IV. Surrogate Worth Tradeoff

PROBLEM IV.1: Selecting the Location of a New University Bus Stop

A university plans to add a shuttle stop near the stadium, aimed at helping students to easily walk to their classrooms and a dining hall.

DESCRIPTION

If the stadium is the center of the coordinates, the classroom building is at (100, 300) and the dining hall is at (200, 400). We assume that 1) students can walk directly to either of them from the bus stop, and that 2) the stadium area has the space for it. Where should we locate this new bus stop so that the walking distance can be shortest?

METHODOLOGY

The objective of this problem is to minimize the walking distance from the bus stop. We use the Surrogate Worth Tradeoff (SWT) method to determine a bus stop location that will satisfy both destinations.

The objective function is:

$$\min\begin{Bmatrix} f_1(x_1,x_2)=\sqrt{(x_1-100)^2+(x_2-300)^2} \\ f_2(x_1,x_2)=\sqrt{(x_1-200)^2+(x_2-400)^2} \end{Bmatrix} \tag{IV.1.1a}$$

Because the number in the square-root operation is always equal to or larger than zero, the objective function is actually the same as:

$$\min\begin{Bmatrix} f_1(x_1,x_2)=(x_1-100)^2+(x_2-300)^2 \\ f_2(x_1,x_2)=(x_1-200)^2+(x_2-400)^2 \end{Bmatrix} \tag{IV.1.1b}$$

SOLUTION

First convert Eq. (1) into the ε-constraint form presented by Eq. (2):

Subject to

$$\left.\begin{matrix} \min f_1(x_1,x_2) \\ f_2(x_1,x_2)\le\varepsilon_2 \end{matrix}\right\} \tag{IV.1.2}$$

Form the Lagrangian function:

$$L(x_1, x_2, \lambda_{12}) = (x_1 - 100)^2 + (x_2 - 300)^2 + \lambda_{12}[(x_1 - 200)^2 + (x_2 - 400)^2 - \varepsilon_2] \quad \text{(IV.1.3)}$$

By Kuhn-Tucker necessary conditions, derive Eqs. (4) to (8):

$$\frac{\partial L(\cdot)}{\partial x_1} = 2(x_1 - 100) + 2\lambda_{12}(x_1 - 200) = 0 \quad \text{(IV.1.4)}$$

$$\frac{\partial L(\cdot)}{\partial x_2} = 2(x_2 - 300) + 2\lambda_{12}(x_2 - 400) = 0 \quad \text{(IV.1.5)}$$

$$\frac{\partial L(\cdot)}{\partial \lambda_{12}} = (x_1 - 200)^2 + (x_2 - 400)^2 - \varepsilon_2 \le 0 \quad \text{(IV.1.6)}$$

$$\lambda_{12}[(x_1 - 200)^2 + (x_2 - 400)^2 - \varepsilon_2] = 0 \quad \text{(IV.1.7)}$$

$$\lambda_{12} \ge 0$$

Equation (IV.1.4) yields:

$$\lambda_{12} = -\frac{x_1 - 100}{x_1 - 200} \quad \text{(IV.1.8)}$$

Equation (IV.1.5) yields

$$\lambda_{12} = -\frac{x_2 - 300}{x_2 - 400} \quad \text{(IV.1.9)}$$

From Eqs, (IV.1.8) and (IV.1.9) we get:

$$\lambda_{12} = -\frac{x_1 - 100}{x_1 - 200} = -\frac{x_2 - 300}{x_2 - 400}$$

$$x_2 = x_1 + 200 \quad \text{(IV.1.10)}$$

Upper and lower limits on x_1 and x_2 may easily be derived by satisfying Eqs. (IV.1.8) and (IV.1.9), as follows:

$$100 < x_1 < 200$$
$$300 < x_2 < 400$$

Samples of Pareto-optimal solutions are shown in Table IV.1.1.

Table IV.1.1. Noninferior Solutions and Tradeoff Values

x_1	x_2	$f_1(x_1, x_2)$	$f_2(x_1, x_2)$	λ_{12}
110	310	200	16200	0.1111
120	320	800	12800	0.2500
140	340	3200	7200	0.6667
160	360	7200	3200	1.5000
180	380	12800	800	4.0000

The values of surrogate worth functions generated by the decisionmaker selecting this bus stop are tabulated as W_{12} in the following table:

Table IV.1.2. Noninferior Solutions, Tradeoff Values, and Surrogate Worth Function Values

x_1	x_2	$f_1(x_1, x_2)$	$f_2(x_1, x_2)$	λ_{12}	W_{12}
110	310	200	16200	0.1111	+8
120	320	800	12800	0.2500	+6
140	340	3200	7200	0.6667	+3
150	350	5000	5000	1	0
160	360	7200	3200	1.5000	-3
180	380	12800	800	4.0000	-6

When $W_{12} = 0$, we get a preferred solution: $x_1 = 150, x_2 = 350$.

