$$

Taking the derivative gives us:

$$\frac{\partial f_2(x,\hat{\alpha})}{\partial x} = 10.125x - 290.25 = 0 \qquad \text{(V.1.8)}$$

$$\therefore \hat{x} = 28.67$$

Now we want to minimize the two objectives together. To do that, we will use the ε -constraint form.

$$\min\begin{bmatrix} f_1(x,\hat{\alpha}) = 4x^2 - 45x + 690 \\ f_2(x,\hat{\alpha}) = 5.0625x^2 - 290.25x + 4160.25 \end{bmatrix} \qquad \text{(V.1.9)}$$

Thus, we have:

$$\min\left[f_1(x,\hat{\alpha}) = 4x^2 - 45x + 690\right] \qquad \text{(V.1.10)}$$

Subject to:

$$5.0625x^2 - 290.25x + 4160.25 \le \varepsilon_2 \qquad \text{(V.1.11)}$$

Thus, we can formulate the Lagrangian equation:

$$L(x,\hat{\alpha},\lambda_{12}) = 4x^2 - 45x + 690 + \lambda_{12}(5.0625x^2 - 290.25x + 4160.25 - \varepsilon_2)$$

(V.1.12)

The Kuhn-Tucker necessary conditions yield:

$$\frac{\partial L(\cdot)}{\partial x} = 8x - 45 + \lambda_{12}(10.125x - 290.25) = 0 \qquad \text{(V.1.13)}$$

$$\frac{\partial L(\cdot)}{\partial \lambda_{12}} = 5.0625x^2 - 290.25x + 4160.25 - \varepsilon_2 \le 0 \qquad \text{(V.1.14)}$$

$$\lambda_{12}\left[5.0625x^2 - 290.25x + 4160.25 - \varepsilon_2\right] = 0 \qquad \text{(V.1.15)}$$

$$\lambda_{12} \ge 0 \qquad \text{(V.1.16)}$$

Using the partial Lagrangian function with respect to x from (V.1.13), we have:

$$\therefore \lambda_{12} = \frac{45 - 8x}{(10.125x - 290.25)} \qquad \text{(V.1.17)}$$

Table V.1.1 displays the results.

Table V.1.1. Non-inferior Solutions and Tradeoff Values

x	$f_1(x,\hat{\alpha})$	$f_2(x,\hat{\alpha})$	λ_{12}
5.625	563.44	2687.77	0.0000
10.00	640.00	1764.00	0.1852
15.00	915.00	945.56	0.5420
20.00	1390.00	380.25	1.3105
25.00	2065.00	68.06	4.1751
28.67	2687.11	0.00	∞

To dramatize the tradeoffs between the sensitivity objective function and the optimality objective function, the latter is evaluated at x^* and at $\hat{x}$ as a function of α.

$$f_1(x^*,\hat{\alpha}) = 4\left(\frac{45}{8}\right)^2 - 2.25\alpha(\frac{45}{8} - 2) + 1.5\alpha^2 = 126.5625 - 8.15625\alpha + 1.5\alpha^2 \tag{V.1.18}$$

$$f_1(x^*,\hat{\alpha}) = 4(28.67)^2 - 2.25\alpha(28.67 - 2) + 1.5\alpha^2 = 821.\overline{7} - 60\alpha + 1.5\alpha^2 \tag{V.1.19}$$

Given the nominal value of $\hat{\alpha} = 20$:

$$\frac{\partial f_1(x,\hat{\alpha})}{\partial \alpha}_{|\alpha=\hat{\alpha}} = 3\hat{\alpha} - 8.15625 = 51.84375 \tag{V.1.20}$$

$$\frac{\partial f_2(x,\hat{\alpha})}{\partial \alpha}_{|\alpha=\hat{\alpha}} = 3\hat{\alpha} - 60 = 0 \tag{V.1.21}$$

Furthermore, we can analyze the changes that take place in $f_1(x^*, \alpha)$ and $f_1(\hat{x}, \alpha)$ when the nominal value, $\hat{\alpha}$, is perturbed by $\Delta\alpha = 5$. The corresponding variations are given below:

$$f_1(x^*,\hat{\alpha}) = 4\left(\frac{45}{8}\right)^2 - 2.25(20)\left(\frac{45}{8} - 2\right) + 1.5(20)^2 = 563.4375 \tag{V.1.22}$$

$$f_1(x^*,\hat{\alpha} - 5) = 4\left(\frac{45}{8}\right)^2 - 2.25(20-5)\left(\frac{45}{8} - 2\right) + 1.5(20-5)^2 = 341.71875 \tag{V.1.23}$$

$$\left|f_1(x^*,\hat{\alpha}) - f_1(x^*,\hat{\alpha} - 5)\right| = 221.719 \tag{V.1.24}$$

Let $\eta(x^*, 0.75\hat{\alpha})$ denote the percentage change in $f_1(x^*, \alpha\hat{})$ with a perturbation of 25% in $\hat{\alpha}$. Then:

$$\eta(x^*, 0.75\hat{\alpha}) = 39.35\%$$

Similarly,

$$f_1(\hat{x},\hat{\alpha}) = 4(28.67)^2 - 45(28.67) + 600 = 2687.1111 \quad \text{(V.1.25)}$$

$$f_1(\hat{x},\hat{\alpha}-5) = 4(43/3)^2 - 33.75(43/3) + 405 = 2724.6111 \quad \text{(V.1.26)}$$

$$\left| f_1(\hat{x},\hat{\alpha}) - f_1(x^*,\hat{\alpha}-5) \right| = 37.5 \quad \text{(V.1.27)}$$

Let $\eta(\hat{x},0.75\hat{\alpha})$ denote the percentage change in $f_1(\hat{x},\hat{\alpha})$ with a perturbation of 25% in $\hat{\alpha}$. Then

$$\eta(\hat{x},0.75\hat{\alpha}) = 1.40\%$$

ANALYSIS

The results given in Figure V.1.2 indicate that following a conservative policy that trades optimality (cost objective) for a less sensitive outcome provides a very stable solution (3% versus 50% changes in $f_1(\cdot)$ with a deviation of 25% from the nominal value $\hat{\alpha}$). Clearly, neither the solution x^* nor $\hat{x}$ is likely to be recommended. From the use of Table V.1.1 and the SWT method, with an interaction with a decisionmaker, a preferred level of x should be selected, where:

$$5.63 \le x \le 28.67$$

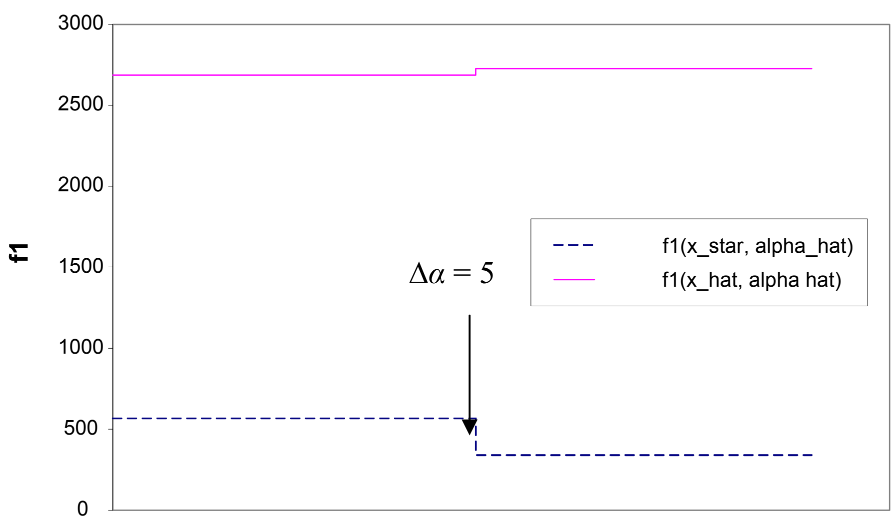

Figure V.1.2. The Functions $f_1(x^*, \alpha)$ and $f_1(\hat{x}, \alpha)$ versus Perturbation in α

PROBLEM V.2: Structural Remodeling

We seek to minimize the stress-related deformation of cantilevered beams on a silicon substrate resulting from stress orientations in the pre-etched thin film. The question is whether or not to replace the current beam with a longer beam.

DESCRIPTION

The question of replacing the current beam with a longer beam can be determined by minimizing the following functions:

1) To minimize the height of maximum deformation.
2) To minimize the sensitivity of the deformation with respect to the thin film stress orientation.

METHODOLOGY

We use the Uncertainty Sensitivity Index Method (USIM) to solve the problem, as follows:

The stress orientation is given as the parameter α, which denotes the angular difference between the stress axes and the etching film orientation. The system output, $f_1(x,\alpha)$, is measured as the degree of deformation of the etched cantilevered beam from true horizontal with respect to the substrate wafer. Our decision variable, x, is the length of the beam.

