A Bidding Model of Perfect Competition

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I demonstrate in this paper that price formation via the procedure of competitive bidding satisfies a version of the law of large numbers, in both the probabilistic sense and the economic sense. That is, if in a sealed-tender auction a seller offers to sell at the highest bid an item having a definite but unknown monetary value, and each of many bidders submits a bid based only on his private sample information about the value, where the bidders' samples are independent and identically distributed conditional on the value, then the maximum bid is almost surely equal to the true value. Thus, no bidder knows the true value of the item, yet it is essentially certain that the seller will receive that value as the sale price. Certain regularity assumptions are needed to prove this proposition. I present three examples, two for which the result is valid and another for which it is not.

1. THE FINITE CASE

Suppose first that the number n of bidders is finite and $n \geqq 2$. Allowing some extra generality, I assume that the payoff to a bidder i is zero if he loses and it is $u(s_i, v)-b_i$ if he observes the " sample " s_i, obtains the item with a winning bid of b_i, and subsequently the " value " v is revealed. Although the function u may reflect aversion to risk about the value, the linearity of the payoff in the bid price excludes aversion to risk about winning or losing the auction. We shall see that the latter is of no consequence when there are many bidders.

A key feature is that the value v is not observed by any bidder and the sample s_i is observed only by bidder i. Consequently, the bid b_i of bidder i depends only on his sample observation s_i and on the number n of bidders, say $b_i = p_n(s_i)$. Thus a strategy is a function p_n which specifies for a bidder i that his bid is $p_n(s_i)$ if he observes the sample s_i and he is one among n bidders.

I assume throughout that the bidders are symmetric, so that each one uses the same strategy. In particular, the sample s_i observed by bidder i is taken to be the realization of a real-valued random variable S_i, and I assume that conditional on the value v the random variables $\{S_i \mid i = 1, 2, \ldots\}$ are mutually independent and identically distributed, each with the distribution function $F(\cdot \mid v)$ where $F(s \mid v) = \text{prob}\,\{S_i \leqq s \mid v\}$. Moreover, since no bidder observes the value v, I assume that each one construes v to be the realization of a real-valued random variable V to which each assigns the same a-priori distribution function G.

The above construction defines a non-cooperative game with incomplete information. Because the game is symmetric, an equilibrium strategy in this game is one that is optimal for each bidder if each other bidder uses it. The results to be demonstrated in Section 2 are based upon the following characterization.

Theorem 1. *If an equilibrium strategy p_n is differentiable and strictly increasing ($p_n' > 0$) on an interval $[s', s'']$, and each $F(\cdot \mid v)$ has a density there, then*

$$p_n(s) = \int_{s'}^{s} \bar{u}_n(t)\,dH_n(t \mid s)^{n-1} + p_n(s')H_n(s' \mid s)^{n-1} \qquad \ldots(1.1)$$

for $s' \leqq s \leqq s''$, where $\bar{u}_n$ and H_n are defined below and have the interpretations that $\bar{u}_n$ specifies a natural upper bound on the bid and $H_n(\cdot \mid s)$ is a distribution function on the interval $(-\infty, s]$.

Proof. Let ΣV be the support of G and for each $v \in \Sigma V$ let $\Sigma S(v)$ be the support of $F(\cdot \mid v)$. Also define $\Sigma V(s) = \Sigma S^{-1}(s) = \{v \in \Sigma V \mid s \in \Sigma S(v)\}$. Then $\Sigma = \bigcup_{v \in \Sigma V} \Sigma S(v)$ is the domain of p_n. To alleviate notational complexities I will construct the proof only for the case in which $[s', s''] = \Sigma$. The hypothesis then states that the equilibrium strategy has $p_n' > 0$ on Σ and, for each $v \in \Sigma V$, $F(\cdot \mid v)$ has a density $f(\cdot \mid v)$ on $\Sigma S(v)$. Note that p_n has an inverse function σ_n with $\sigma_n' > 0$. Hence, if a bidder uses any strategy p^* while each other bidder uses p_n then his conditional probability given $V = v$ of submitting the highest bid when he observes s is $Q_n(\sigma_n(p^*(s)) \mid v)$, where we define

$$Q_n(s \mid v) = F(s \mid v)^{n-1}. \qquad \ldots(1.2)$$

His expected payoff is therefore

$$\int_{\Sigma V} \int_{\Sigma S(v)} [u(s, v) - p^*(s)] Q_n(\sigma_n(p^*(s)) \mid v) dF(s \mid v) dG(v). \qquad \ldots(1.3)$$

