Auctions of Shares

Robert Wilson

The Quarterly Journal of Economics, Vol. 93, No. 4. (Nov., 1979), pp. 675-689.

Stable URL:
http://links.jstor.org/sici?sici=0033-5533%28197911%2993%3A4%3C675%3AAOS%3E2.0.CO%3B2-N

The Quarterly Journal of Economics is currently published by The MIT Press.

AUCTIONS OF SHARES

ROBERT WILSON

This paper compares the sale prices resulting from two different types of auctions. In an ordinary "unit" auction an item is sold, to the bidder submitting the highest bid, at a sale price equal to the highest bid. In a "share" auction the bidders receive fractional shares of the item at a sale price that equates the demand and supply of shares. Several examples are studied in detail in order to obtain exact comparisons of the sale prices.

The main conclusion derived from this study is that a share auction can yield a significantly lower sale price. In some cases the share-auction sale price is only half of the unit-auction sale price.

UNIT AND SHARE AUCTIONS

In a unit auction there is a single indivisible item that is to be sold to some one of the bidders. Each bidder submits to the seller a sealed tender specifying a price bid for the entirety of the item. The seller then awards the item to the bidder submitting the highest bid price at a sale price equal to the highest bid price. This is the type of auction used by the Department of Interior to sell leases of tracts on the Outer Continental Shelf for oil and gas exploration and development.

In a share auction there is an item of which shares are to be sold to several of the bidders. Each bidder submits a sealed tender specifying a schedule of prices bid for varying fractional shares of the item. An alternative, equivalent format is a schedule that for each possible price per share specifies the number of shares requested. The seller then selects that sale price such that the total of the shares requested by all of the bidders matches the available supply of shares. Each bidder receives the number of shares he requested at the sale price and for these he pays the sale price per share. This type of auction is a significant feature of the Phillips' Plan for selling OCS leases. According to this proposal, the "item" to be sold in the auction would be the collection of leases of all the tracts covering a specified geological structure; namely, a "unitized" lease of the entire structure. The shares of the item to be sold in the auction would be working-

The Quarterly Journal of Economics, November 1979 0033-5533/79/0093-0675$01.00

interest shares of the unitized lease. The proposal is designed to enable smaller firms and more risk-averse firms to participate in the auctions of highly risky leases by allowing them to bid for fractional working-interest shares, thereby reducing their capital requirements for payment of the sale price, and also reducing their exposure to risk.

It is an important matter for public policy to determine whether the adoption of a share-auction system for selling leases would be likely to increase or decrease government revenues. Of course, some increase can be expected if a share auction attracts more bidders and allows each bidder to limit his exposure to risk; also, there are well-documented benefits to be expected from the greater productive efficiency of unitized leases. The question remains, however, of whether a share auction would enable the major firms (with ample capital and negligible aversion to risk) to exploit the system by adopting a bidding strategy that would reduce the sale price. If this kind of exploitation is possible, then the benefits from a share auction may be more than offset by the resulting loss of revenue to the government.

In the next sections the methods of game theory are used to study share auctions and to compare them with unit auctions. Several examples are examined in detail in order to obtain exact characterizations of the sale prices resulting from the two types of auctions.

The main conclusion derived from this study is that a share auction is subject to manipulation by the bidders, with the result that the sale price is reduced significantly. In some cases the seller may lose up to half of the unit-auction sale price by adopting a share auction.

FORMULATION

I assume that the number of bidders is known beforehand by all participants to be a fixed number $n \geqq 2$. Moreover, in this paper I consider only situations in which the bidders are entirely alike in their characteristics; that is, the bidders are symmetric. In such a situation whatever is an optimal bidding strategy for any one bidder must also be an optimal strategy for any other bidder.[1] Thus, each bidder uses the same optimal strategy in preparing his bid. The subsequent analysis, therefore, is aimed at determining that strategy which is optimal for any one bidder if each other bidder is using it. An optimal strategy of this kind is a prevailing standard of behavior among the

1. That is, an optimal strategy is a symmetric Nash equilibrium.

bidders that is the commonly used rule or procedure for preparing bids.