SOLUTION

Two objective functions are given as follows:

$$f_1(x,\alpha) = 3x^2 - 2x\alpha + 5\alpha^2 \tag{IV.2.1}$$

$$f_2(x,\alpha) = \left(\frac{\partial f_1(x,\alpha)}{\partial \alpha}\right)^2 = (-2x + 10\alpha)^2 = 4x^2 - 40\alpha x + 100\alpha^2 \tag{IV.2.2}$$

By applying the ε-constraint form, we can formulate the Lagrangian equation:

min $f_1(\cdot)$
s.t. $f_2 <= \varepsilon_2$

$$L(x,\alpha,\lambda) = f_1(x,\alpha) + \lambda_{12}(f_2(x,\alpha) - \varepsilon) \tag{IV.2.3}$$

The Kuhn-Tucker necessary conditions for stationarity yield:

$$\frac{\partial L(x,\alpha,\lambda_{12})}{\partial x} = 6x - 2\alpha + 8\lambda_{12}x - 40\alpha\lambda_{12} = 0 \tag{IV.2.4}$$

$$\frac{\partial L(x,\alpha,\lambda_{12})}{\partial \lambda_{12}} = 4x^2 - 40\alpha x + 100\alpha^2 \leq 0 \qquad \text{(IV.2.5)}$$

$$\lambda_{12}\frac{\partial L(x,\alpha,\lambda_{12})}{\partial \lambda_{12}} = 4x^2 - 40\alpha x + 100\alpha^2 = 0 \qquad \text{(IV.2.6)}$$

$$\lambda_{12} \geq 0 \qquad \text{(IV.2.7)}$$

Let the nominal value of α be equal to 1, and substituting the value into (IV.2.4):

$$\frac{\partial L(x,\hat{\alpha},\lambda)}{\partial x} = 6x - 2 + 8\lambda_{12}x - 40\lambda_{12} = 0 \qquad \text{(IV.2.8)}$$

$$\lambda_{12} = \frac{1-3x}{4x-20} \qquad \text{(IV.2.9)}$$

Plugging this result into various values of *lambda* we arrive at the Pareto-optimal curve shown in Figure V.2.1:

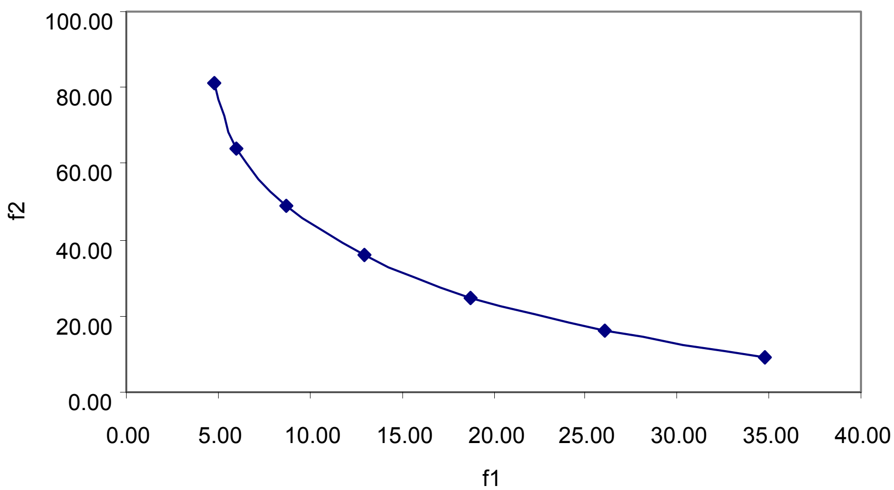

Figure V.2.1. Noninferior Solution in Function Space

We can also look at the sensitivity of the response to changes in the value of *α,* as depicted in Figure V.2.2 below:

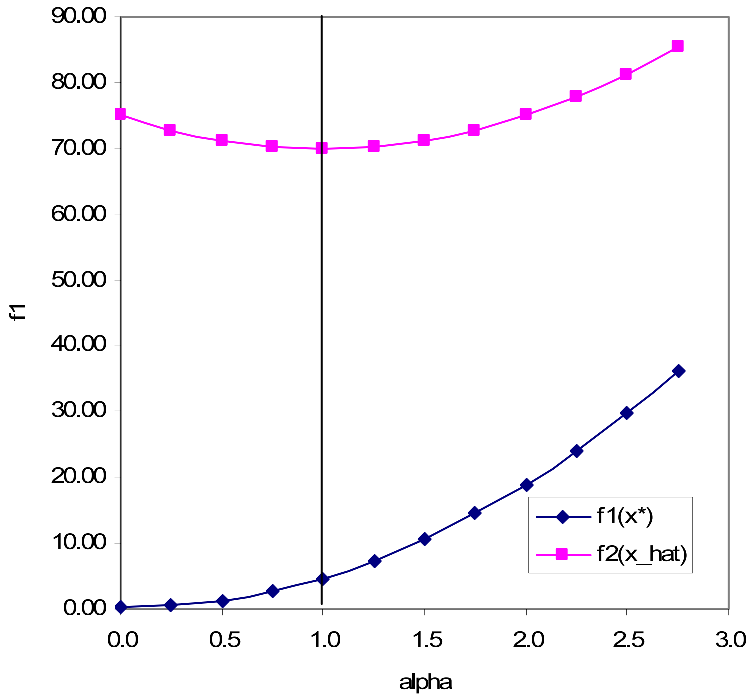

Figure V.2.2. The Functions $f_1(x^*, \alpha)$ and $f_1(\hat{x}, \alpha)$ versus Perturbation in α

ANALYSIS

As can be seen, the deformation for the *conservative* decision is significantly higher than for the *business–as-usual* option; however it is less sensitive to changes in α. This basically says that large values of x, resulting in longer cantilever beams, have a consistently higher deformation, but are also less sensitive to changes in α. Therefore, if the level of deformation offered by the longer beam is acceptable, then this should be used.

PROBLEM V.3: The Cost of Buying and Maintaining a Car

A graduate student on a limited budget would like to buy a car with high gas mileage, so that he does not have to spend so much on fuel. Although financing is available to the student, he does not want to invest more than $50,000 in higher-priced fuel-efficient models.

DESCRIPTION

To determine whether he can afford to maintain a car, the student must weigh the cost of the car against the uncertain costs of fuel.

METHODOLOGY

The Uncertainty Sensitivity Index Method (USIM) can help decide which car to buy. The two objectives are to: (1) minimize car price; and (2) maximize miles per gallon.

SOLUTION

Through previous research we have determined that at the Pareto-optimal frontier of the two objectives, the relationship can be approximated by the following formula applicable over the range for which car price is at most $50,000:

P (car price in dollars) = 1,000
M (miles per gallon) = 5

Note that this model is valid for $0 \leq P \leq 50{,}000$ and $5 \leq M \leq 55$ (In practice, P is negatively related to M even considering the prices of hybrid cars.)

Since the student would like to minimize his costs, it is convenient to express M also in terms of dollars which will make the problem a single-objective optimization. The conversion is shown below:

Cost ($) = [fuel cost ($/gal)] * [average miles per day] * [days in use] / M

***Given fuel cost of $2.15/gal; 30 miles average per day and 1,500 days if he stays for his Ph.D.:

Cost ($) = $96,750 / M

However, since there is a lot of uncertainty about the price of fuel over the coming years, we will hold that variable out and denote it by α. Then, the fuel cost over the time the student will own the car will be:

Cost ($) = α 45,000 / M

His total cost of car ownership will then be

$$P + \text{Cost} = 1{,}000\,(M - 5) + \alpha\, 45{,}000 \,/\, M$$

The objective for this problem is to minimize this function:

$$\text{Min } \{f_1(M, \alpha\,) = 1{,}000\, M - 5{,}000 + \alpha\, 45{,}000 \,/\, M\,\}$$
$$\text{s.t. } 5 \le M \le 55$$

The minimum of this function can be found by taking the derivative with respect to *M* and equating it to zero.

$$\frac{\partial f(\cdot)}{\partial M} = 1{,}000 - \alpha\, 45{,}000 \,/\, M^2 = 0$$
$$M = \sqrt{45\alpha}$$

At the nominal value of $\hat{\alpha}$ = 2.15, the optimal solution with respect to price would be at $M \approx 9.84$, which would have a total ownership cost of about $14,672.

However, this solution does not take into consideration the uncertainty surrounding fuel prices. The sensitivity index is defined as the square of the derivative of the output function with respect to α:

$$f_2 = (45{,}000 \,/\, M)^2 = 2.025 \times 10^9 \,/\, M^2.$$

This function is minimized when *M* is maximized. This is intuitive because if the car has higher gas mileage, it will use less gas and the ownership cost will be less susceptible to fuel-price changes. In this case, the maximum valid *M* for our model is 55. At $M = 55$, the sensitivity index is 669,421.5. The optimal value for minimum ownership cost (at $\alpha = 2.15$) has a sensitivity index of 20,930,232.5, a much higher value.

Moreover, since we cannot minimize both the cost of ownership function and the sensitivity index, we are left with a multiobjective optimization problem given by:

$$\text{Min } \{1{,}000\, M - 5{,}000 + \alpha\, 45{,}000 \,/\, M\,\} \text{ and Min } \{2.025 \text{ x } 10^9 \,/\, M^2\}$$

The two objective functions are plotted below and we can see that the minima do not coincide. However, the two graphs intersect.