Reversing the order of integration in this formula and then, for each $s \in \Sigma$, differentiating with respect to $p^* = p^*(s)$ yields the necessary condition for an optimum that

$$0 = \int_{\Sigma V(s)} \{[u(s, v) - p^*] Q_n'(\sigma_n(p^*) \mid v) f(s \mid v) \sigma_n'(p^*) - Q_n(\sigma_n(p^*) \mid v) f(s \mid v)\} dG(v) \qquad \ldots(1.4)$$

for each $s \in \Sigma$. Now, if p_n is to be an equilibrium strategy then it is necessary that $p^* = p_n$ satisfies (1.4). Using the relations $\sigma_n(p_n(s)) = s$ and $\sigma_n'(p_n(s)) = 1/p_n'(s)$, this yields the following linear differential equation for p_n as a necessary condition:

$$0 = \int_{\Sigma V(s)} \{[u(s, v) - p_n(s)] Q_n'(s \mid v) f(s \mid v) - p_n'(s) Q_n(s \mid v) f(s \mid v)\} dG(v). \qquad \ldots(1.5)$$

An alternative form of this differential equation is

$$0 = [\bar{u}_n(s) - p_n(s)] \hat{\phi}_n(s) - p_n'(s)/(n-1), \qquad \ldots(1.6)$$

where, if the conditional distribution of V given $\max_{i \leqq n} S_i = s$ is

$$G_n(v \mid s) = \frac{\int_{\inf \Sigma V(s)}^{v} Q_n(s \mid w) f(s \mid w) dG(w)}{\int_{\Sigma V(s)} Q_n(s \mid w) f(s \mid w) dG(w)}, \qquad \ldots(1.7)$$

and it is $\bar{G}_n$ if Q_n' replaces Q_n in (1.7), then

$$\bar{u}_n(s) = \int_{\Sigma V(s)} u(s, v) d\bar{G}_n(v \mid s) \qquad \ldots(1.8)$$

and

$$\hat{\phi}_n(s) = \int_{\Sigma V(s)} \phi(s \mid v) dG_n(v \mid s), \qquad \ldots(1.9)$$

where $\phi(s \mid v) = f(s \mid v)/F(s \mid v)$. Thus $\bar{u}_n(s)$ is the conditional expectation of the gross payoff $u(s, v)$ given that (1) the bidder's sample is s, and (2) the maximum among his competitors' samples is s. (The nature of this upper bound on the bid will be developed in greater detail as we proceed; however, observe here that $\bar{u}_n$ reflects the fact that winning the auction is itself an informative event, namely, it reveals that the other bidders observed less favourable samples.) It is straightforward to verify that the family of solutions of (1.6) has the required form (1.1). In particular,

$$H_n(t \mid s) = \exp\left\{-\int_t^s \hat{\phi}_n(\tau)d\tau\right\}, \quad s' \leqq t \leqq s \leqq s'. \qquad \ldots(1.10)$$

The interpretation of the distribution function $H_n(\cdot \mid s)$ is admittedly opaque, although if $F(\cdot \mid v)$ is independent of v then $H_n(t \mid s) = F(t)/F(s)$ and therefore $H_n(t \mid s)^{n-1} = Q_n(t)/Q_n(s)$ is the conditional distribution of the second-highest sample given that s is maximal. More revealing perhaps is the asymptotic form of H_n which I derive in Section 2; c.f. (2.5).

In view of Theorem 1 it is useful to identify sufficient conditions for the formula (1.1) to be valid. I assert the following.

Theorem 2. *Sufficient conditions for the validity of the formula* (1.1) *are that*

(a) $\bar{u}_n$ *is continuous and strictly increasing on* $[s', s'']$, *and*

(b) $H_n(\cdot \mid \cdot)$ *is differentiable and strictly increasing in* t *for* $t \leqq s \in [s', s'']$.

In turn, the following assumptions are sufficient for (a) and (b) to hold on all of Σ.