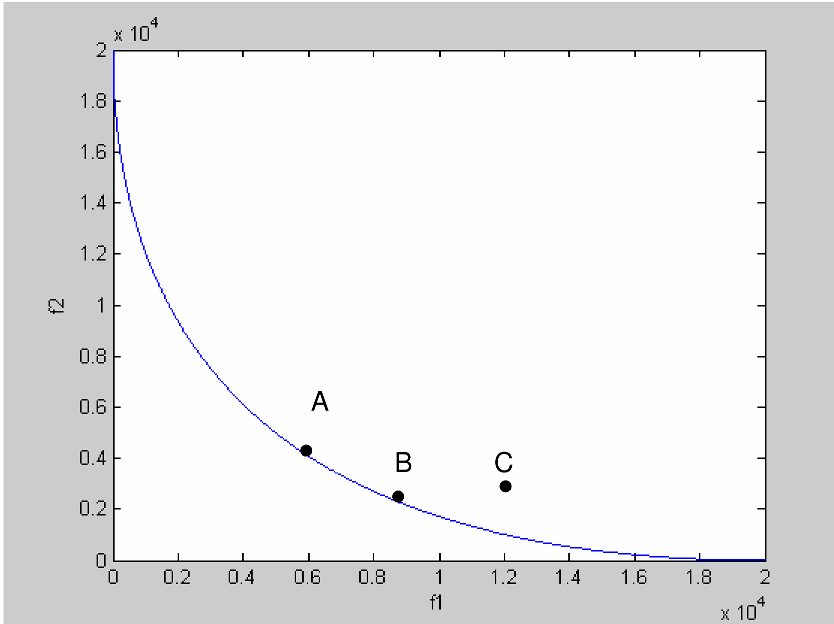

Figure IV.1.1. Noninferior Solution in the Functional Space

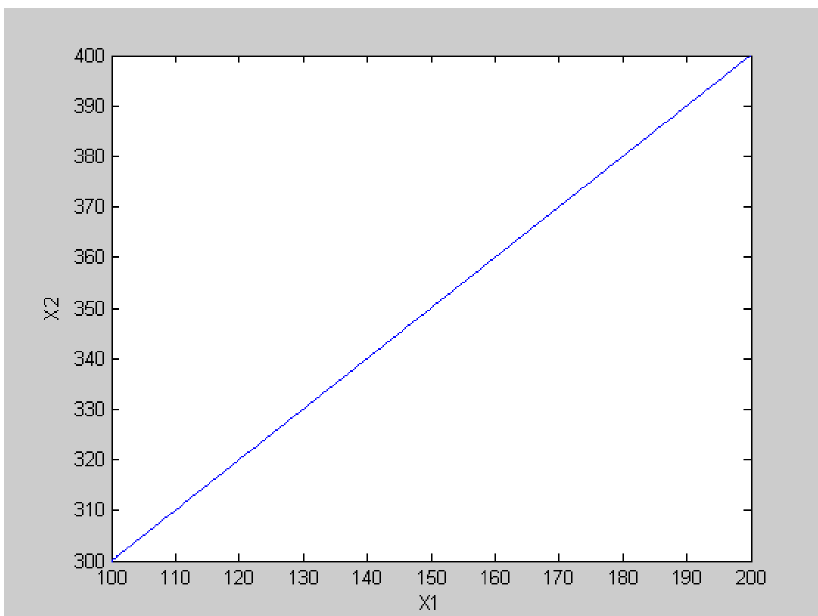

Figure IV.1.2. Noninferior Solution in the Decision Space

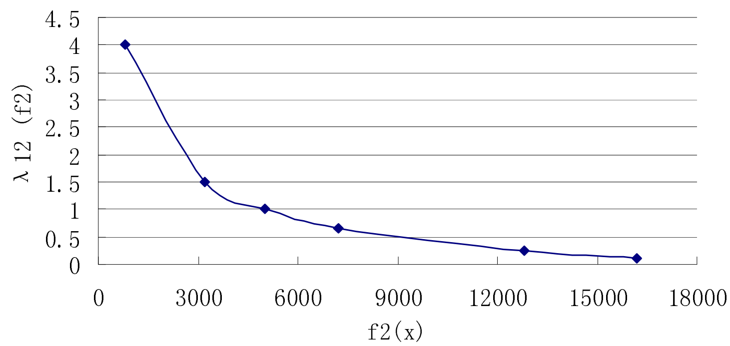

Figure IV.1.3. Tradeoff Function λ_{12} (f_2) versus f_2(x)

ANALYSIS

From the plots in Figures IV.1.1 to IV.1.3, if the decision is on the Pareto frontier and we want a smaller f_2(x)—that is, a shorter distance to the classroom building, both λ_{12} and f_1 will be increased. As shown in Figure IV.1.1, if we want a smaller f_2 value than that of Point A, it could be Point B (on the Pareto frontier) or Point C (not on the Pareto frontier), and Points A and B both have larger f_1 values. This means we could not obtain more benefit (a shorter distance to classes) from f_2 without sacrificing f_1 (i.e., students will have a longer walk to the dining hall). Similarly, on the Pareto frontier we could not obtain more benefit from f_1 without sacrificing f_2. This validates the Pareto solution yielded above.

PROBLEM IV.2: Investing in Stock Market and Small Family Business

The purpose of this problem is to maximize profit by allocating a person's time and budget on the stock market and a small family business.

DESCRIPTION

Todd has a total of \$20,000 to split up between investing in the stock market and reinvesting in a small family business. He has 8 hours available every day to divide between researching stocks and working on his small business. It goes without saying that he wants to maximize the profit of both his stock investment and his business pursuit.

METHODOLOGY

We use the Surrogate Worth Tradeoff Method (SWT) to solve this multiobjective problem.

Let x_1 denote the hours Todd would spend in researching stocks, and x_2 the amount of money (measured in thousands of dollars) he would invest in the stock market. Also, let $f_1(x_1, x_2)$ denote the profit he could make through stock investment in 5 years, and $f_2(x_1, x_2)$ denote the profit he could make from his business in 5 years. Now let us describe $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ by the following simplified mathematical relations:

$$f_1(x_1, x_2) = 10x_2(1 - e^{-x_1/8}) \tag{IV.2.1}$$

$$f_2(x_1, x_2) = (8 - x_1)(20 - x_2) \tag{IV.2.2}$$

Hence, we can formulate our problem in the following fashion:

$$\text{Max } f_1(x_1, x_2) \tag{IV.2.3}$$

$$\text{Max } f_2(x_1, x_2) \tag{IV.2.4}$$

$$x_i \geq 0,\ \ x_1 \frac{\partial L}{\partial x_1} = 0,\ \ x_2 \frac{\partial L}{\partial x_2} = 0$$

SOLUTION

This is a typical multiobjective tradeoff problem. We need to find the noninferior solution as well as the Pareto-optimal one via analysis based on the Kuhn-Tucker theorem. To facilitate our analysis, we first reformulate the model in the standard ε-constraint form:

$$\text{Min } \text{-}f_1(x_1, x_2) \tag{IV.2.5}$$

$$\text{st} \quad \text{-}f_2(x_1, x_2) \le \varepsilon_2 \tag{IV.2.6}$$

$$x_1 \ge 0, x_2 \ge 0$$

Equivalently, if we define $\tilde{f}_1(x_1, x_2) = -f_1(x_1, x_2)$, $\tilde{f}_2(x_1, x_2) = -f_2(x_1, x_2)$, then we can rewrite (IV.2.5) and (IV.2.6) as follows:

$$\text{Min } \tilde{f}_1(x_1, x_2) \tag{IV.2.5'}$$

$$\text{st} \quad \tilde{f}_2(x_1, x_2) \le \varepsilon_2 \tag{IV.2.6'}$$

$$x_1 \ge 0, x_2 \ge 0$$

The Lagrangian formulation is defined as:

$$\begin{aligned} L &= \tilde{f}_1(x_1, x_2) + \lambda_{12}\left(\tilde{f}_2(x_1, x_2) - \varepsilon_2\right) \\ &= 10x_2(e^{-x_1/8} - 1) + \lambda_{12}\left((x_1 - 8)(20 - x_2) - \varepsilon_2\right) \end{aligned} \tag{IV.2.7}$$