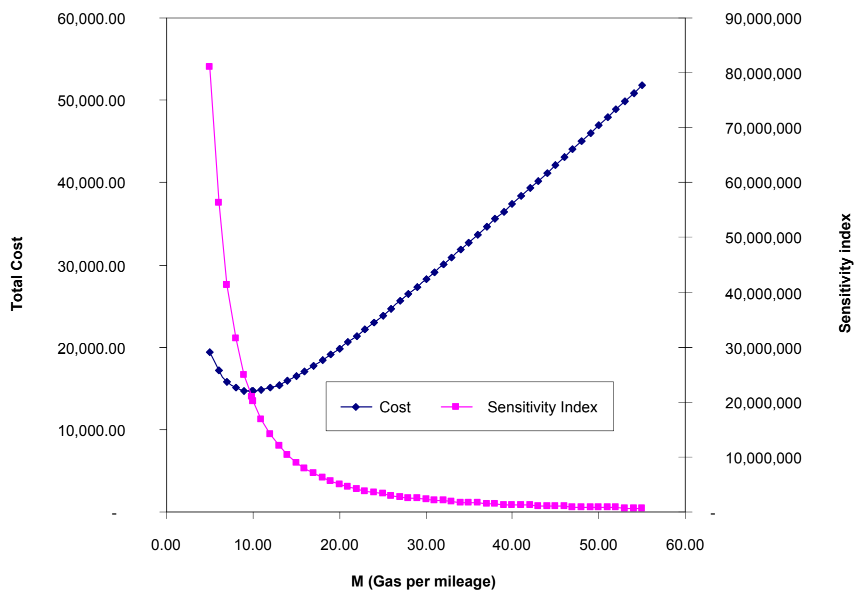

Figure V.3.1. Cost versus sensitivity index

According to the Surrogate Worth Trade-off (SWT) method, the trade-off (or Lagrangian multiplier) between these two objectives is given by the negative ratio of their derivatives:

$$\lambda = -(1{,}000 - \alpha\, 45{,}000 / M^2) / (-4.050 \text{ x } 10^9 / M^3)$$

The table below shows additional values for our objectives given different choices in M for $\alpha = 2.15$:

Table V.3.1. Noninferior Solutions and Trade-off Values

M	Ownership	Sensitivity Index ($\text{x}10^3$)	lamdba
9.84	14,672.32	20,913.89	0.000018
20.00	19,837.50	5,062.50	0.150
30.00	28,225.00	2,250.00	0.595
40.00	37,418.75	1,265.63	1.485
55.00	51,759.09	669.42	3.977

The possible solutions tabulated above are also shown on the graph below:

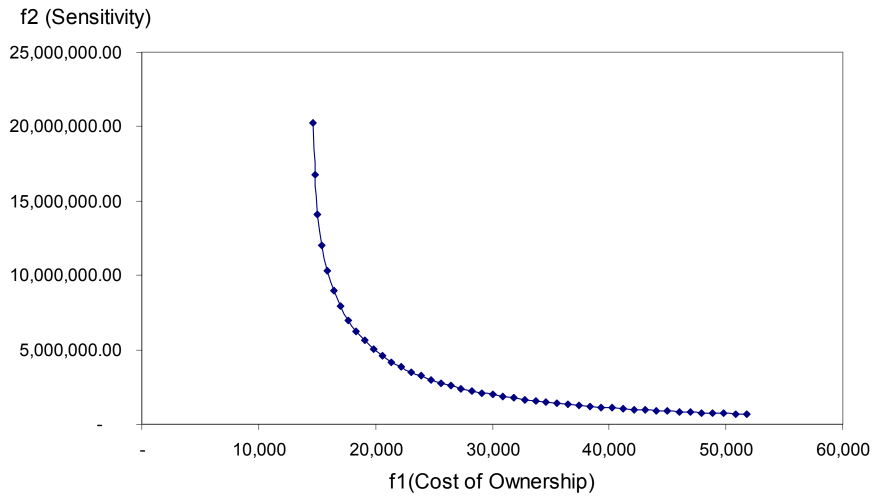

Figure V.3.2. Noninferior solution in the functional space

ANALYSIS

Let us consider again the two most extreme scenarios, the *M* that will minimize the total cost of ownership given a fuel cost of \$2.15/gal and at the same time minimize the sensitivity of the cost of ownership to changes in the cost of fuel. We will refer to them as business-as-usual and conservative, respectively. The total cost of ownership for each scenario is plotted below against some possible costs of fuel. As we can see from this chart, the conservative approach is more costly, but it changes less with changes in fuel costs. On the other hand, gas price fluctuation has a relatively higher influence on the cost of ownership in the business-as-usual approach. The second chart is similar but is based on percent changes in fuel price.

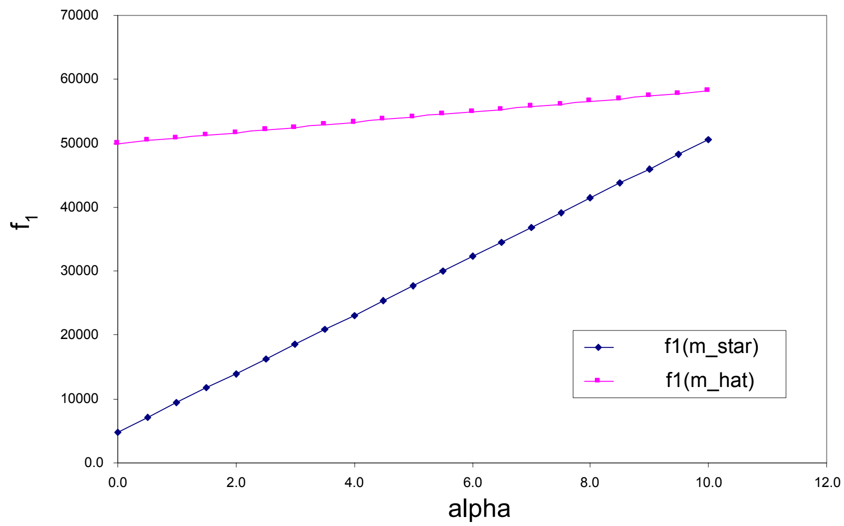

Figure V.3.3. Changes in cost for *business-as-usual* and *conservative* cases versus α

Note: Cost of fuel $= \dfrac{45000\alpha}{M}$, so given M values (e.g., 9.84 and 55), cost is positively related to α in this case.

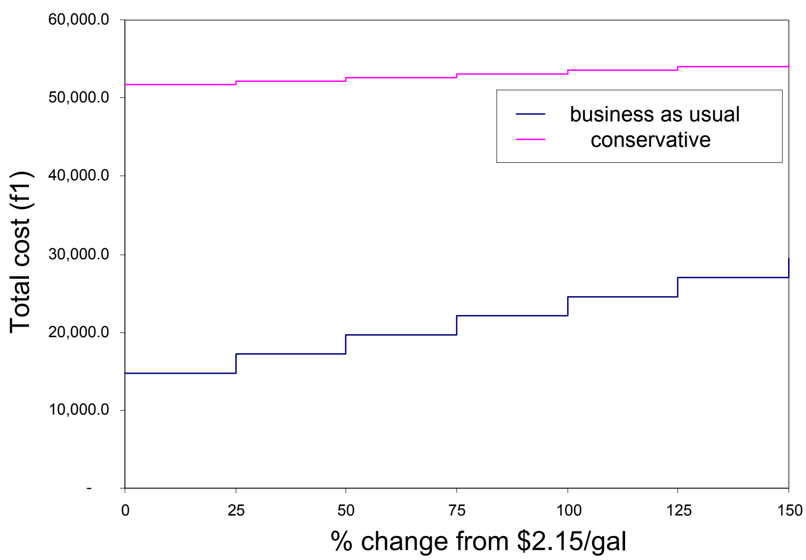

Figure V.3.4. Changes in cost with perturbation in fuel price

Despite the risk of an increased cost of ownership, the student has decided to follow the business-as-usual approach since the dominant factor is the cost of the car itself, not the cost of the fuel.

PROBLEM V.4: Four USIM EXERCISES

The following four generic exercises demonstrate the Uncertainty Sensitivity Index Methodology (USIM).

DESCRIPTION

Problems A through D each illustrate a different approach and present the solution and analysis of the results.

EXERCISE A: USIM with One Objective

Consider a system with the objective function:

$$\min f(x,\alpha) = x^2 + (1-\alpha^2)x + \alpha^2$$

where α is a parameter with nominal value $\overline{\alpha} = 0.5$ and changing from $\underline{\alpha} = 0$ to $\overline{\alpha} = 1.0$.

Consider a second objective based on the sensitivity function resulting from the above function. Then construct and solve a multiobjective optimization problem.

SOLUTION

Use USIM to analyze the problem, which takes into account the sensitivity to optimal solutions.