Assumption 1. ΣV *is compact and convex.*

Assumption 2. ΣS: $\Sigma V \rightarrow \Sigma$ is a continuous, compact-valued, and convex-value correspondence.

Assumption 3. $F(\cdot \mid v)$ is stochastically strictly ordered by v on ΣV and it has a density $f(\cdot \mid v)$ which is strictly positive on $\Sigma S(v)$.

Assumption 4. u: $\Sigma \times \Sigma V \rightarrow R$ is continuous and weakly increasing $[(s', v') \gg (s, v)$ implies $u(s', v') > u(s, v)]$.

This result is not needed directly for the topics in Section 2 and therefore I will not undertake its lengthy proof here. I will, however, assume hereafter that (a), (b) and Assumptions 1-4 are satisfied.

The final result which is needed is a characterization of the " initial condition " for the differential equation (1.6).

Theorem 3. *If the seller sets a reservation price* p_n^0, *then there exists an interval* $[c_n', c_n] \subset \Sigma$ *such that*

(i) $p_n(s) < p_n^0$ if $s < c_n'$,

(ii) $p_n(s) = p_n^0$ if $c_n' \leqq s \leqq c_n$, $\qquad \ldots(1.11)$

(iii) $p_n(s) = \int_{c_n}^s \bar{u}_n(t)dH_n(t \mid s)^{n-1} + p_n^0 H_n(c_n \mid s)^{n-1}$ if $c_n \leqq s$.

Moreover, $\bar{u}_n(c_n) \geqq p_n^0$.

Proof. The argument is a repetition of the proof of Theorem 1 except that if a bidder is to have any chance of winning he must bid at least p_n^0. Hence the bid p^* must be optimized subject to the constraint that $p^* \geqq p_n^0$. This leaves unchanged the necessary condition (1.4) for any interval $[s', s'']$ in which $p^* > p_n^0$, and otherwise (1.4) is replaced by an inequality. The previous monotonicity assumptions assure, therefore, that the equilibrium strategy has the form (1.11). Note that the conditional probability of winning is again $Q_n(s \mid v)$ if $s > c_n$, but it is zero if $s < c_n'$, and it is

$$\textstyle\sum_{k=0}^{k=n-1} \frac{1}{k+1}\binom{n-1}{k} F(c_n' \mid v)^{n-1-k}[F(c_n \mid v) - F(c_n' \mid v)]^k \qquad \ldots(1.12)$$

if $c'_n \leqq s \leqq c_n$. Lastly, it must be that $\bar{u}(c_n) \geqq p^0_n$ since otherwise $p_n(c_n) = p^0_n > \bar{u}_n(c_n)$, whereas I claim that $p_n(s) \leqq \bar{u}_n(s)$ is necessary whenever $p^0_n \leqq p_n(s)$. This is because the formula, e.g. (1.3), for a bidder's expected payoff shows that his expected payoff conditional on any s is non-negative if and only if $p_n(s) \leqq \hat{u}_n(s)$, where

$$\hat{u}_n(s) = \int_{\Sigma V(s)} u(s, v) dG_n(v \mid s) \qquad \text{...(1.13)}$$

is the expected valuation of the item conditional on winning. The monotonicity assumptions assure that $\hat{u}_n(s) \leqq \bar{u}_n(s)$. If his conditional expected payoff were negative he would prefer to submit a bid $p^*(s) < p^0_n$ having no chance to win.

In various examples one finds that c'_n is determined via the condition that $\hat{u}_n(c'_n) = p^0_n$. Unfortunately I have not found any simple method to determine c_n in the general case. If $p^0_n = \inf \{u(s, v)\}$ then clearly it is required that $c'_n = c_n = \inf \Sigma$.

The following three examples illustrate Theorem 1 and they will be used again later to illustrate the results of Section 2. Several other examples are given in R. Wilson [2].

Example 1. Suppose that $u(s, v) = s$, $G(v) = v$ for $v \in \Sigma V = [0, 1]$, and $F(s \mid v) = s/2v$ for $s \in \Sigma S(v) = [0, 2v]$. Then $\hat{u}_n(s) = \bar{u}_n(s) = s$ and $H_n(t \mid s) = t/s$ for $0 \leqq t \leqq s \leqq 2$. Assume that $p^0_n = 0$ so that $c_n = 0$. Then $p_n(s) = [(n-1)/n]\hat{u}_n(s)$.