The Kuhn-Tucker conditions lead to:

$$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 10x_2(-\tfrac{1}{8})e^{-\frac{x_1}{8}} + \lambda_{12}(20 - x_2) = 0$$

$$\Rightarrow \lambda_{12} = \frac{10x_2 e^{-\frac{x_1}{8}}}{8(20 - x_2)} \tag{IV.2.8}$$

$$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 10(e^{-\frac{x_1}{8}} - 1) + \lambda_{12}(8 - x_1) = 0$$

$$\Rightarrow \lambda_{12} = \frac{10(1 - e^{-\frac{x_1}{8}})}{8 - x_1} \tag{IV.2.9}$$

$$\frac{\partial L}{\partial \lambda_{12}} \le 0 \Rightarrow (x_1 - 8)(20 - x_2) - \varepsilon_2 \le 0 \tag{IV.2.10}$$

$$\lambda_{12}\frac{\partial L}{\partial \lambda_{12}} = 0 \Rightarrow \lambda_{12}((x_1 - 8)(20 - x_2) - \varepsilon_2) = 0 \tag{IV.2.11}$$

$$\lambda_{12} > 0 \tag{IV.2.12}$$

Since $\lambda_{12} > 0$ guarantees a noninferior solution, (IV.2.10)–(IV.2.12) leads to

$$(x_1 - 8)(20 - x_2) - \varepsilon_2 = 0$$
$$\lambda_{12} > 0$$

Eq. (IV.2.12) requires:

$$20 - x_2 > 0 \quad \Rightarrow \quad 0 < x_2 < 20 \qquad \text{(IV.2.13)}$$
$$8 - x_1 > 0 \quad \Rightarrow \quad 0 < x_1 < 8 \qquad \text{(IV.2.14)}$$

From (IV.2.8) and (IV.2.9) we know that:

$$\frac{10 x_2 e^{-\frac{x_1}{8}}}{8(20 - x_2)} = \frac{10(1 - e^{-\frac{x_1}{8}})}{8 - x_1}$$

This leads to:

$$x_2 = \frac{160(1 - e^{-\frac{x_1}{8}})}{8 - x_1 e^{-\frac{x_1}{8}}} \qquad \text{(IV.2.15)}$$

Eqs. (IV.2.13), (IV.2.14), and (IV.2.15) together determine the Pareto-optimal solution to our problem. Table IV.2.1 shows the data which is depicted in Figure IV.2.1 (the noninferior solution in the decision space). Figure IV.2.2 depicts the solution in the functional space. Needless to say, the negativeness of the slope represents the tradeoff, i.e., $\lambda_{12} = -\frac{f_2(\cdot)}{f_1(\cdot)}$.

Table IV.2.1: Noninferior Solutions and Tradeoff Values

x_1	x_2	$f_1(x_1, x_2)$	$f_2(x_1, x_2)$	λ_{12}
0.5	1.29	0.78	140.35	0.08
1.0	2.64	3.10	121.51	0.17
1.5	4.05	6.92	103.68	0.26
2.0	5.49	12.15	87.04	0.37
2.5	6.96	18.68	71.73	0.49
3.0	8.43	26.35	57.87	0.63
3.5	9.88	35.00	45.55	0.79
4.0	11.29	44.44	34.82	0.98
4.5	12.66	54.48	25.68	1.23
5.0	13.97	64.91	18.10	1.55
5.5	15.20	75.55	12.01	1.99
6.0	16.34	86.23	7.32	2.64
6.5	17.40	96.78	3.90	3.71
7.0	18.36	107.06	1.64	5.83
7.5	19.23	116.97	0.39	12.17
7.99	19.99	126.24	0.00	631.66

ANALYSIS

In this example, we have discovered the noninferior solution and the Pareto optimum for a multiobjective tradeoff problem (see Figure IV.2.1). Figure IV.2.2 shows the tradeoffs that Todd has to make to arrive at his decision.

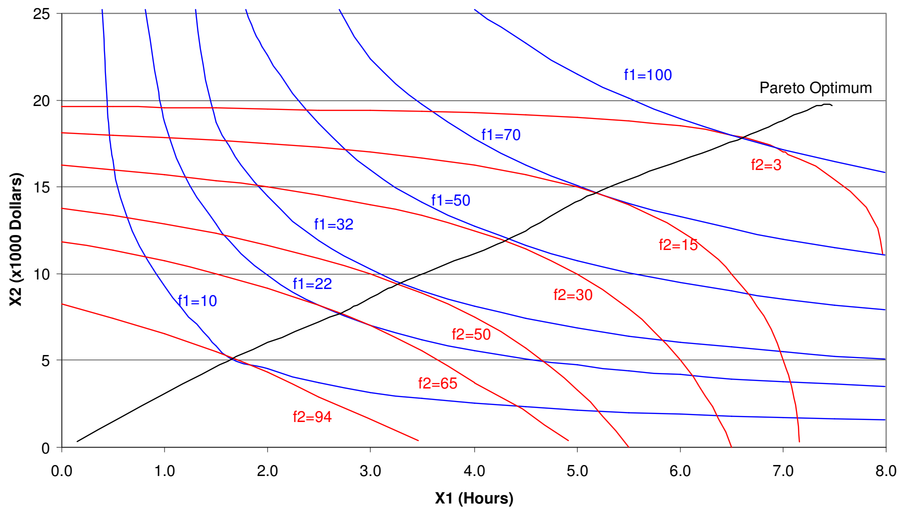

Figure IV.2.1. Noninferior solutions in the decision space

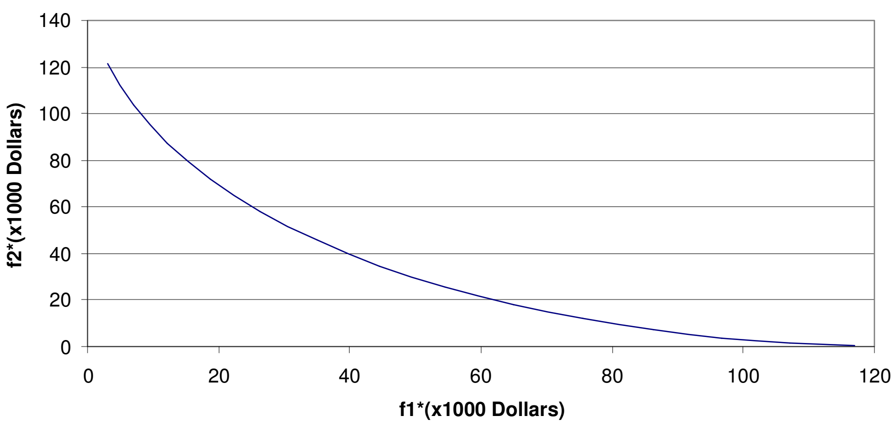

Figure IV.2.2. Noninferior solutions in the functional space

PROBLEM IV.3: Finding a Student Apartment

The purpose of this problem is to find an apartment that provides easier access to a particular university and requires the minimum possible rent cost.

DESCRIPTION

A student is looking for an apartment near the university. For an apartment to be "optimal," it must be inexpensive and allow for a quick, easy commute to campus. Thus, two of the student's objectives are to minimize cost and to minimize commute time.

METHODOLOGY

For this multiobjective problem, we use the Surrogate Worth Tradeoff (SWT) method to calculate the tradeoffs between the two objectives to be minimized: rent cost (f_1) and commuting time (f_2).

The two decision variables used to optimize these objectives are: distance (x_1, in kilometers) and access (x_2, a subjective score of how easy the commute is with a smaller absolute value being much easier). Off-campus housing does not start until three-quarters of a kilometer away from school, and commute time is measured in minutes.