Let:

$$f_1(x,\hat{\alpha}) = f(x,\hat{\alpha}) = x^2 + \frac{3}{4}x + \frac{1}{4}$$

$$f_2(x,\hat{\alpha}) = \left(\frac{\partial f}{\partial \alpha}\right)^2 \Big|_{\alpha=\hat{\alpha}} = x^2 - 2x + 1$$

Now that we have a multiple objective problem, the SWT method can be applied to solve it.

$$\min \begin{Bmatrix} f_1(x,\hat{\alpha}) = x^2 + \frac{3}{4}x + \frac{1}{4} \\ f_2(x,\hat{\alpha}) = x^2 - 2x + 1 \end{Bmatrix}$$

i) $\min f_1(x,\hat{\alpha})$: $\quad x_1{}^* = -\frac{3}{8} \qquad f_1{}^* = \frac{7}{64}$

$\min f_2(x,\hat{\alpha})$: $\quad x_2{}^* = 1 \qquad f_2{}^* = 0$

ii) Form an ε-constraint problem:

$\min f_1(x,\hat{\alpha})$

s.t. $\quad f_2(x,\hat{\alpha}) \le \varepsilon$

$$\varepsilon \geq f_2{}^* = 0$$

iii) Form the Lagrangian of the problem:

$$L(x,\lambda_{12}) = f_1(x,\hat{\alpha}) + \lambda_{12}(f_2(x,\hat{\alpha}) - \varepsilon) \qquad \lambda_{12} \geq 0$$

$$\lambda_{12} = -\frac{2x + \frac{3}{4}}{2x-2} \geq 0$$

$$-\frac{3}{8} \leq x \leq 1$$

To obtain a satisfactory λ_{12}, we need to calculate x by using

$$\lambda_{12} = -\frac{2x + \frac{3}{4}}{2x-2} \quad \text{e.g. } \lambda_{12} = 1,\ x^* = \frac{5}{16}$$

i) When α changes from $\hat{\alpha}$

$$f_1(x,\alpha) = x^2 + (1-\alpha^2)x + \alpha^2$$

$$f_2(x,\alpha) = 4\alpha^2(x-1)^2$$

$$\min\begin{Bmatrix} f_1(x,\alpha) \\ f_2(x,\alpha) \end{Bmatrix}$$

Using the SWT method:

$$\lambda_{12} = -\frac{\partial f_1(x,\alpha)}{\partial f_2(x,\alpha)} \qquad \alpha \text{ is fixed}$$

$$\lambda_{12} = -\frac{2x+1-\alpha^2}{8\alpha^2(x-1)} \geq 0$$

$$= \begin{Bmatrix} x \geq \dfrac{(\alpha^2-1)}{2} \\ x \geq 1 \end{Bmatrix}$$

We can see that when $\alpha > \sqrt{3}$, there is no trade-off between f_1 and f_2 since non-negativity of λ_{12} does not hold, given the range of x. If we know $\alpha \in [0,1]$, then no matter how it will change, any solution of $x \in [0,1]$ is non-inferior (i.e., $\overline{\underline{X}}^* = [0,1].$).

In general, when $f_1(x,\alpha)$ satisfies certain conditions, we have:

$$\underline{\overline{X}}^* = \underline{\overline{X}}_{\alpha} \bigcap \underline{\overline{X}}_{\overline{\alpha}}$$

$$= \left\{ x \mid \lambda_{12} = -\frac{\partial f_1}{\partial f_2} \geq 0 \text{ to any } \alpha \in [\underline{\alpha}, \overline{\alpha}] \right\}$$

EXERCISE B: Envelope Solution Approach

Using the original function in Exercise A, solve the multiobjective optimization problem using the envelope approach.

SOLUTION

Use the functions:

$$f_1(x,\alpha) = x^2 + (1-\alpha^2)x + \alpha^2$$

$$f_2(x,\alpha) = \left(\frac{\partial f_1}{\partial \alpha}\right)^2 = 4\alpha^2(x-1)^2$$

$$\frac{\partial f_1}{\partial x} = 2x + 1 - \alpha^2 \qquad \frac{\partial f_1}{\partial \alpha} = -2\alpha(x-1)$$

$$\frac{\partial f_2}{\partial x} = 8\alpha^2(x-1) \qquad \frac{\partial f_2}{\partial \alpha} = 8\alpha(x-1)^2$$

$$\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial \alpha} - \frac{\partial f_1}{\partial \alpha}\frac{\partial f_2}{\partial x} = 0$$

$$2x + 1 + \alpha^2 = 0 \quad \text{or} \quad \alpha^2 = -2x - 1$$

(In the above problem, x must be less than or equal to –1/2 and α has a real solution, but if $x \geq -1/2$, then there is no trade-off between f_1 and f_2.) Figure V.4.1 graphically shows the tradeoffs between f_1 and f_2.

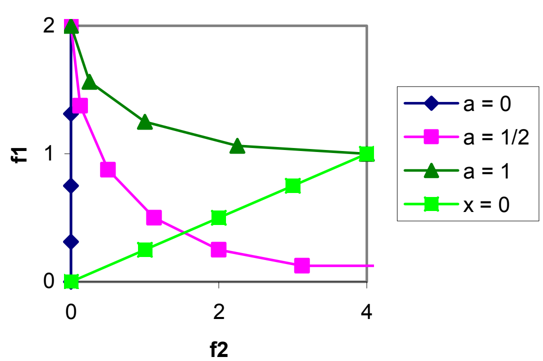

Figure V.4.1. f_1 **versus** f_2

EXERCISE C: USIM Problem Using Envelope Solution Approach

Consider the original function used in Exercise B, modified as follows:

$$f_1(x,\alpha) = x^2 + (1+\alpha^2)x - \alpha^2$$

Solve the multiobjective optimization problem using the envelope approach.

SOLUTION

$$f_1(x,\alpha) = x^2 + (1+\alpha^2)x - \alpha^2$$

$$f_2(x,\alpha) = \left(\frac{\partial f_1}{\partial \alpha}\right)^2 = 4\alpha^2(x-1)^2$$

From $$\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial \alpha} - \frac{\partial f_1}{\partial \alpha}\frac{\partial f_2}{\partial x} = 0$$

$$\alpha^2 = 2x+1 \qquad (x \geq -1/2)$$

So $$f_1[x,\alpha(x)] = x^2 + (1+\alpha^2)x - \alpha^2 = 3x^2 - 1$$

$$f_2[x,\alpha(x)] = 4(2x+1)(x-1)^2$$

We could get $x = F(f_1)$ then plug it in $f_2(x)$, so we have the envelope $f_1 \sim f_2$.)

Generally, given a desired λ_{12} , we can choose the most compromised solution x^* and α^* (parameter design) in the following way:

$$\lambda_{12} = -\frac{2x+1+\alpha^2}{8\alpha^2(x-1)} = \text{ given value}$$

$$2x+1 = \alpha^2$$

(two equations with two unknowns)

Simplifying it:

$$\lambda_{12} = -\frac{1}{4(x-1)}$$

Actually, on the envelope we can easily calculate the trade-off, from the envelope equation:

$$\frac{\partial f_1}{\partial \alpha}\frac{\partial f_2}{\partial \alpha} = \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial x_2}$$

So $$\lambda_{12} = -\frac{\partial f_1}{\partial f_2} = -\frac{\frac{\partial f_1}{\partial x}}{\frac{\partial f_2}{\partial x}} = -\frac{\frac{\partial f_1}{\partial \alpha}}{\frac{\partial f_2}{\partial \alpha}} = -\frac{\frac{\partial f_1}{\partial \alpha}}{2\frac{\partial f_1}{\partial \alpha}\frac{\partial^2 f_1}{\partial \alpha^2}} = -\frac{1}{2\frac{\partial^2 f_1}{\partial \alpha}}$$

The necessary condition for the Pareto optimum is:

$$\lambda_{12} \geq 0 \text{ or } \frac{\partial^2 f_1}{\partial \alpha^2} \leq 0$$

This can be used to determine the existence of the envelope on the Pareto frontier (see Figure V.4.2).

In the above example:

Original: $\frac{\partial^2 f_1}{\partial \alpha^2} = 2(1-x) \geq 0$ no envelope $\left(\frac{\alpha^2 - 1}{2} \leq x \leq 1\right)$

Modified: $\frac{\partial^2 f_1}{\partial \alpha^2} = 2(x-1) \leq 0$ envelope, since $\left(-\frac{\alpha^2 + 1}{2} \leq x \leq 1\right)$

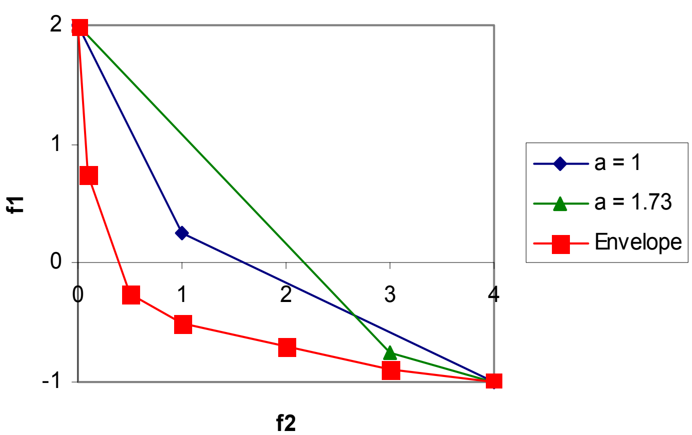

Figure V.4.2. f_1 **versus** f_2

EXERCISE D: USIM Extension

Consider a system with the following objective function:

$$f(\underline{x}, \alpha) = (x_3 - 1)^2 + \frac{1}{2}(1-\alpha^2)x_1 + \alpha^2$$

s.t. $h(\underline{x}) = \beta x_1 + \beta^2 x_2 + x_3 - 1 = 0$

α, β have nominal values 0.5 and 1, respectively.