Example 2. Suppose that $u(s, v) = v$, $G(v) = 1+v$ for $v \in \Sigma V = [-1, 0]$, and $F(s \mid v) = -sv$ for $s \in \Sigma S(v) = [0, -1/v]$. (Negative values have the interpretation that the "seller" is purchasing an item from the "lowest-cost" bidder.) One finds that $\hat{u}_n(s) = \bar{u}_n(s) = -[(n+1)/(n+2)].\text{Min}\,(1, 1/s)$ and $H_n(t \mid s) = t/s$ for $0 \leqq t \leqq s$. If $p^0_n = -1$ then $c_n = 0$ and therefore (1.1) or (1.11) yields

$$p_n(s) = \hat{u}_n(s) . \frac{n-1-\min\,(1, (1/s))^{n-2}}{n-2} \qquad \text{...(1.14)}$$

if $n > 2$.

Example 3. Suppose that $u(s, v) = v$, $G(v) = 1 - e^{-\lambda v}$ for $v \in \Sigma V = [0, \infty)$, and $F(s \mid v) = e^{vs}$ for $s \in \Sigma = (-\infty, 0]$. Then $\hat{u}_n(s) = 2/(\lambda - ns)$, $\bar{u}_n(s) = 3/(\lambda - ns)$, and $H_n(t \mid s) = [(\lambda - nt)^{-2/n}]/[(\lambda - ns)^{-2/n}]$ for $t \leqq s \leqq 0$. Also, if $p^0_n = 0$ then $c_n = -\infty$, and therefore (1.11) yields

$$p_n(s) = \frac{1}{1+(1/3(n-1))} \hat{u}_n(s). \qquad \text{...(1.15)}$$

No one of these examples entirely satisfies Assumptions 1-3. For example, $F(\cdot \mid 0)$ is not well-defined in each case. Their equilibrium strategies are, however, quite regular; in particular, in each example $[p_n - \hat{u}_n] \to 0$ as $n \to \infty$, indicating that a bidder's expected payoff declines to zero as the number of other bidders increases. One can verify in Examples 2 and 3, moreover, that the expectation of the maximum bid converges to the expectation of V as n increases. Nevertheless, Examples 1 and 2 differ fundamentally from Example 3 in the following respect. In Example 1 the maximum bid converges almost surely, conditional on $V = v$, to $2v$, and in Example 2 to v, but in Example 3 the maximum bid converges to a random variable. The latter can be seen from the fact that in Example 3 the random variables

$$\max_{i \leqq n} \hat{u}_n(S_i) \quad \text{and} \quad \max_{i \leqq n} \bar{u}_n(S_i)$$

have distributions which are independent of n. Thus, in Example 3 the sale price converges to a non-degenerate random variable as the number of bidders increases. Another way to see this is to observe that the bidders' maximum sample converges almost surely to the upper bound of zero, but since this is equally true for every possible value, there is no way that the inferential process yielding $\hat{u}_n$ and $\bar{u}_n$ can distinguish the true realization of V.

In the next section I examine this problem in greater detail. Among the several possible approaches to assure that the sale price converges, I adopt as a sufficient condition the assumption that the upper bound of the samples is in a one-to-one correspondence with the value, as in Examples 1 and 2.

2. THE ASYMPTOTIC CASE

I now address the main topic of this paper, which is the determination of the sale price when there are many bidders. I will assume throughout that the equilibrium strategy has the form (1.11) derived in Theorem 3. The principal additional assumption that will be used to exclude the phenomenon in Example 3 is the following. Let $b(v) = \sup \Sigma S(v)$ for each $v \in \Sigma V$. That is, $b(v)$ is the maximum possible sample if $V = v$, and according to Assumption 1 and Assumption 2, b is a continuous function on the compact interval ΣV.

Assumption 5. $b\colon \Sigma V \rightarrow \Sigma$ is a strictly increasing function.

I remark that the assumption included in Assumption 3 that $f(b(v)|\, v) > 0$ will play a central role in the following development, although presumably a more elaborate argument could eliminate it.