SOLUTION

In functional form, the objective functions are as follows:

$$\min \begin{cases} f_1(x_1, x_2) = \dfrac{500}{x_1^{\,2}} + 75x_2 \\ f_2(x_1, x_2) = 30x_1 + 5x_2^{\,2} \end{cases}$$

Translating this into the ε-constraint form gives:

$$\begin{aligned} &\min \quad f_1(x_1, x_2) \\ &s.t. \\ &f_2(x_1, x_2) \le \varepsilon_2 \qquad x_1,\, x_2 \ge 0 \end{aligned}$$

where f_2 is the minimum of the second objective function.

The Lagrangian function for this optimization problem is:

$$L(x_1, x_2, \lambda_{12}) = \frac{500}{x_1^{\,2}} + 75x_2 + \lambda_{12}\left[30x_1 + 5x_2^{\,2} - \varepsilon_2\right]$$

Taking the partial derivatives from this function and using the Kuhn-Tucker conditions yields:

$$\frac{\partial L}{\partial x_1} = \frac{-1000}{x_1^{\ 3}} + 30\lambda_{12} = 0 \qquad \text{(IV.3.1)}$$

$$\frac{\partial L}{\partial x_2} = 75 + 10\lambda_{12} x_2 = 0 \qquad \text{(IV.3.2)}$$

$$\frac{\partial L}{\partial \lambda_{12}} = 30x_1 + 5x_2^{\ 2} - \varepsilon_2 = 0$$, where $\lambda_{12} > 0$ ensures Pareto optimality.

Solving (IV.3.1) and (IV.3.2) for lambda yields:

$\lambda_{12} = \dfrac{100}{3x_1^{\ 3}}$ from (IV.2.1) and $\lambda_{12} = \dfrac{-75}{10x_2}$ from (IV.2.2).

This will lead to: $x_2 = -\dfrac{225}{1000} x_1^{\ 3}$.

ANALYSIS

Plotting this in the decision space gives the student a picture of how the two decision variables relate to each other, as shown in Figure IV.3.1.

Plotting the values of the two objective functions gives a picture of the Pareto-optimal frontier, as shown in Figure IV.3.2.

Figures IV.3.1 and IV.3.2 show that the student can expect to pay a lower rent if the residence is farther away from the campus. However, this benefit can be offset by the longer commute. Thus, these two factors construct a noninferior solution frontier. Considering this frontier would help the student select his or her optimal accommodation.

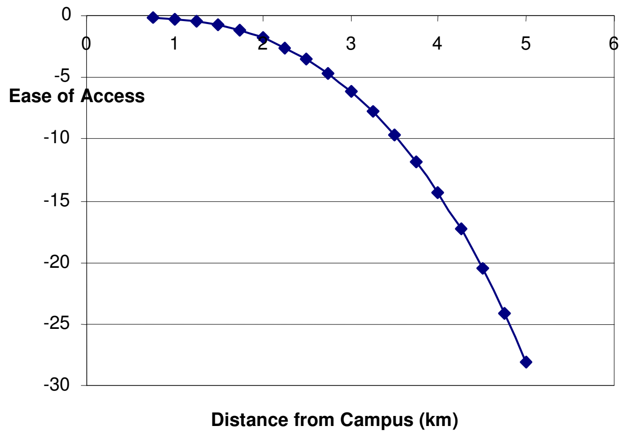

Figure IV.3.1. Noninferior Solution in Decision Space

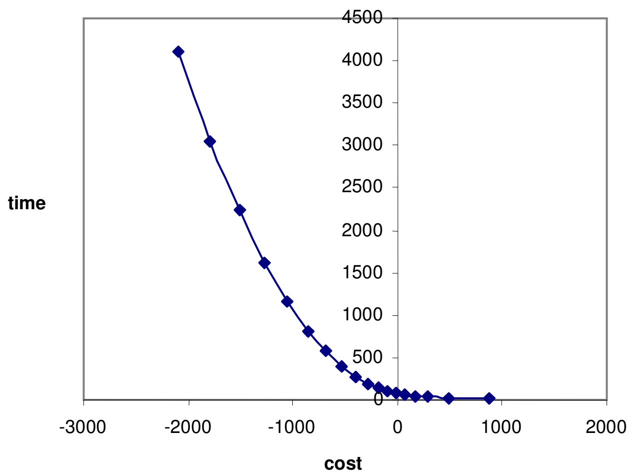

Figure IV.3.2. Noninferior Solution in Functional Space

.

PROBLEM IV.4: Balancing Exercise and Sleep

A student wants to exercise every morning, but also likes to sleep longer.

DESCRIPTION

A student likes to work out for 60 minutes in the morning, but she only has that much free time before classes begin. The physiological benefit of exercise per minute increases the longer she works out. Because she is usually up late studying, she would also like to be able to sleep longer. Obviously, the longer she sleeps, the less time she can spend at the gym. In other words, the increase in rest per minute sleeping in will decrease the time for working out.

METHODOLOGY

In order to analyze the tradeoff between the two objectives of increased rest and increased benefit from exercise, we utilized the Surrogate Worth Tradeoff (SWT) method.

SOLUTION

The first step is to develop the following model:

Definition of variables:

R = rest (units of well-being)
E = benefit of exercise (units of health)
r = sleeping in (activity)
e = exercising (activity)
T_i = time spent on activity i (min), where i = {r, e}

Decision variables

f_R = objective function, maximize rest
f_E = objective function, maximize benefit from exercise

Model:

$$E = (T_E/30)^2$$
$$R = (T_R/15)^{1/2}$$
$$f_R = \max\{R\} = \max\{(T_R/15)^{1/2}\}$$
$$f_E = \max\{E\} = \max\{(T_E/30)^2\}$$
$$\text{s.t. } T_R + T_E \leq 60$$
$$T_R \geq 0$$
$$T_E \geq 0$$

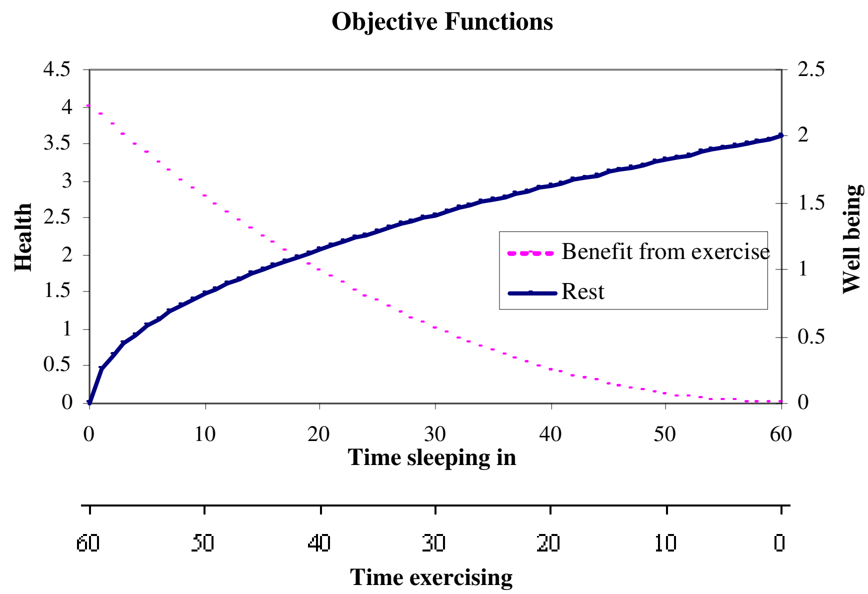

Figure IV.4.1. Well-being and Health in the Decision Space

Therefore, if $T_R + T_E < 60$, one objective could be improved without compromising the other. However, this situation cannot be Pareto optimal because time spent sleeping in plus time spent exercising must equal 60 minutes.