SOLUTION

Consider minimizing the sensitivity of the constraint:

Let $f_3(\underline{x}) = \left.\left(\dfrac{\partial h(\underline{x},\beta)}{\partial \beta}\right)^2\right|_{\beta=\hat{\beta}} = (x_1 + 2x_2)^2$

Minimize it under $h(\underline{x},\beta) = 0$

$$\min f_3(\underline{x}) = (x_1 + 2x_2)^2$$
$$\text{s.t. } x_1 + x_2 + x_3 - 1 = 0$$

Solve the problem and get:

$$x_3 = 1 - x_2$$
$$x_1 = -2x_2$$

Substitute these values in $f(\underline{x},\alpha)$ and calculate:

$$f(\underline{x},\alpha) = x_2{}^2 + (1-\alpha^2)x_2 + \alpha^2$$

Since this is exactly the same function that was minimized in Exercise A, the remainder of the USIM solution is the same as before.

PROBLEM V.5: Budget Allotment for Cyber-Security

A large company is considering a change in the amount of money to budget for cyber-security for the upcoming year.

DESCRIPTION

Changing the budget (negative means reduce it, positive means increase) could have an effect on the number of cyber attacks during that year, and can be calculated using the following equation:

$$f_1(x,\alpha) = y(x,\alpha) = 3x^2 - 4x(\alpha - 2) - \alpha^2$$

METHODOLOGY

Since the company is uncertain as to which fiscal policy to adopt, they use the Uncertainty Sensitivity Index Method (USIM) to help guide their decision.

The decision variable, *x*, represents the change in budget for the upcoming year. This is used to calculate the change in number of cyber attacks for that time period. Thus, a negative value for the objective function means there will be a reduction in the number of attacks. While increasing the budget will generally reduce the number of attacks, at some point attacks will begin increasing as the budget increases. Also, a decrease in the budget may still cause a decrease in attacks. This happens because allocating money to cyber-security takes away from (or gives more to) facility security allocations, thus more (or fewer) attacks will occur. The objective is to find the change in budget that minimizes the number of attacks. The USIM should be applied because the model parameter is unknown. All budget values are in units of $1 million. All attack values are in units of tens (i.e., -9 means 90 fewer attacks).

SOLUTION

An alpha value of 3 was determined using a systems identification procedure.

Given $\hat{\alpha} = 3$

$$y(x,\hat{\alpha}) = 3x^2 - 4x - 9 \tag{V.5.1}$$

$$f_2(x,\hat{\alpha}) = [-4x - 2\alpha]^2 = 16x^2 + 16\alpha x + 4\alpha^2 \tag{V.5.2}$$

(V.5.1) and (V.5.2) can be written as a joint optimality and sensitivity problem as follows:

$$\min \begin{bmatrix} f_1(x,\hat{\alpha}) \\ f_2(x,\hat{\alpha}) \end{bmatrix} \tag{V.5.3}$$

$$\min\begin{bmatrix} f_1(x,\hat{\alpha}) = 3x^2 - 4x - 9 \\ f_2(x,\hat{\alpha}) = 16x^2 + 48x + 36 \end{bmatrix} \tag{V.5.4}$$

Use the ε-constraint form:

$$\min[3x^2 - 4x - 9] \tag{V.5.5}$$

$$16x^2 + 48x + 36 \le \varepsilon_2 \tag{V.5.6}$$

From (V.5.5) and (V.5.6) formulate the Lagrangian function:

$$L(x,\hat{\alpha},\lambda_{12}) = 3x^2 - 4x - 9 + \lambda_{12}[16x^2 + 48x + 36 - \varepsilon_2] \tag{V.5.7}$$

According to the Kuhn-Tucker necessary conditions, (V.5.7) can be solved:

$$\frac{\partial L}{\partial x} = 6x - 4 + \lambda_{12}[32x + 48] = 0 \tag{V.5.8}$$

$$\frac{\partial L}{\partial \lambda_{12}} = 16x^2 + 48x + 36 - \varepsilon_2 \le 0 \tag{V.5.9}$$

$$\lambda_{12}[16x^2 + 48x + 36 - \varepsilon_2] = 0,\ \lambda_{12} \ge 0 \tag{V.5.10}$$

From (V.5.8) solve for λ_{12}:

$$\lambda_{12} = \frac{4 - 6x}{32x + 48} \tag{V.5.11}$$

ANALYSIS

Given equations (V.5.8) through (V.5.11), Table V.5.1 and Figure V.5.1 show noninferior solutions and trade-off values.

Table V.5.1. Noninferior Solutions & Trade-off Values

x	$f_1(x,\hat{\alpha})$	$f_2(x,\hat{\alpha})$	λ_{12}
0.67	-10.33	75.11	0.00
0.5	-10.25	64	0.02
0.4	-10.12	57.76	0.03
0.3	-9.93	51.84	0.04
0.2	-9.68	46.24	0.05
0.1	-9.37	40.96	0.07
0	-9	36	0.08
-0.1	-8.57	31.36	0.10

x	$f_1(x,\hat{\alpha})$	$f_2(x,\hat{\alpha})$	λ_{12}
-0.2	-8.08	27.04	0.13
-0.3	-7.53	23.04	0.15
-0.4	-6.92	19.36	0.18
-0.5	-6.25	16	0.22
-0.6	-5.52	12.96	0.26
-0.7	-4.73	10.24	0.32
-0.8	-3.88	7.84	0.39
-0.9	-2.97	5.76	0.49
-1	-2	4	0.63
-1.1	-0.97	2.56	0.83
-1.2	0.12	1.44	1.17
-1.3	1.27	0.64	1.84
-1.4	2.48	0.16	3.88
-1.5	3.75	0	∞

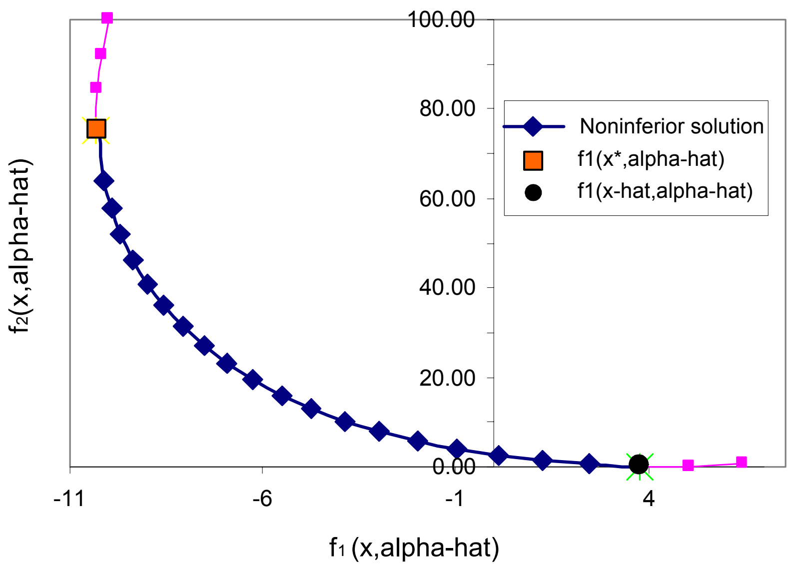

Figure V.5.1. Noninferior solution in functional space

$$\min f_1(x,\hat{\alpha}) = f_1(x^*,\hat{\alpha}) = -10\frac{1}{3} \tag{V.5.12}$$

$$\min f_2(x,\hat{\alpha}) = f_2(\hat{x},\hat{\alpha}) = 0 \tag{V.5.13}$$

From (V.5.12) and (V.5.13) two critical values are computed as:

$$x^* = \frac{2}{3} \text{ (Business-as-usual policy)}$$

$$\hat{x} = -1.5 \text{ (Most conservative policy)}$$

Given x^* and $\hat{x}$, two functions are derived with respect to α and are plotted versus α :

$$f_1(x^*, \alpha) = \frac{4}{3} - \frac{8}{3}(\alpha - 2) - \alpha^2 \qquad \text{(V.5.14)}$$

$$f_1(\hat{x}, \alpha) = \frac{27}{4} + 6(\alpha - 2) - \alpha^2 \qquad \text{(V.5.15)}$$

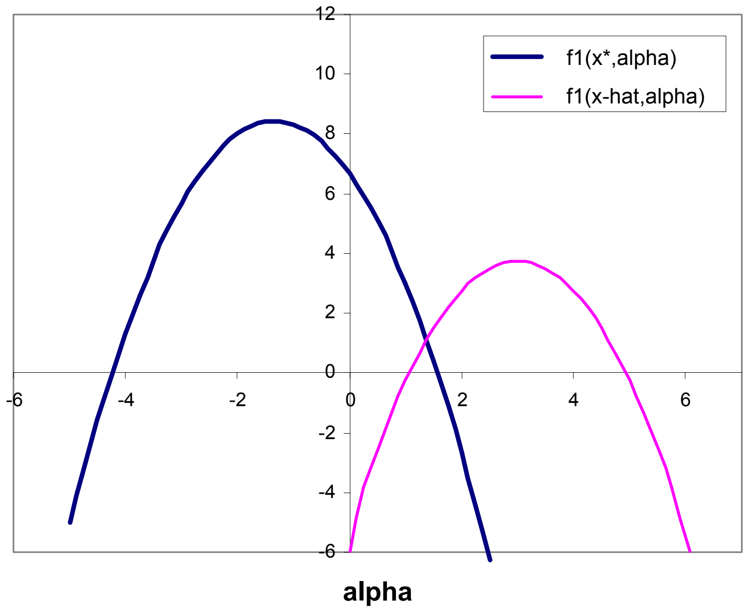

Figure V.5.2. The functions $f_1(x^*, \alpha)$ **and** $f_1(\hat{x}, \alpha)$

$$\left.\frac{\partial f_1(x^*, \alpha)}{\partial \alpha}\right|_{\alpha = \hat{\alpha}} = -\frac{8}{3} - 2\hat{\alpha} = -\frac{26}{3} \qquad \text{(V.5.16)}$$

$$\left.\frac{\partial f_1(\hat{x}, \alpha)}{\partial \alpha}\right|_{\alpha = \hat{\alpha}} = 6 - 2\hat{\alpha} = 0 \qquad \text{(V.5.17)}$$

From (V.5.16) and (V.5.17), we can distinguish stabilities for two objective functions. A most conservative policy will lead to a more stable state than a business-as-usual policy.