The actual value of the item is assumed to be some number v. The value v is the same for each bidder. In the case of an OCS lease, for instance, the value is the discounted present value of the stream of revenues obtained minus the costs incurred. Ordinarily the value is not known with certainty at the time of the auction by any of the bidders. In the next section, however, I shall first illustrate the analysis by studying the special case in which the value is known with perfect certainty, or at least no bidder has any proprietary information about the value.

No Proprietary Information

The first example to be studied is the one in which no bidder has any proprietary information about the value. This example includes two different situations. In one situation each bidder knows the value with certainty. In the other situation each bidder is uncertain but lacking any proprietary information each bidder assesses the same certainty equivalent for the value (namely, that certain amount which he would accept in lieu of the uncertain true value).

Consider first the situation in which the value v is known with certainty by each bidder. If a unit auction is used, it is evident that the only optimal strategy is for each bidder to submit a bid price equal to the value. Consequently, some one of the bidders will receive the item at a sale price p^*, which is equal to the value, namely $p^* = v$, and the seller receives the full value.

I claim that if a share auction is used, then it is possible for the seller to receive a sale price p^0, which is only half of the value, namely $p^0 = v/2$. This claim is substantiated by exhibiting an optimal strategy that results in such a sale price. A strategy must specify a schedule of bids for shares. Consequently, a strategy is a function, say $x(p)$, which specifies that if the sale price is $p^0 = p$, then the bidder requests a fraction $x(p)$ of the available shares. The sale price p^0 is then that price p such that $nx(p^0) = 1$, since there are n bidders submitting the same schedules and the available supply of shares is 1. I assert that an optimal strategy is to submit a schedule that at each price p requests

$$x(p) = \frac{1 - 2p/nv}{n - 1}$$

shares. Note first that if each bidder uses this strategy, then the sale price, which satisfies $nx(p^0) = 1$, is $p^0 = v/2$, and the fraction of the shares received by each bidder is $x(p^0) = 1/n$. In order to demonstrate that this is an optimal strategy, we must show that it is optimal for any one bidder if each other bidder uses it. If $n - 1$ other bidders are using it and the remaining bidder submits a schedule $y(p)$, then the clearing price p^0 will be the one satisfying

$$(n - 1)x(p^0) + y(p^0) = 1,$$

and his profit will be

$$[v - p^0]y(p^0) = [v - p^0]2p^0/nv.$$

The price p^0 that maximizes his profit is $p^0 = v/2$. Since this is precisely the sale price that will result if he submits the schedule x, it follows that in fact x is an optimal schedule to submit.

The above analysis is also valid if the value is uncertain, no bidder has proprietary information, and no bidder is risk-averse. For one can take v to represent the certainty equivalent of the value, which is just its expectation.

Now suppose that the value is uncertain, and there is no proprietary information, but each bidder is risk-averse. Since this situation is rather complicated mathematically, it will suffice for illustrative purposes to consider the special case of two bidders, $n = 2$, each having a constant Arrow-Pratt measure of risk aversion $r > 0$ corresponding to an exponential utility function $U(z) = -(1/r)\exp(-rz)$. Also, assume that the value has a normal probability distribution with mean m and variance s^2. From these assumptions one can show that the certainty equivalent of a bidder's profit $[v - p]x$ when the sale price is p, he receives x shares, and v is uncertain, is

$$[m - p]x - (r/2)(xs)^2.$$

I claim that in this example an optimal strategy is to request at each price p that number x of shares which satisfies the equation,

$$p = m + s^2[rx - r/2 - c\sqrt{x/(1 - x)}],$$

where c is a specified positive constant. Such a schedule is depicted in Figure I as the function $p(x)$. The intersection of the two curves $p(x)$ and $p(1 - x)$ at $x = 1/2$ shows how the sale price $p^0 = p(1/2)$ is determined. Note that the sale price can be made arbitrarily small by increasing the undetermined constant c to sufficiently large values.[2] It is worth noting that this is also an optimal strategy even if r