Given $T_R + T_E = 60$, the student can solve for T_E and substitute the solution into the first objective function. The two objectives then become:

$f_R = \max \{R\} = \max \{(T_R/15)^{1/2} \}$
$f_E = \max \{E\} = \max \{((60\text{-}T_R)/30)^2\}$
s.t. $0 \le T_R \le 60$

Multiobjective problems like this one can also be restructured as a single objective problem by converting the other objectives into constraints. This is done by mandating that the function meet some minimal requirement. By doing this, a Lagrangian function can be formed that incorporates the minimum requirements for additional objectives into one objective function. According to the Surrogate Worth Tradeoff Method (SWT), the value of the Lagrangian multiplier can be found by taking the negative of the derivative of one function divided by the derivative of the other. The Lagrangian multiplier, denoted by $\lambda_{E,R}$, is the cost or value added to the objective function per unit of change in the constraint or second objective function. In this case, the Lagrangian multiplier is as follows:

$$L = \left(\frac{(60 - T_R)}{30} \right)^2 + \lambda_{E,R} \left(\left(\frac{T_R}{15} \right)^{1/2} - \varepsilon_E \right)$$

$$\frac{\partial L}{\partial T_R} = -\frac{2}{30}\left(\frac{60 - T_R}{30}\right) + \lambda_{E,R}\left(\frac{\sqrt{15}}{2\sqrt{T_R}}\right) = 0$$

$$\lambda_{E,R} = \frac{\frac{120 - 2T_R}{900}}{\frac{\sqrt{15}}{2\sqrt{T_R}}} = \frac{\sqrt{T_R}\,(60 - T_R)}{225\sqrt{15}}$$

ANALYSIS

In the SWT, $\lambda_{E,R}$ must not be negative. Examining this equation, it can be seen that the feasible region for a noninferior solution is $0 \le T_R \le 60$. This confirms the bounds previously established for time spent in sleeping.

The following Table IV.4.1 and Figure IV.4.2, IV.4.3 and IV.4.4 show and illustrate noninferior solutions to this multiobjective problem:

Table IV.4.1. Noninferior Solution and Tradeoff Values

T_E	T_R	f_E	f_R	$\lambda_{E,R}$
0	60	0	2	0
15	45	0.25	1.73	0.12
30	30	1	1.41	0.19
45	15	2.25	1	0.2
60	0	4	0	∞