Along with perturbation in α**,** another plot will help us gain better understanding of a situation involving uncertainty:

$$f_1(x^*,\hat{\alpha}) = -10.33 \tag{V.5.18}$$
$$f_1(x^*,\hat{\alpha}-.5) = -6.25 \tag{V.5.19}$$
$$\left|f_1(x^*,\hat{\alpha}) - f_1(x^*,\hat{\alpha}-.5)\right| = -4.08 \tag{V.5.20}$$

$\eta(x^*,.75\hat{\alpha}) = 39.5\%$

$$f_1(\hat{x},\hat{\alpha}) = 3.75 \tag{V.5.21}$$
$$f_1(\hat{x},\hat{\alpha}-.5) = 3.5 \tag{V.5.22}$$
$$\left|f_1(\hat{x},\hat{\alpha}) - f_1(x^*,\hat{\alpha}-.5)\right| = .25 \tag{V.5.23}$$

$\eta(\hat{x},.75\hat{\alpha}) = 6.7\%$

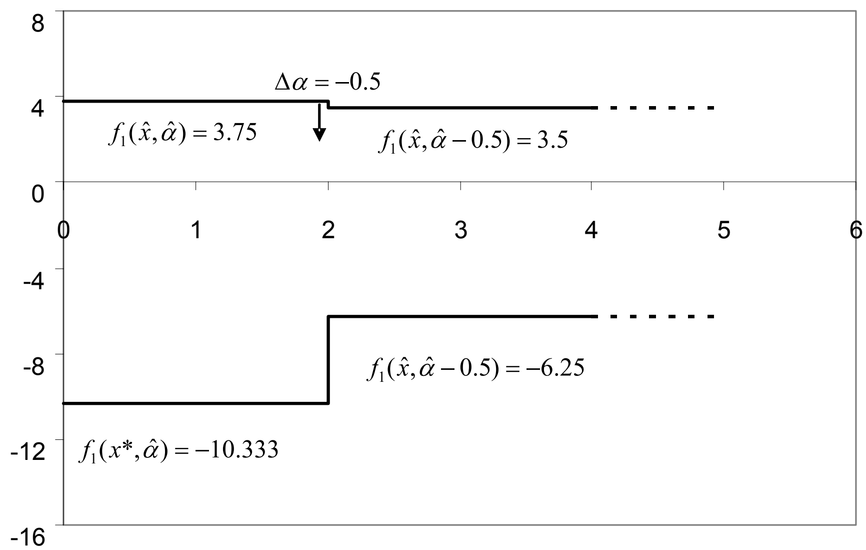

Figure V.5.3. The functions $f_1(x^*,\alpha)$ and $f_1(\hat{x},\alpha)$ versus perturbation in α

The results given in Figure V.5.3 indicate that following a conservative policy that trades optimality for a less sensitive outcome provides a very stable solution (6.7% versus 39.5%). Using the Surrogate Worth Trade-off (SWT) method, and talking to the person in the company who is in charge of this decision, the preferred change in budget should be between –$1.5 million and $670,000. It does seem logical to choose a value that is a reduction in budget that also causes a reduction in attacks. Thus, it may make sense to choose a budget change between –$1.1 million and no change.

PROBLEM V.6: Art Museum Temperature Maintenance

In order to keep the artworks housed in an art museum in their best condition, the interior temperature of the building must be closely controlled and monitored. The problem is to determine the desired temperature at an optimal cost within the given climate of the museum.

DESCRIPTION

Suppose that the goal is to set the temperature at the low 60s degrees. Let x represent the temperature and $y(x,\alpha)$ denote the cost. The cost (in thousands of dollars) is a function of both the temperature x and a parameter α and can be written as follows:

$$y(x,\alpha) = (x-60)^2 - \alpha x - \alpha^2$$

METHODOLOGY

Use the Uncertainty Sensitivity Index Method (USIM) to solve this problem.

Let the cost objective function be redefined as $f_1(x,\alpha)$, and we wish to minimize it:

$$f_1(x,\alpha) = y(x,\alpha)$$

or

$$f_1(x,\alpha) = (x-60)^2 - \alpha x - \alpha^2$$

Let the nominal value of α be $\hat{\alpha}$ where $\hat{\alpha}$ = 10. Then $y(x,\hat{\alpha})$ can be rewritten as $y(x,\hat{\alpha}) = x^2 - 130x + 3500$. Since

$$\frac{\partial y(x,\alpha)}{\partial \alpha} = -x - 2\alpha \ ,$$

we define the sensitivity index function f_2 to be

$$f_2(x,\alpha) = x^2 + 4\alpha x + 4\alpha^2 \ .$$

Substituting α with $\hat{\alpha}$, we have

$$f_1(x,\alpha) = x^2 - 130x - 3500 \qquad \text{(V.6.1)}$$
$$f_2(x,\alpha) = x^2 + 40x + 400 \qquad \text{(V.6.2)}$$

Suppose there are no constraints on x. The joint optimality and sensitivity problem can be written in a multiobjective framework as follows:

$$\text{Min } f_1(x,\alpha) = x^2 - 130x - 3500 \tag{V.6.3}$$
$$\text{Min } f_2(x,\alpha) = x^2 + 40x + 400 \tag{V.6.4}$$

Solve via the SWT method:

The first phase is converting the second objective f_2 into the ε -constraint as follows:

$$\text{Min } f_1(x,\hat{\alpha}) \tag{V.6.5}$$
$$\text{s.t. } f_2(x,\hat{\alpha}) \le \varepsilon_2 \tag{V.6.6}$$

The problem can be written as:

$$\text{Min } x^2 - 130x - 3500 \tag{V.6.7}$$
$$\text{s.t. } x^2 + 40x + 400 \le \varepsilon_2 \tag{V.6.8}$$

Form the Lagrangian function,

$$L(x,\hat{\alpha},\lambda_{12}) = x^2 - 130x - 3500 + \lambda_{12}\left(x^2 + 40x + 400 - \varepsilon_2\right) \tag{V.6.9}$$

The Kuhn-Tucker necessary conditions for stationarity are as follows:

$$\frac{\partial L(\cdot)}{\partial x} = 2x - 130 + \lambda_{12}(2x + 40) = 0 \tag{V.6.10}$$
$$\frac{\partial L(\cdot)}{\partial \lambda_{12}} = x^2 + 40x + 400 - \varepsilon_2 \le 0 \tag{V.6.11}$$
$$\lambda_{12}\left(x^2\, 40x + 400 - \varepsilon_2\right) = 0 \tag{V.6.12}$$
$$\lambda_{12} \ge 0 \tag{V.6.13}$$

Solving Equation (V.6.10) yields

$$\lambda_{12} = \frac{130 - 2x}{2x + 40} \tag{V.6.14}$$

See Table V.6.1 for several noninferior solutions with the corresponding tradeoff values.

Table V.6.1. Noninferior Solutions and Tradeoff Values

x	$f_1(x,\hat{\alpha})$	$f_2(x,\hat{\alpha})$	λ_{12}
-19	6331	1	84
-10	4900	100	7.5
0	3500	400	3.25
10	2300	900	1.833
20	1300	1600	1.125
30	500	2500	0.7
40	-100	3600	0.41667
50	-500	4900	0.214286
60	-700	6400	0.0625