2. The constant c must satisfy $c \geq r3\sqrt{3}/8$ in order to ensure that the schedule is a non-increasing function.

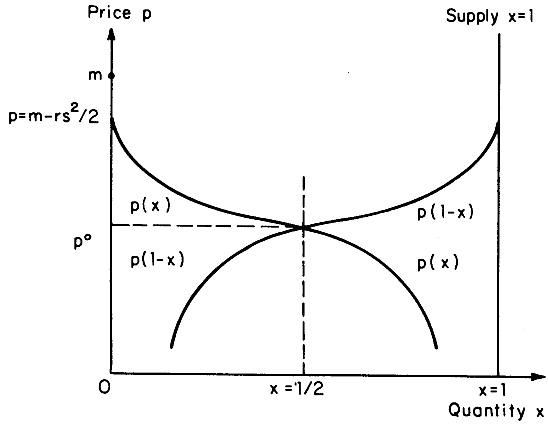

FIGURE I

$= 0$, corresponding to the case of no risk aversion.

In order to demonstrate that this is an optimal strategy, we must show that if one bidder uses it, then it is optimal for the other bidder also. If one bidder uses it, then the second bidder wants to submit a schedule that will maximize his certainty equivalent at the sale price. If we regard p as a function of x, he wants to maximize

$$[m - p(1 - x)]x - (r/2)s^2x^2$$

by choosing x, since the bidder knows that if he is to receive x shares, then the sale price must be $p^0 = p(1 - x)$. The share x that accomplishes this maximization is in fact $x = 1/2$, which is precisely the share that the bidder would receive if he submitted the same schedule as the other bidder. Consequently, the strategy is an optimal one for each bidder.

From these two examples, both of which assume that no bidder has any proprietary information, we see that share auctions may be distinctly unfavorable to the seller. In the first example the seller receives only half of the value no matter how many bidders there are. In the second example, which allows uncertainty about the value and risk aversion among the bidders, matters may be even worse for the seller: there are many different optimal strategies corresponding to the continuum of possible values of the arbitrary constant c, and if c is sufficiently large, then the sale price, which is $p^0 = m - cs^2$, may be less than the sum of the two bidders' certainty equivalents, which is $m - (r/4)s^2$, by an amount $s^2[c - r/4]$. Since c can be made as large as m/s^2 without driving the sale price below zero, we see that the seller's loss in revenue may be quite large if the mean m is sufficiently large compared to the variance s^2.

In the next section I shall demonstrate that these unfavorable characteristics remain essentially unaltered when the bidders have proprietary information.

PROPRIETARY INFORMATION

In this section I compare unit auctions and share auctions in situations in which the bidders have access to proprietary information about the value. The actual value v is assumed to be uncertain. In particular, each bidder supposes that the actual value v is the realization of a random variable V which has the distribution function $G(v) = \text{prob}\{V \leqq v\}$. Each bidder i ($i = 1, \ldots, n$) has, however, been able to obtain sample information about the value that is summarized in an estimate or statistic s_i. Again, each sample s_i is considered to be the realization of a random variable S_i, which has the conditional distribution function $F(s_i;v) = \text{prob}\{S_i \leqq s_i \mid V = v\}$ given that $V = v$. The bidders' samples are assumed to be independently and identically distributed given V. Bidder i's sample s_i is proprietary information that only he among the participants is able to observe.

For a bidder participating in a share auction, a strategy is described by a function $x(p;s_i)$ of both the price p and his sample s_i, such that after he obtains the information that $S_i = s_i$, then he submits a schedule specifying that at each price p he requests $x(p;s_i)$ shares. As before, an optimal strategy is one that is optimal for any one bidder when each other bidder is using it.