Theorem 4. *For each value $v \in \Sigma V$, if as $n \rightarrow \infty$ the seller's reservation price allows higher bids [i.e. c_n is bounded strictly below $b(v)$], then the maximum bid converges almost surely to $u(b(v), v)$.*

Proof. I will assume that u is everywhere positive: this loses no generality since $\Sigma \times \Sigma V$ is compact and any increasing affine transformation of u transforms $\hat{u}_n$, $\bar{u}_n$, and p_n similarly. Define the random variables

$$M_n = \max_{i \leqq n} S_i \quad \text{and} \quad P_n = \max_{i \leqq n} p_n(S_i)$$

for each n. Since p_n is strictly increasing, $P_n = p_n(M_n)$. Conditional on $V = v$, $M_n \rightarrow b(v)$ almost surely as $n \rightarrow \infty$. Consequently, the proposition to be proved is that if $\{m_n\}$ is a non-decreasing sequence converging to $b(v)$ then $p_n(m_n)$ converges to $u(b(v), v)$. Notation is simplified by letting $\beta(v) = u(b(v), v)$ and by defining $\alpha(s) = u(s, a(s))$ where $a(s) = \inf \Sigma V(s)$ for each $s \in \Sigma$. Note that $a\colon \Sigma \rightarrow \Sigma V$ is continuous and non-decreasing, and strictly increasing on the range of b, where it is the inverse of b by Assumption 5. The first step is to establish the following

Lemma A. (a) $\hat{u}_n(s) \rightarrow \alpha(s)$; $\bar{u}_n(s) \rightarrow \alpha(s)$; $\hat{\phi}_n(s) \rightarrow \phi(s \mid a(s))$;

(b) $\hat{u}_n(m_n) \rightarrow \beta(v)$; $\bar{u}_n(m_n) \rightarrow \beta(v)$; $\hat{\phi}_n(m_n) \rightarrow \phi(b(v)|\, v)$.

The various parts of this proposition have similar proofs. I will demonstrate mainly that $\hat{\phi}_n(m_n) \rightarrow \phi(b(v)|\, v)$ since the parts involving u are simplified by the fact that u is monotone increasing whereas ϕ need not be. Now if $a(s) < z < w \leqq \sup \Sigma V(s)$ then the stochastic ordering assumed in Assumption 3 assures that $F(s \mid w) < F(s \mid z) < F(s \mid a(s)) = 1$. Hence, if $y = [a(s)+z]/2$ then

$$\frac{\int_z^\infty \phi(s \mid w)dG_n(w \mid s)}{\int_{a(s)}^y \phi(s \mid w)dG_n(w \mid s)} \leqq \frac{F(s \mid z)^{n-1} \int_z^\infty \phi(s \mid w) f(s \mid w)dG(w)}{F(s \mid y)^{n-1} \int_{a(s)}^y \phi(s \mid w) f(s \mid w)dG(w)} = \delta \varepsilon^{n-1}, \quad \ldots(2.1)$$

where $\varepsilon = F(s \mid z)/F(s \mid y) < 1$ and $\delta > 0$. Therefore,

$$\hat{\phi}_n(s) \leqq \int_{a(s)}^{z} \phi(s \mid w) dG_n(w \mid s) + \delta\varepsilon^{n-1} \int_{a(s)}^{y} \phi(s \mid w) dG_n(w \mid s)$$

$$\leqq \frac{(1+\delta\varepsilon^{n-1}) \int_{a(s)}^{z} \phi(s \mid w) dG_n(w \mid s)}{G_n(z \mid s)}. \quad \ldots(2.2)$$

Similarly, an alternative to (1.9) is

$$\frac{1}{\hat{\phi}_n(s)} = \int_{\Sigma V(s)} \left[\frac{1}{\phi(s \mid w)}\right] d\bar{G}_n(w \mid s), \quad \ldots(2.3)$$

and therefore

$$\frac{1}{\hat{\phi}_n(s)} \leqq (1+\delta'\varepsilon^{n-2}) \int_{a(s)}^{z} \left[\frac{1}{\phi(s \mid w)}\right] \frac{d\bar{G}_n(w \mid s)}{\bar{G}_n(z \mid s)}. \quad \ldots(2.4)$$