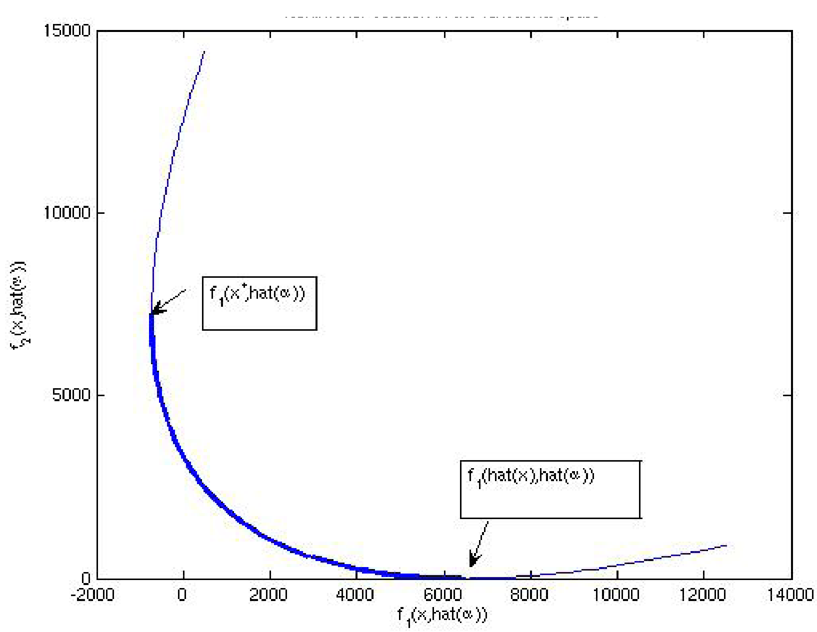

Figure V.6.1. Noninferior Solution in the Functional Space

Figure V.6.1 depicts the noninferior solution in the functional spaces f_1 and f_2. Let x^* and $\hat{x}$ denote the decision variables which minimize $f_1(x,\hat{\alpha})$ and $f_2(x,\hat{\alpha})$. In other words:

$$\text{Min } f_1(x,\hat{\alpha}) = f_1(x^*,\hat{\alpha}) \tag{V.6.15}$$

$$\text{Min } f_2(x,\hat{\alpha}) = f_1(\hat{x},\hat{\alpha}) \tag{V.6.16}$$

Then we can compute x^* and $\hat{x}$ with a straightforward method of looking for stationary points in the respective functions to yield:

$$x^* = 65$$
$$\hat{x} = -20$$

To study the tradeoffs between the sensitivity objective function f_2 and the optimality objective function f_1, the latter is evaluated at x^* and $\hat{x}$ as a function of α. The resulting functions $f_1(x^*, \alpha)$ and $f_2(\hat{x}, \alpha)$ are plotted in Figure V.6.2. The functions are as follows:

$$f_1(x^*, \alpha) = -\alpha^2 - 65\alpha + 25 \quad \text{(V.6.17)}$$
$$f_2(\hat{x}, \alpha) = -\alpha^2 + 20\alpha + 6400 \quad \text{(V.6.18)}$$

Note that at the nominal value of α, $f_1(x^*, \hat{\alpha})$ changes rapidly with a slope equal to -85.
$f_2(\hat{x}, \alpha)$ has a rate of zero at the nominal value of α ,10.

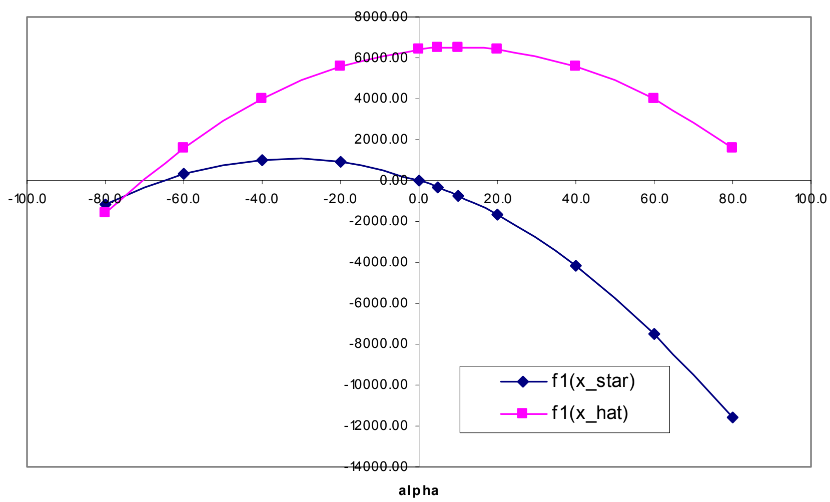

Figure V.6.2. Sensitivity as a function of the Parameter α

Now we focus on the changes that take place in $f_1(x^*, \hat{\alpha})$ and $f_1(\hat{x}, \hat{\alpha})$ when the nominal value α is perturbed by the amount $\Delta\alpha = -5$. Then as a result we have:

$$f_1(x^*, \hat{\alpha}) = -725$$

$$f_1(x^*, \hat{\alpha} - 5) = -325$$
$$\left| f_1(x^*, \hat{\alpha}) - f_1(x^*, \hat{\alpha} - 5) \right| = 400$$

ANALYSIS

Further analysis is performed to determine the performance of the cost function $f_1(x^*, \hat{\alpha})$ relative to $f_1(\hat{x}, \hat{\alpha})$.

Let $\eta(x^*, \hat{\alpha} - 5)$ denote the percentage of change in $f_1(x^*, \hat{\alpha})$ with a perturbation of 50% in $\hat{\alpha}$. Then

$$\eta(x^*, 0.5\hat{\alpha}) = 55\%$$

Similarly,

$$f_1(\hat{x}, \hat{\alpha}) = 6500$$
$$f_1(\hat{x}, \hat{\alpha} - 5) = 6475$$
$$\left| f_1(\hat{x}, \hat{\alpha}) - f_1(x^*, \hat{\alpha} - 5) \right| = 2500$$

and

$$\eta(\hat{x}, 0.5\hat{\alpha}) = 0.38\%.$$

See Figure V.6.3 for the comparison of $\eta(x^*, 0.5\hat{\alpha})$ and $\eta(\hat{x}, 0.5\hat{\alpha})$. It is clear that the conservative policy that trades optimality for a less sensitive outcome provides an extremely stable solution. In the case of $\eta(x^*, 0.5\hat{\alpha})$, we have a 50% deviation given a 50% perturbation in the nominal value of α. In the latter case, the deviation is basically ignorable given the same perturbation. Therefore, if the nominal value of α is incorrectly assessed, the result would be rather disastrous if we choose x^* over $\hat{x}$, even though x^* would give us the better value for the optimality problem. On the other hand, $\hat{x}$ makes the problem very parameter-insensitive but the objective value is not so good. In the end, however, we still need to interact with a decisionmaker about which preferred x is chosen with $\hat{x} \leq x \leq x^*$.

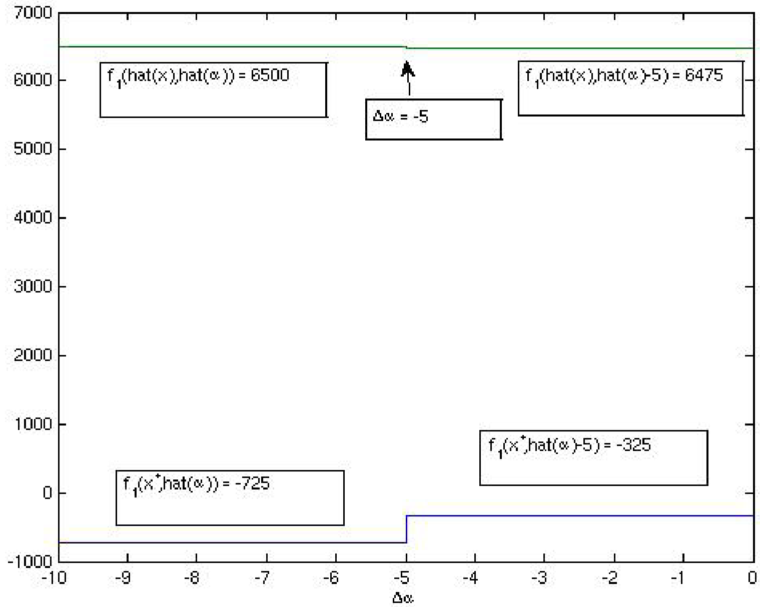

Figure V.6.3. Cost as a function of the Parameter α

PROBLEM V.7: Multiobjective Optimization and Sensitivity Analysis

This problem demonstrates how to integrate sensitivity analysis with multiobjective optimization. Solve the following multiobjective optimization problem using the Uncertainty Sensitivity Index Method (USIM) and analyze your results.

$$f_1 = (x-2)^2 + 2x\alpha + \alpha^2$$

$$f_2 = \left(\frac{\partial f_1}{\partial \alpha}\right)^2 = (2x+2\alpha)^2 = 4(x+\alpha)^2$$

Assume the nominal value $\hat{\alpha} = 1$

PROBLEM V.8: Earthquake-Proofing a Building

How can a building be structurally fortified to counteract the vibration caused by an earthquake? The vibration of a building caused by an earthquake may be dampened by placing shock-absorbing materials under and around its foundation, as can be seen in Figure V.8.1. Vibration risk can be related to the "work" exerted by the building structure to counteract the forces acting on it. Suppose that greater magnitudes of "work" lessen the susceptibility of the building to vibration risk, which consequently leads to less structural stress. A simple schematic of the problem is depicted in the given diagram consisting of only two active forces: (i) weight of the structure; and (ii) "spring" force.

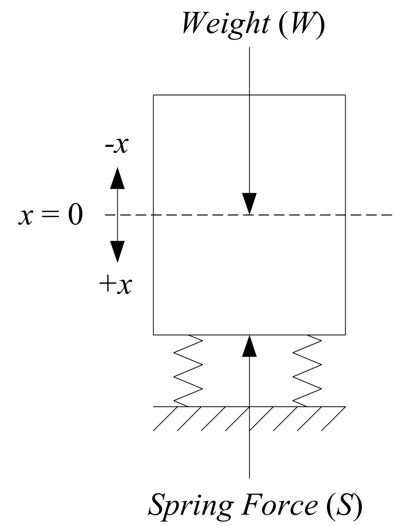

Figure V.8.1. Demonstration of shock absorbing materials

Consider the following model describing the "work" ω exerted by the building due to the forces present in the above diagram:

$$\omega = \omega_S + \omega_W = -0.5\alpha x^2 + Wx$$

where:

- x: vibration-triggered vertical displacement of the building in meter (m), measured relative to an equilibrium position (i.e., "initial deformation" of the spring)
- $\omega_S = -0.5\alpha x^2$: "work" component due to the spring force S where α is the Hooke's Law spring constant, whose nominal value is $\hat{\alpha}$ = 2×10^9 Newton per meter (N/m).
- $\omega_W = Wx$: "work" component due to the weight of the building (W = 2×10^9 N)

Use the Uncertainty Sensitivity Index Method (USIM) and the Surrogate Worth Trade-off (SWT) method in order to incorporate the uncertainty of the parameters.