An optimal strategy can be characterized mathematically in the following way. If each one other than bidder i is using the strategy x, then when i uses a strategy y the sale price p^0 will be that price for which

$$\sum_{j \neq i} x(p^0;s_j) + y(p^0;s_i) = 1.$$

Note that the sale price p^0 depends upon all of the samples (s_j), whereas bidder i knows only his own sample s_i. Hence, for him the sale price remains uncertain. He can, however, assess the probability distribution function of the sale price, which conditional on V is a function,

$$H(p;v,y) = \text{prob}\{p^0 \leqq p \,|\, V = v, y(p;s_i) = y\}$$

$$= \text{prob}\left\{\sum_{j \neq i} x(p;S_j) \leqq 1 - y(p;s_i) \,|\, V = v\right\}.$$

Using this function, i may express his expected utility when he uses the strategy y as

$$E\left\{\int_0^\infty U([V-p]y(p;S_i))dH(p;V,y(p;S_i))\right\},$$

and it is this expression that an optimal strategy must maximize by choosing $y = x$. A solution to this maximization problem can be characterized by using the Euler condition from the calculus of variations. For the particular problem at hand the Euler condition takes the following simple form:

$$0 = E\{U'[(V-p)H_p + x(p;s_i)H_y] \,|\, S_i = s_i\},$$

where the marginal utility is $U' = U'([V-p]x(p;s_i))$ and H_p and H_y are the two partial derivatives of H evaluated at $y = x(p;s_i)$. In addition to the Euler condition the calculus of variations requires also that various transversality conditions are satisfied, but I shall not spell these out here.

The Euler condition has an appealing intuitive interpretation that it is worth emphasizing. It can be shown that the Euler condition is, in this case, equivalent to the requirement that among all pairs (p,y) the one that maximizes the conditional expected utility,

$$E\left\{U([V-p]y) \,|\, S_i = s_i, \sum_{j \neq i} x(p;S_j) = 1 - x(p;s_i)\right\},$$

is the choice $(p,y) = (p, x(p;s_i))$. This property expresses two features of an optimal strategy. First, the bidder i uses the information, that if the sale price turns out to be $p^0 = p$, then $\Sigma_{j \neq i} x(p;s_j) = 1 - x(p;s_i)$, to infer information about the other bidders' sample observations. Second, he recognizes that the schedule he submits will affect the sale price by moving it along the locus of price-quantity pairs (p,y) satisfying $\Sigma_{j \neq i} x(p;s_j) = 1 - y$; thus, he takes account of his schedule's effect on the sale price.

This interpretation is illustrated most clearly in the case of two bidders ($n = 2$) with linear utilities ($U' = 1$). In this special case $H(p;v,y) = F(s(p,1-y);v)$, where $s(p,1-y)$ is that sample that would lead the other bidder to request $1 - y$ shares at the price p. Consider bidder $i = 1$ and let the other bidder be $j = 2$. Let $s_2 = s(p,1 - x(p;s_1))$

be the other bidder's sample, and let $p(1-y;s_2)$ be the price at which bidder 2 requests $1-y$ shares. Then the Euler condition can be recast in the form,

$$0 = E\{V|S_1 = s_1, S_2 = s_2\} - p(1-x;s_2) + xp'(1-x;s_2),$$

where $x = x(p;s_1)$. This expresses the fact that bidder 1 infers bidder 2's sample observation s_2 from the price p, because he knows that $p = p(1-x;s_2)$ if p is the sale price; and, that the choice of y that maximizes his conditional expected profit

$$[E\{V|S_1 = s_1, S_2 = s_2\} - p(1-y;s_2)]y$$

along bidder 2's schedule is the choice $y = x(p;s_1)$.

With these general results as background, I now turn to the detailed study of some examples.