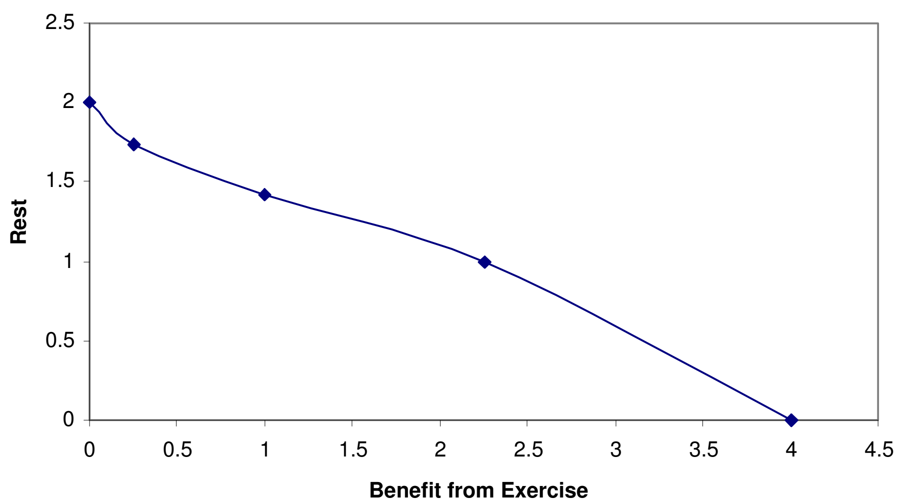

Figure IV.4.2. Noninferior Solution in the Functional Space

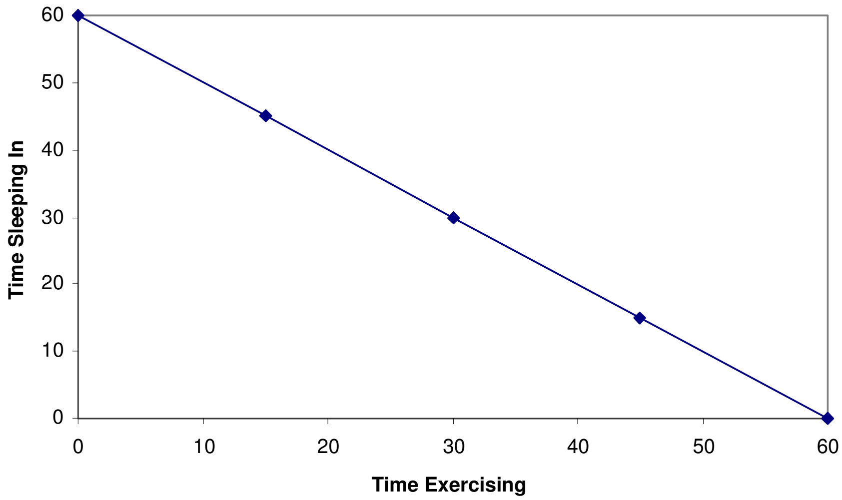

Figure IV.4.3. Noninferior Solution in the Decision Space

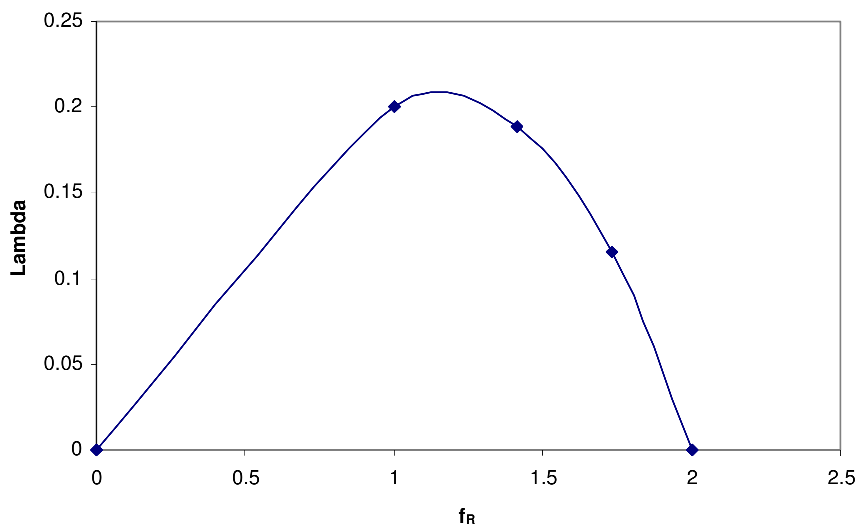

Figure IV.4.4. Tradeoff Function $\lambda_{E,R}$ versus f_R

PROBLEM IV.5: Survival and Colonization in Space

Planet Zysk has had space travel for several centuries and its inhabitants have traveled to all the planets in its solar system. Their science is extremely well-advanced, and the planet's scientists and top administrators have been aware for some time that their sun would become a supernova sometime within the next 300–700 years. The projected date of the disaster is still somewhat unclear, but for the past 100 years, the planet's scientists have been trying desperately to develop a faster-than-light (FTL) spaceship to transport its population to other solar systems so that they can survive the coming catastrophe. They have finally succeeded. The spaceship has been tested and is successful over fairly short distances (several light-years), and the planet's leaders are now turning their attention to planning colonizing expeditions.

DESCRIPTION

As soon as practicable, Planet Zysk wants to send off as many expeditions as possible so that they can learn which colonies are most successful in order to plan full-scale evacuations. From their previous planetary expeditions**,** Zysk's scientists have developed a so-called survival index. This helps them to estimate the chances of survival of both the colony ship and the colony itself. It is not known whether this index is completely applicable to long-range FTL expeditions, but this must be used in lieu of any other information.

METHODOLOGY

From empirical data, the scientists have also formed a function which predicts the amount of time necessary to prepare expeditions of different sizes. They hope to minimize their time function while maximizing their survival index. Like those on Earth, Planet Zysk's risk analysts have developed the Surrogate Worth Trade-off (SWT) method. They will use this to help them decide how long they should take to prepare and how many Zyskians to send on the expeditions. Because the SWT method is meant to minimize two functions, Zysk's scientists realize that maximizing the positive survival index is the same as minimizing the negative survival index. Both survival index and time are functions of the numbers of Zyskians and the number of pounds of equipment that will be sent.

A panel of three leading scientists will decide on the worth of any choices given: whether time should be minimized at the expense of the survival index; or the opposite. There is a general consensus, before looking at the figures, that one Zysk year (1003 of their days) is probably a reasonable time to shoot for, but there is considerable disagreement on what the Negative Survival Index (NSI) should be—figures range from -2000 to as high (or low) as -9000.

Decision Variables:

x_1 = number of Zyskians to be on any one ship

x_2 = number of pounds of equipment to be sent on any ship

Problem Statement:

These are the functions which are to be minimized:

Time = $f_1 = x_1^2 + 3x_2^2 + 40$ (in Zyskian days)

Negative Survival Index = $f_2 = -5(x_1 - 50)^2 - \frac{1}{2}(x_2 - 10)^2$ (in Zyskian pounds)

SOLUTION

1) Put in ε -constraint form:

Minimize $f_1 = x_1^2 + 3x_2^2 + 40$

$$\text{s.t.} \quad f_2 = -5(x_1 - 50)^2 - \frac{1}{2}(x_2 - 10)^2 \le \varepsilon_2$$
$$\varepsilon_2 \ge f_2 + \delta_2$$
$$\delta_2 \ge 0$$

2) Lagrangian:

$$L(x_1, x_2, \lambda_{12}) = f_1 + \lambda_{12}(f_2 - \varepsilon_2)$$
$$= x_1^2 + 3x_2^2 + 40 + \lambda_{12}[-5(x_1 - 50)^2 - \frac{1}{2}(x_2 - 10)^2 - \varepsilon_2]$$

3) Apply Kuhn-Tucker necessary conditions:

$$\frac{\partial L}{\partial x_1} = 2x_1 - 10\lambda_{12}(x_1 - 50) = 0$$
$$\frac{\partial L}{\partial x_2} = 6x_2 - \lambda_{12}(x_2 - 10) = 0$$

4) Solve for λ_{12} in each and set them equal:

$$x_2 = \frac{-10x_1}{29x_1 - 1500}$$

The data are illustrated in the following tables and figures.

Table IV.5.1. Noninferior Solutions

x_1	x_2	Time $x_1^2 + 3x_2^2 + 40$	Negative Survivability $-5(x_1-5)^2 - \frac{1}{2}(x_2-10)^2$
50.1	10.63694	2,889.44	-10,170
50.2	11.35747	2,947.02	-10,216
50.3	12.17918	3,015.09	-10,263
50.4	13.125	3,096.96	-10,311
50.5	14.22535	3,197.33	-10,360
50.6	15.52147	3,323.11	-10,412
50.7	17.07071	3,484.72	-10,467
50.8	18.95522	3,698.54	-10,528
50.9	21.29707	3,991.51	-10,598
51.0	24.28571	4,410.39	-10,682
51.1	28.23204	5,042.35	-10,792
51.2	33.68421	6,065.32	-10,953
51.3	41.70732	7,890.19	-11,221
51.4	54.68085	11,651.95	-11,763
51.5	79.23077	21,524.79	-13,208
51.6	143.3333	64,335.89	-19,747
51.7	738.5714	1,639,176.16	-276,313

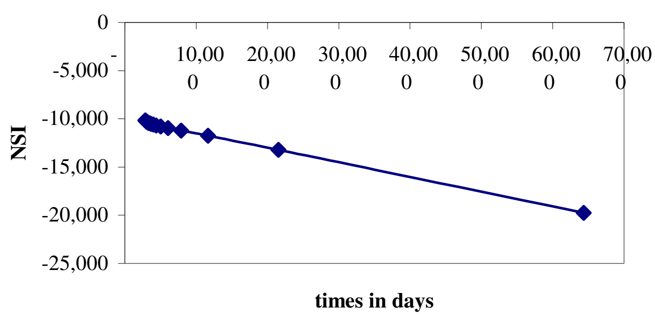

Figure IV.5.1. Noninferior solution in the functional space

ANALYSIS

Table IV.5.2. Noninferior Solutions and Trade-off Values

x_1	x_2	f_1	f_2	λ_{12}	Worth
50.1	10.63694	2,889.44	-10,170	100.2	5
50.5	14.22535	3,197.33	-10,360	20.2	4
51	24.28571	4,410.39	-10,682	10.2	3
51.2	33.68421	6,065.32	-10,953	8.533333	1
51.5	79.23077	21,524.79	-13,208	6.866667	-2
51.7	738.5714	1,639,176.16	-276,313	6.082353	-4