PROBLEM V.9: Evaluating Investments for a Portfolio

Risk must be evaluated in a portfolio of two investments. Portfolio risk can be assessed using the variance metric (denoted here by f_1). For the case of two investments:

$$f_1 = \sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \rho\sigma_1\sigma_2 x_1 x_2$$

Note that in general, the above expression is derived as follows:

$$\begin{aligned} f_1 = Variance &= Var(x_1 Inv_1 + x_2 Inv_2) \\ &= x_1^2 Var(Inv_1) + x_2^2 Var(Inv_2) + 2x_1x_2 Cov(Inv_1, Inv_2) \\ &= x_1^2\sigma_1^2 + x_2^2\sigma_2^2 + 2x_1x_2 Cov(Inv_1, Inv_2) \end{aligned}$$

For portfolio selection consisting of two investments:

$$Cov(Inv_1, Inv_2) = \frac{1}{2}\rho\sigma_1\sigma_2$$

where:

f_1 = portfolio risk (this risk is measured in terms of variance of portfolio returns, hence f_1 is unitless)
σ_1 = standard deviation (or volatility) of returns of Investment 1 (Inv_1)
σ_2 = standard deviation (or volatility) of returns of Investment 2 (Inv_2)
ρ = correlation of returns of Investments 1 and 2
x_1 = portfolio weight to allocate to Investment 1
$x_2 = 1 - x_1$ = portfolio weight to allocate to Investment 2

Use the Uncertainty Sensitivity Index Method (USIM) to answer the following questions:

(a) Derive $f_1(x_1,\alpha)$, given the following parameters:

$\sigma_1 = 0.2$
$\sigma_2 = \alpha$
$\rho = -0.8$

(b) Derive the sensitivity function $f_2(x_1,\alpha)$.

(c) Plot the noninferior solution in the function space using the functions obtained in Steps (a) and (b). Use a nominal value of $\hat{\alpha} = 0.3$.

(d) Analyze the results and discuss the sensitivity of portfolio risk to different values of α.

PROBLEM V.10: Preventing West Nile Viral Disease

The West Nile virus is spread to humans through a bite from the *Culex* species of mosquito. Once in the bloodstream, the virus can reach the brain and cause encephalitis—a brain inflammation that can affect the entire nervous system. Unfortunately, there is no specific treatment for West Nile encephalitis other than supportive therapy (such as hospitalization, intravenous fluids, and respiratory support) for severe cases. Antibiotics will not work because a virus, not bacteria, causes West Nile disease. No vaccine for the virus is currently available.

Applying a DEET-based insect repellent (DEET is short for N,N-diethyl-m-toluamide) is recommended to minimize the risk of acquiring the disease. The downside is that such repellents have been thought to cause adverse skin reactions when used in excessive quantities (especially when combined with sunscreen). The question is: how much DEET can a person apply to avoid the risk of contracting West Nile disease without suffering an adverse skin reaction?

This problem can be solved using the Uncertainty Sensitivity Index Method (USIM), as follows:

Consider a health risk function which has the following form:

$$f_1(x;\alpha) = 1 - \alpha x^{\alpha-1}\exp(-x^{\alpha}) \qquad\qquad x > 0$$

where:

$f_1(x;\alpha)$ = health risk function, which is normalized such that a value of 1 means maximum health risk and 0 means minimum health risk.
x = amount of repellent applied (in grams per square inch of skin)
α = concentration of active ingredient (in parts per 10)

When $\hat{\alpha} = 2$ (i.e., the nominal value of α), the health risk function behaves like a bathtub curve as depicted below.

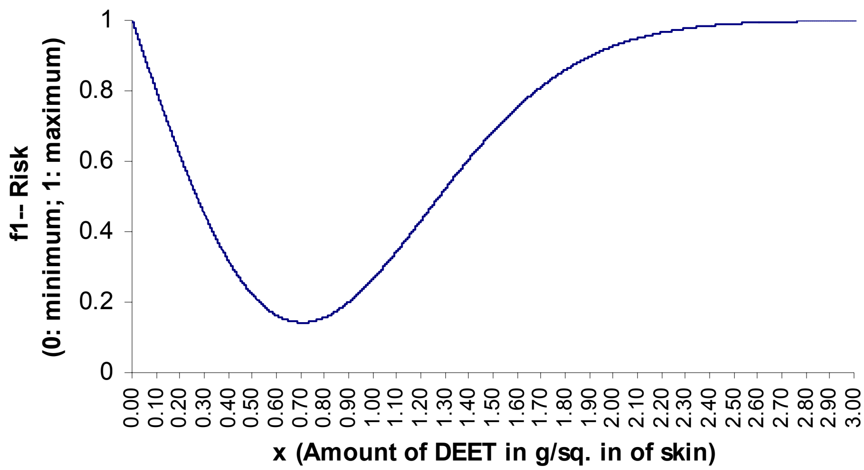

Figure V.10.1. Risk Function for Amount Of DEET Applied to Skin

Conduct USIM and analyze the results. (Note: You may have to resort to numerical methods when generating noninferior solutions for the multiobjective problem comprising $f_1(x; \hat{\alpha})$ and its corresponding sensitivity function $f_2(x; \hat{\alpha})$).

PROBLEM V.11: Optimizing Amount of Catalyst in a Chemical Substance

This problem examines the sensitivity of a given chemical substance as a function of the amount of reaction time given the amount of catalyst.

This exercise is concerned with an industrial process in which the reaction time of a chemical substance depends on the temperature and the amount of catalyst. We are interested in minimizing the reaction time and its sensitivity to the temperature. Currently we are not satisfied with the specified reaction time of a catalyst we are using. We have to determine the amount of catalyst that will give us the least reaction time. The difference in the reaction time (from the original reaction time) is given by the following equation:

$$y(x,\alpha) = \frac{3}{2}x^2 + \frac{3}{2}x(1-\alpha) - \frac{2}{3}\alpha^2$$

Solve the problem using the Uncertainty Sensitivity Index Method (USIM).

A negative value of y indicates a reduction in the original reaction time, while a positive value indicates an increase. In short, the greater the negative value we obtain the better, because we are reducing our original reaction time.

- $y(x,\alpha)$ denotes the difference in reaction time
- x denotes the difference from the original amount of catalyst. (A negative value denotes a reduction of the original amount, while a positive value denotes an increase.)
- α denotes the model's parameter (temperature). Assume a nominal value of $\hat{\alpha} = 2$.

PROBLEM V.12: Uncertainty Regarding Costs in a Widget Factory

A company that produces widgets needs to balance the costs of labor and materials. The company has two objectives: to minimize costs in general, and to minimize the fluctuating costs of labor.

For each widget, the factory must use a specific type of expensive paint. There is a cost function that depends on the amount of paint used, $f_1(x, \alpha) = (x\text{-}6)^2 - \alpha^2 (x\text{-}3) - (\alpha\text{-}4)^2$. Let x represent the amount of paint used, and α represent some price fluctuation in the cost of labor. Assume that $\hat{\alpha} = 6$.

Use the Uncertainty Sensitivity Index Method (USIM) to solve this problem.

PROBLEM V.13: Determining Safe Proportions of Chemical Components

Electrostatic deposition, also known as electroplating, is a manufacturing process wherein a metal is deposited onto the surface of a plastic or another metal to affect the latter's physical, mechanical, or chemical property. Prior to the actual deposition process, the surface of the material requires extensive cleaning to assure proper adhesion. To avoid environmental hazards, the process engineer wants to determine the correct amounts of two chemicals that can be used as cleansing agents.

In the cleaning process, the electroplating plant must minimize toxic fumes and totally avoid producing heavy metal. Two major chemicals— X_1 and X_2 —are commonly present in the cleansing agent and are also available in 50% solutions. These produce toxic fumes in the reduction process given by:

$$y = 0.0003{x_1}^2\beta - 0.02x_2\alpha^2$$

where: y is the amount of toxic fumes

x_1 is the amount of Chemical X_1

x_2 is the amount of Chemical X_2

α is the concentration of Chemical X_1

β is the concentration of Chemical X_2

Furthermore, the cleaning process causes the two chemicals to react with the material being cleaned to produce a heavy metal. As there is no economically feasible way of filtering this out from the cleansing agent, producing the heavy metal must be totally avoided. The heavy metal production is given by the chemical process:

$${x_1}^2(1-\beta) + x_2\beta = 0.00008$$

Given two objective functions, we can assume the variability of α and β, so the Uncertainty Sensitivity Index Method (USIM) is applied to solve this problem. The new functions of representing sensitivities need to be addressed as well. Assume nominal parameter values of $\hat{\alpha} = 0.5$ and $\hat{\beta} = 0.5$.