These inequalities are valid for z arbitrarily close to $a(s)$, so $\hat{\phi}_n(s) \to \phi(s \mid a(s))$ as $n \to \infty$, for each fixed $s \in \Sigma$. Similarly, $\hat{\phi}_n(m_n) \to \phi(b(v) \mid v)$ since $a(m_n) \leqq v$ and $a(m_n) \to a(b(v)) = v$, so the above inequalities hold for each $z > v$. The proofs for $\hat{u}_n$ and $\bar{u}_n$ are entirely similar except that instead of (2.3) and (2.4) one uses the monotonicity of u to establish that $\hat{u}_n(s) \geqq \alpha(s)$ and $\bar{u}_n(s) \geqq \alpha(s)$, and of course $\alpha(m_n) \leqq \beta(v)$ and $\alpha(m_n) \to \beta(v)$. This completes the proof of the lemma. An immediate corollary which we shall use is that

$$H_n(t \mid s) \to H(t \mid s) \equiv \exp\left\{-\int_t^s \phi(\tau \mid a(\tau)) d\tau\right\}, \quad \text{and} \quad H_n(t \mid m_n) \to H(t \mid b(v)). \quad \ldots(2.5)$$

Note in particular that $H(\cdot \mid s)$ is strictly increasing since $\phi(s \mid a(s))$ is assumed to be strictly positive for each $s \in \Sigma$.

The second step in the proof is to establish the convergence of the equilibrium strategy.

Lemma B. *$p_n(s) \to \alpha(s)$ for each $s > \sup\{c_n\}$ in Σ.*

The proof derives from the equality

$$\int_c^s [p_n(t) - \bar{u}_n(t)] dH_n(t \mid s) = \frac{-1}{n-2}\left[p_n(s) - \int_c^s \bar{u}_n(t) dH_n(t \mid s) - p_n(c) H_n(c \mid s)\right] \quad \ldots(2.6)$$

for each interval $[c, s]$ with $c > c_n$, $n > 2$. Using (1.1) and the fact that

$$dH_n(\tau \mid t) \cdot H_n(t \mid s) = dH_n(\tau \mid s),$$

one has

$$\int_c^s p_n(t) dH_n(t \mid s) = \int_c^s \left\{\int_c^t \bar{u}_n(\tau) dH_n(\tau \mid t)^{n-1} + p_n(c) H_n(c \mid t)^{n-1}\right\} dH_n(t \mid s)$$

$$= \int_c^s \bar{u}_n(\tau) \int_\tau^s \frac{dH_n(t \mid s)}{H_n(t \mid s)^{n-1}} dH_n(\tau \mid s)^{n-1} + p_n(c) H_n(c \mid s)^{n-1} \int_c^s \frac{dH_n(t \mid s)}{H_n(t \mid s)^{n-1}}$$

$$= \frac{1}{n-2}\left\{\int_c^s \bar{u}_n(\tau)[H_n(\tau \mid s)^{-(n-2)} - 1] dH_n(\tau \mid s)^{n-1} + p_n(c) H_n(c \mid s)^{n-1}[H_n(c \mid s)^{-(n-2)} - 1]\right\}$$

$$= \frac{-1}{n-2} p_n(s) + \frac{n-1}{n-2} \int_c^s \bar{u}_n(\tau) dH_n(\tau \mid s) + \frac{1}{n-2} p_n(c) H_n(c \mid s), \quad \ldots(2.7)$$

which is precisely (2.6). Observe that the right side of (2.6) converges to zero. On the left,

$$dH_n(t \mid s) = H_n(t \mid s)\hat{\phi}_n(t)dt \quad \text{and} \quad H_n(t \mid s) \to H(t, s) > 0 \quad \text{and} \quad \hat{\phi}_n(t) \to \phi(t \mid a(t)) > 0.$$

Consequently, $[p_n(t) - \bar{u}_n(t)] \to 0$ for almost every t in each such interval $[c, s]$, and therefore $p_n(s) \to \lim_{n\to\infty} \bar{u}_n(s) = \alpha(s)$ for almost every $s > \sup\{c_n\}$. Lastly, the qualifier " almost every " can be excluded since each strategy p_n is differentiable and strictly increasing, and $\bar{u}_n$ and α are continuous and non-decreasing.