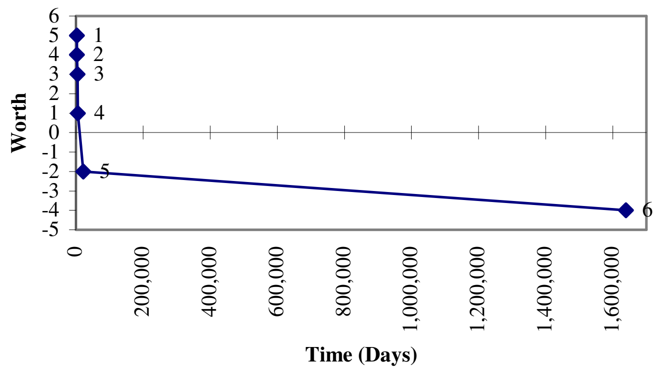

Figure IV.5.2. Surrogate worth trade-off curve

1) Time is minimal here but the negative survival index (NSI) is not desirable—its absolute value is too low to ensure much chance of survival at all (remember, we are minimizing the NSI so we want it to be as negative as possible—in other words, to have as great an absolute value as possible).
2) Time has increased to over three Zysk years, which is still well within the desired time frame; NSI is still not desirable for those who wish the expeditions to be greater in quality than quantity, so the general consensus is to allow a longer interval for preparation.
3) Time is now over four Zysk years, and the NSI is better, but the panel's average worth function still indicates that a further trade-off of time for a negative NSI is fairly desirable.
4) At this point, some of the panel obviously feels that an increase in NSI negativity is getting less and less worth the increase in time; however, the

average worth function still slightly favors allowing more time for trip preparation.

5) Here the average worth function clearly indicates that any further trade-off of time for NSI is undesirable; we have passed the minimum that even the most conservative members of the panel felt was necessary, and it is more important to minimize time.
6) This point is clearly desirable to all.

There are several technologies currently under development that might cause Zyskians to change their evolution of worth at the points above. For instance, there is great hope that a way can be found to transport embryos so that reproductive capability need not be a prime consideration of whom to send. If so, fewer adults would need to be sent for breeding purposes, freeing up space for much-needed specialists in various fields. This might change the NSI sufficiently to alter the relative worth of some points on the curve. Also, if an error has been found in calculating when the sun will become a supernova, or if new information on that event becomes available, a change in the number of years left before the catastrophe would alter the trade-off decisions. For example, if the sun were to become a supernova in 100 years instead of 300, time would become increasingly dear.

PROBLEM IV.6: Solving a Problem with Two Non-linear Objectives

A student is asked to develop the non-inferior solutions and a table of trade-off values to minimize two objective functions simultaneously.

Using the data, the student also is asked to plot the non-inferior solution in both the decision space and the functional space.

The two objective functions are as follows:

$$\text{Minimize} \quad \begin{aligned} f_1(x_1, x_2) &= x_1^2 + 2(x_2 - 1)^2 + 4 \\ f_2(x_1, x_2) &= (x_1 - 4)^2 + x_2^2 + 3 \end{aligned}$$

PROBLEM IV.7: Employment and Inflation

Consider the relationship between employment (or unemployment) and inflation in a national economy:

This example problem involves at least two objective functions. It is desirable that both of them, unemployment and inflation, be minimized. Determine a preferred solution for a computer company that needs to reduce its staff without contributing to an increase in inflation.

Use the Surrogate Worth Trade-off (SWT) method to generate Pareto-optimal solutions (at least 5) and their associated trade-offs.

Let $U(x_1,x_2)$ be the *unemployment* function. Let $I(x_1,x_2)$ be the *inflation* function where x_1 = *money supply* and x_2 = *government spending*. (Assume that the federal bank of this government has no independence; therefore the money supply is also controlled by the executive branch. Also, inflation is measured by the price index. Dividing by 100 gives us the inflation rate.)

Decision Variables:

x_1 = money supply
x_2 = government spending

Problem Statement:

These are the functions which are to be minimized:

Unemployment = $U(x_1, x_2) = 100 - x_1^2 - x_2^2$

Inflation = $I(x_1, x_2) = x_1^3 + x_2^3$

Use the Surrogate Worth Trade-off (SWT) method to generate Pareto-optimal solutions (at least 5) and their associated trade-offs.

PROBLEM IV.8: Risk-Return Tradeoff for an Investment Decision

Investors do not usually hold a single asset; they hold groups of assets referred to as a portfolio. The following is a partial list of the possible financial instruments that investors can consider:

- Money market: US Treasury bills (T-bills), Bonds
- Currency market
- Stock market

Portfolio investing assumes that for a given level of return, an investor would prefer less risk. Similarly, an investor would prefer a higher return for a given level of risk. Some financial assets, such as T-bills, bonds and notes, are considered to be risk-free because they yield a fixed rate of return. Unlike these, stocks have an element of risk. This risk-return tradeoff is the motivation for this analysis to focus on stock investments.

Several simplifications were employed in the analysis. As mentioned above, the scope was limited to the stock market only, and to two stocks in particular – Compaq (CPQ) and Microsoft (MSFT). Additionally, the analyses were based on the following assumptions:

- In this problem, a stock's return is based purely on the price changes for several time periods.
- There is no lending and borrowing.
- Transaction costs are not taken into consideration.
- Uninvested money earns no return (for simplification purposes), $x_1 + x_2 \leq 1$.

Two stocks were selected for the analysis—COMPAQ (CPQ) and Microsoft (MSFT). Five-years (1995-1999) of data on their annual rates of return (AROR) were recorded, as shown in Table IV.8.1.

Table IV.8.1. AROR of Stocks CPQ and MSFT for Years 1995-1999

		1995	1996	1997	1998	1999	r
Stock i=1	(CPQ)	0.55	0.90	0.49	-0.36	-0.08	0.30
Stock i=2	(MSFT)	0.88	0.56	1.15	0.68	-0.09	0.64

r is the average rate of return (AROR) for the stocks, computed as:

$r = \frac{1}{5}\sum_{j=1}^{5} r_{ij}$, where i = Stocks 1 (CPQ) and 2 (MSFT), and j = Periods 1 to 5.