Example 1. In this example I assume that the bidders have no risk aversion, so that the utility function U is linear and $U' = 1$. The value V is assumed to have a gamma distribution with parameters m and k, namely with mean m/k and variance m/k^2. Conditional on $V = v$ each sample S_i has a Weibull distribution with the distribution function $F(s;v) = e^{-vs^{-b}}$ (the subsequent analysis can easily be generalized to allow S_i^{-b} to have a gamma distribution whose second parameter is v). Actually, to simplify matters, I shall replace $-s^{-b}$ by s so that each sample S_i has the distribution function $F(s;v) = e^{vs}$ for $s \leqq 0$.

I claim that in this example an optimal strategy for a share auction is

$$x(p;s_i) = \left[1 - 2p\frac{k - ns_i}{n(n+m)}\right] \Big/ [n-1].$$

Assuming that each bidder $j \neq i$ uses this strategy one finds that the Euler condition reduces to

$$0 = \left[\frac{n+m}{k - s_i - z} - p\right]\frac{(n-1)(n+m)}{2p^2}y - y\frac{(n-1)(n+m)}{2p},$$

where

$$z = (n-1)[k/n - (n+m)y/2p],$$

for bidder i's optimal choice of y, from which it follows that the indicated strategy is optimal.

When each bidder uses the optimal strategy, the sale price is

$$p^0 = \frac{1}{2}\frac{m+n}{k - \sum_i s_i} = \frac{1}{2}E\{V|S_1 = s_1, \ldots, S_n = s_n\}.$$

That is, regardless of the number of bidders the seller receives half of the conditional expectation of the value given all of the bidders' information. The share received by the ith bidder is

$$x(p^0;s_i) = \left[1 - \frac{k - ns_i}{\sum_j (k - ns_j)}\right] \Big/ [n - 1],$$

which one can show to be always nonnegative, and his expected profit is this same fraction of the sale price.

An important feature of this example is the fact that the seller's part of the conditional expectation of the value that he receives as the sale price stays at one-half for any number of bidders. He benefits not at all from increased competition among the bidders as their number increases.

This feature can be compared with the outcome of a corresponding unit auction. Suppose that all of the shares are to be awarded to the bidder submitting the highest bid price. A strategy in such a unit auction is a function $p(s_i)$, which specifies for bidder i that if he observes $S_i = s_i$, then he submits the bid price $p(s_i)$. I claim that the optimal strategy for the corresponding unit auction is

$$\begin{aligned} p(s_i) &= \frac{m+2}{m+1+n/(n-1)} \cdot \frac{m+1}{k - ns_i} \\ &= \frac{m+2}{m+1+n/(n-1)} E\left\{V \,|\, S_i = s_i \geqq \max_{j \neq i} S_j\right\}. \end{aligned}$$

This is verified by supposing that each bidder other than i uses this strategy and then showing that bidder i's best response is to use the same strategy. If each other bidder uses the strategy and bidder i submits the bid price q after observing that $S_i = s_i$, then he will be awarded the shares if $\max_{j \neq i} p(s_j) < q$, in which case his profit is $v - q$ when $V = v$; or he will receive no shares and zero profit. Conditional on V his probability of winning is

$$\text{prob}\left\{\max_{j \neq i} p(S_j) \leqq q \,|\, V = v\right\} = F\left(\frac{k - a/q}{n};v\right)^{n-1},$$

where $a = (m+1)(m+2)/(m+1+n/(n-1))$. His expected profit is therefore

$$\begin{aligned} &E\left\{[V - q]F\left(\frac{k - a/q}{n};V\right)^{n-1} \,\Big|\, S_i = s_i\right\} \\ &= \left[\frac{m+1}{k - s_i(n-1)(k - a/q)/n} - q\right]\left[\frac{k - s_i}{k - s_i(n-1)(k - a/q)/n}\right]^{m+1}, \end{aligned}$$

and it is this quantity that his optimal choice of the bid price q must maximize. In fact, one verifies, by setting the derivative of the expected profit equal to zero, that the optimal choice is $q = p(s_i)$. Thus, the strategy is an optimal one for every bidder.