The remainder of the proof of Theorem 4 consists of showing that $p_n(m_n) \to \beta(v)$. Integrating (1.11) (iii) by parts,

$$p_n(m_n) = \bar{u}_n(m_n) - [\bar{u}_n(c_n) - p_n^0]H_n(c_n \mid m_n)^{n-1} - \int_{c_n}^{m_n} H_n(t \mid m_n)dp_n(t)$$
$$\leqq \bar{u}_n(m_n), \qquad \ldots(2.8)$$

which merely verifies the obvious. Therefore $\lim_{n\to\infty} \sup_{k \geqq n} p_k(m_k) \leqq \beta(v)$ by Lemma A. On the other hand, using the equality (2.6) from Lemma B,

$$\lim_{n\to\infty} \inf_{k \geqq n} \frac{\int_c^{m_k} [p_k(t) - \bar{u}_k(t)]dH_k(t \mid m_k)}{[1 - H_k(c \mid m_k)]} \geqq 0$$

for each $c > \sup\{c_n\}$ with $c < b(v)$. Therefore, $\lim_{n\to\infty} \inf_{k \geqq n} p_k(m_k) \geqq \lim_{n\to\infty} \bar{u}_n(m_n) = \beta(v)$. Thus $p_n(m_n) \to \beta(v)$, and therefore $P_n \to u(b(v), v)$ almost surely for each $v \in \Sigma V$.

The assumption that the density $f(\cdot \mid v)$ is positive on the boundary of $\Sigma S(v)$ plays a crucial role in the above proof. It is hardly plausible that this is actually crucial. Presumably an alternative proof could be constructed by considering a sequence $\{\varepsilon_l\}$ with $\varepsilon_l \to 0$ as $l \to \infty$ in which for each l one truncates ΣS at $b - \varepsilon_l$.

The Assumption 5 that b is strictly increasing at v is, however, crucial to the required continuity of a at $b(v)$. This is evident in Example 3, where b is actually constant (the lack of compactness is not important in Example 3; Wilson [2] gives another, more complicated, example satisfying compactness).

If $u(s, v) = v$ then the compactness of ΣS is not essential since one can always achieve this via an appropriate transformation. In this case, the form that Theorem 4 would take if ΣV were unbounded below presumably corresponds to an appropriate generalization of Gnedenko's famous theorem on extreme-value distributions (cf. W. Feller [1]).

Lastly, the assumption that a bidder's payoff is linear in the bid is apparently irrelevant, since Lemma B shows that asymptotically each bidder is indifferent whether or not he wins the item, and the probability $1/n$ of winning converges to zero. One can also see that a bidder's expected payoff is of the order of $1/n^2$ since (2.6) shows that his conditional expected payoff given that he wins is of the order of $1/n$ (recall that $p_n(s) \leqq \hat{u}_n(s) \leqq \bar{u}_n(s)$). For instance, in Examples 1, 2 and 3 a bidder aims for a conditional expected " profit " percentage given that he wins of $1/n$, $[1 - \min(1, 1/s)^{n-2}]/(n-2)$, and $1/(3n-2)$ respectively, and of course this is also the seller's expected percentage " loss " in Examples 2 and 3.

3. CONCLUSIONS

I have shown that, with certain regularity conditions satisfied, the sale price converges almost surely to the " true value " as the number of bidders increases, even though each bidder observes only incomplete sample information about the value. In my view this result adds substance to several often-cited ideas. First, there is the notion that a sale price conveys " all " of the relevant information among the agents in an economy. Second is the presumption that a theory of price formation is, or at least is consistent with, a theory of value. And third, there is the premise underlying Walrasian models that an agent in a "perfectly competitive" economy can, or must, regard a prevailing system of prices as parameters. And lastly there is the conjecture underlying studies of Walrasian models that a theory of price formation will some day justify their restrictive assumptions. Of course none of these are established firmly here, but I would guess at least that the latter may ultimately succeed—in some way more realistic than the theory of the core of economies with its untenable assumption of complete information; cf. Wilson [3].

There remains the question of how to deal with cases such as Example 3 in which the sale price persists in the limit as a random variable. I think that it calls for a theory in which, even with perfect competition, prices are possibly random—as for instance in the theory of temporary equilibria.

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