Expected Return (mean) of a Combination of Assets:

The expected return of a combination of *n* assets is just the sum of the expected value of each return.

$$E(r_1 + r_2) = \bar{r}_1 + \bar{r}_2$$

The return on a portfolio of assets r_P is now simply a weighted average of the return on the individual assets, where x_i is the fraction of the investor's funds invested in the i^{th} asset.

$$r_p = x_1 \bar{r}_1 + x_2 \bar{r}_2 \quad \Leftrightarrow \quad r_p = x_1 0.3 + x_2 0.64 \quad \Rightarrow \quad f_1(x_1, x_2)$$

Risk of a Combination of Assets:

The variance is the expected value of the squared deviations from the mean return on the portfolio:

$$\begin{aligned} \sigma_P{}^2 &= E(r_P - \bar{r}_P)^2 \\ &= (x^{'})Q(x)^T(x) \end{aligned}$$

where Q is the covariance of the stocks and is given by:

$$Q = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} = \begin{bmatrix} 0.21 & 0.07 \\ 0.07 & 0.17 \end{bmatrix}$$

Risk ($\sigma_p{}^2$) can be expressed as:

$$\begin{aligned} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} (x_1\rho_{11} + x_2\rho_{21}) & (x_1\rho_{12} + x_2\rho_{22}) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= x_1{}^2\rho_{11} + x_2{}^2\rho_{22} + x_1x_2(\rho_{12} + \rho_{21}) \end{aligned}$$

$$= 0.21x_1{}^2 + 0.17x_2{}^2 + (0.07 + 0.07)x_1x_2 \qquad \Rightarrow \qquad f_2(x_1, x_2)$$

The requirement of this problem is to use the Surrogate Worth Tradeoff (SWT) method to perform multiobjective optimization with respect to the two objectives: f_1 (expected return), and f_2 (risk). Generate a plot of the Pareto optimal solutions in both decision space and functional space.

PROBLEM IV.9: Carcinogenic Chemicals in a New Product

Company X is expanding their product line by producing Product Y. This product's main ingredients are Chemical A and Chemical B, both of which are carcinogenic.

The labor union has requested that an occupational exposure limit (OEL) be established for Chemicals A and B, for the following reasons: (1) both chemicals are listed as a human carcinogen by the International Agency for Research on Cancer (IARC), and (2) neither chemical is regulated by the Occupational Safety and Health Administration (OSHA). Company X's management has tasked its Safety Department with developing an OEL, which would minimize the number of cancer incidents and costs associated with the new product.

Use the Surrogate Worth Tradeoff (SWT) method to analyze risk of cancer incidents with respect to associated costs.

The relationships for i) exposure to incident cancers (f_1) and ii) costs of exposure to Chemicals A and B (f_2) are the following:

$$f_1(D_A, D_B) = 1 + D_A + D_B^2$$
$$f_2(D_A, D_B) = 100 + 16(D_A + D_B) - (D_A + D_B)^2$$

where:

$f_1(D_A, D_B)$ = cancer incidents as a function of exposure
$f_2(D_A, D_B)$ = cost as a function of exposure
D_A = exposure to Chemical A
D_B = exposure to Chemical B

PROBLEM IV.10: Building a New Rubber Manufacturing Plant

A Taiwan company is one of the main rubber-product suppliers in the world. It plans to establish a new plant in a new industry zone located on the southwestern coast of Taiwan. The company is attempting to produce two new rubber products which will make it occupy more market share. However, these two products will surely cause a change in air pollution. Today, most people in Taiwan are very aware of environmental protection problems. Before the local government approved the plant proposal, there was a large-scale citizens' protest in front of city hall and the Mayor promised that the government would supervise the plant closely. The rubber company will pay extra fees to deal with the pollution problem. How can the company management maximize profits while minimizing the cost for environmental protection?

Solve the following multiobjective optimization model using the Surrogate Worth Tradeoff (SWT) method.

(max profit) $f_1(x_1,x_2) = (5x_1-12)^2+(3x_2-5)^2$

(min cost) $f_2(x_1,x_2) = 4x_1^2+3x_2^2+2x_1x_2$

x_1 : tons of product @ 1 per unit per day

x_2 : tons of product @ 2 per unit per day

f_1 : profit model in terms of unit price and cost, which includes fixed and variable components

f_2 : cost model for dealing with environmental protection

PROBLEM IV.11: Providing Security for a College Concert

The purpose of this problem is to minimize the injuries and the cost of the security in a rock star performance at a concert hall.

A university is planning to have a rock star perform at its concert hall. In order for the event to go smoothly and ensure future business, the university would like to minimize the number of injuries at the event (f_1). At the same time, they would like to minimize the cost of security (f_2). Because a majority of injuries occur closest to the stage, those planning the concert would like to determine the optimal amount of security officers to deploy to the stage area (x_1) per hour. In addition, security officers must also be dispersed throughout the rest of the arena to guarantee the safety of the entire audience. This will be another decision variable, x_2, which will also be on a per hour basis. It is known that the concert hall has a maximum capacity of 15,000 attendees, which is reflected in the first objective function. The officers deployed around the stage are in a more chaotic situation and may have a better chance of dealing with injury. Because of this, they are given a little more pay than those officers standing guard throughout the arena. The two objective functions can be seen below:

$$\min \left\{ \begin{array}{l} f_1(x_1, x_2) = 15000 - 20x_1^2 - 7.5x_2^2 \\ f_2(x_1, x_2) = 100x_1 + 75x_2 \end{array} \right\}$$

Solve the multiobjective problem described above and plot the injury and security cost tradeoff using the Surrogate Worth Tradeoff (SWT) Method.

PROBLEM IV.12: Marikina River Overflow Modeling

The Marikina River overflow scenario in the Philippines is a chronic problem. Hence, the analysis and modeling of the impact of the river overflows requires significant attention and management by policymakers.

The objective of this problem is to identify the impact of the current policy decision on future concerns using the multiobjective multi-stage risk impact assessment method. In particular, the channel overflow scenario will be formulated as a multiobjective optimization problem.

The Surrogate Worth Trade-off (SWT) method is useful in multiobjective problems. Faced with multiple objectives, the approach is to select a primary objective and optimize this while constraining the decisions considered so that the other objectives are attained even at minimum levels. A set of Pareto optimal points are generated and trade-off analysis can precede with single or multiple decision makers.

In this particular problem, use SWT to optimize the following two objectives:

- Minimize investment cost (f_1)
- Minimize risk of flood (f_2)

Investment cost: $f_1(x_1, x_2) = (x_1 - 3)^2 + (x_2 - 7)^2 + 2$

Risk of flood damage: $f_2(x_1, x_2) = (x_1 - 8)^2 + (x_2 - 10)^2 + 7$

where: x_1 = number of floodways built
x_2 = number of drainage systems established

PROBLEM IV.13: Designing an Enormous Ice-Cream Cone

The purpose of this problem is to find optimal height and radius for ice-cream cone that maximizes its volume while minimizing its surface.

Every summer, ice-cream companies are likely to do promotions for increasing their sales. One company wants to make the biggest ice-cream cone while it uses the smallest amount of materials just for advertising display. What should the radius and the height of the ice-cream cone be?

Solve this problem using the Surrogate Worth Tradeoff (SWT) method to calculate the volume and surface tradeoffs for the ice-cream cone.

Function for volume:

$$\hat{f}_1(R, H) = \frac{\pi R^2 H}{3}$$

Function for surface area

$$f_2(R, H) = \pi R \sqrt{R^2 + H^2}$$

where R : *radius*, H : *height*