The sale price received by the seller in a unit auction is

$$p^* = \max_i p(s_i) = \frac{m+2}{m+1+n/(n-1)} \cdot \frac{m+1}{k - n(\max_i s_i)}$$

$$= \frac{m+2}{m+1+n/(n-1)} E\left\{V \middle| \max_i S_i = \max_i s_i\right\}.$$

The expectation of the sale price is the corresponding fraction of $E\{V\}$. Observe that as the number n of bidders increases, the expected sale price becomes very nearly equal to the expected value. In contrast, the expected sale price received by the seller in a share auction under the same circumstances is one-half of $E\{V\}$ for any number of bidders. Thus, the seller's expected revenue is greater from a unit auction than from a share auction, at least in this example, and the advantage of a unit auction increases as the number of bidders increases.

Example 2. In this example I illustrate briefly the effects of risk aversion in circumstances analogous to the second example in the previous section. Each bidder has an Arrow-Pratt measure of risk aversion that is a constant $r > 0$. The case of no risk aversion corresponds exactly to $r = 0$ in the subsequent analysis. The value V has a normal distribution with mean m and variance $1/h$ (h is called the "precision"). Conditional on $V = v$, each sample S_i has a normal distribution with mean v and precision h'. Due to the mathematical complexities involved, I present here only the special case of two bidders, $n = 2$. I claim that an optimal strategy is to request at each price p that number $x = x(p;s_i)$ of shares that satisfies the equation,

$$p = \frac{1}{h+2h'}\left[hm + 2h's_1 + rx - \frac{r}{2} - c\sqrt{\frac{x}{1-x}}\right],$$

where c is a positive constant that is arbitrary except for the requirement that $c \geq r3\sqrt{3}/8$ in order to ensure that the schedule is a non-increasing function. The verification that this is an optimal strategy is obtained by checking that it satisfies the Euler condition and the transversality conditions. Two typical schedules are depicted in Figure II to show how the sale price is determined. Each schedule is tangent to the two lines $x = 0$ and $x = 1$, and to one of them at $p = -\infty$; and there is an inflection point at $x = 1/4$.

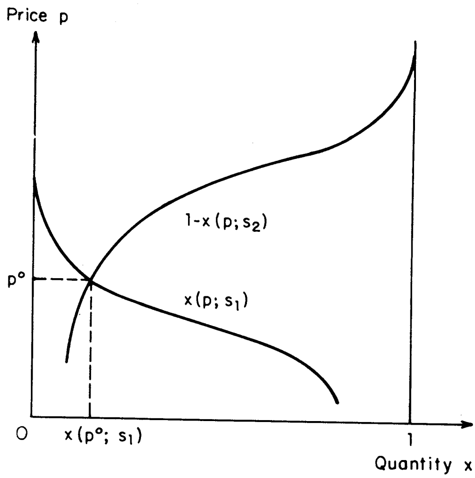

FIGURE II

If both bidders observe the same sample observation $s_i = s$, then the sale price is

$$p^0 = \frac{1}{h + 2h'}[hm + 2h's - c] = E\{V|S_1 = s, S_2 = s\} - \frac{(c/2)}{(h + 2h')}$$

depending on how large c is compared to the posterior precision $h + 2h'$. On the other hand, if bidder 2's sample observation is $s_2 = +\infty$, then the sale price is

$$p^0 = E\{V|S_1 = s_1, S_2 = s_1\} - \frac{(r/2)}{(h + 2h')}$$

depending on how large the risk aversion is, but note here that the conditional expectation of the value is based on $S_2 = s_1$ rather than the actual fact that $S_2 = +\infty$. In general, the allocation of shares is the solution of a cubic equation in x. If the risk aversion is negligible, say $r = 0$, then one can solve explicitly for the sale price to obtain

$$p^0 = E\{V|S_1 = s_1, S_2 = s_2\} - \frac{1}{h + 2h'}\sqrt{|hs_1 - hs_2|^2 + c^2}.$$

The sale price is reduced as the parameter c is increased, and it is further reduced if the risk aversion r is positive.

I have presented the analysis of the corresponding unit auction in an earlier paper [Wilson, 1969]. The relevant feature to be noted here is that the share auction invariably yields a lower sale price to the seller.

In addition to these two examples others can be analyzed by solving the Euler condition, using the transversality conditions to determine the constants of integration, by employing numerical methods to integrate the resulting partial differential equation. I have not succeeded in solving any other examples in explicit algebraic form.

DISCRIMINATORY PRICING

It is sometimes argued that in a share auction the seller can increase his revenue by employing discriminatory pricing of the shares to each bidder. This argument is false, however, since the bidders will respond to this maneuver by altering their strategies.

Suppose that in a share auction with nondiscriminatory pricing a bidder were to submit a schedule specifying that if the sale price is p, then he requests $x(p)$ shares, or equivalently, he requests x shares at the sale price $p(x)$. Similarly, suppose that in a share auction with discriminatory pricing he requests x shares at the sale price $q(x)$. If the pricing is nondiscriminatory, he will pay $p(x)x$ to the seller if the sale price is $p^0 = p(x)$ and he receives x shares. If the pricing is discriminatory, he will pay $\int_0^x q(y)dy$ if the sale price is $q^0 = q(x)$ and he receives x shares. Each bidder will receive the same allocation of shares and pay the same amount to the seller if his two strategies in the two types of auctions are related by the equation,

$$p(x)x = \int_0^x q(y)dy.$$

Differentiating this equation with respect to x yields the relationship

$$q(x) = p(x) + xp'(x).$$

Thus, from his strategy for an auction with nondiscriminatory pricing, he can easily derive a corresponding strategy for an auction with discriminatory pricing. Moreover, it is simple to verify that if $p(x)$ were an optimal strategy with nondiscriminatory pricing, then also $q(x)$ is an optimal strategy with discriminatory pricing. The consequence for the seller is that the sale price is reduced by converting to discriminatory pricing but his revenue remains unchanged, and the allocation of shares among the bidders remains unchanged.

VICKREY AUCTIONS

Another variant of the share auction that has been proposed is one described by William Vickrey [1961]. In a Vickrey auction the seller offers to each bidder a "rebate." The rebate is designed to induce the bidders to submit higher schedules so that the sale price will be equal to the expectation of the value; however, the seller's net revenue is less than the sale price by the amount of the rebates he must pay. I shall show that a Vickrey auction need not increase the seller's net revenue.

In Vickrey's form of a share auction a bidder i who receives x shares when the sale price is p obtains a profit of $[v - p]x + B_i(p)$ if the realized value is $V = v$, where $B_i(p)$ is the amount of the rebate he receives from the seller. The rebate is calculated as follows. Suppose that each bidder $j \neq i$ submits the schedule $x(p;s_j)$. Let p_i be the sale price that would have resulted in the absence of bidder i, namely

$$\sum_{j \neq i} x(p_i;s_j) = 1.$$

Then bidder i's rebate is

$$B_i(p) = p - p_i - \int_{p_i}^{p} \sum_{j \neq i} x(q;s_j)dq.$$

Using the same methods employed before, one can show that an optimal strategy for a bidder in a Vickrey auction must satisfy the following property: at each price p the number of shares requested, namely $x(p;s_i)$, must be the same as the choice of y, which maximizes his conditional expected utility:

$$E\left\{U([V - p]y + B_i(p)) \,|\, S_i = s_i, \sum_{j \neq i} x(p;S_j) = 1 - x(p;s_i)\right\}.$$

In the next paragraphs I examine several examples of Vickrey auctions.

Example 1. This example is the same as the previous Example 1 except that a Vickrey auction is used. Omitting the derivation here, I claim that an optimal strategy is

$$x(p;s_i) = (1/n)[1 - c(k - ns_i - (m + n)/p)],$$

where the parameter c can be any positive constant. If each bidder uses this strategy, then the sale price is

$$p^0 = \frac{m + n}{k - \sum_i s_i} = E\{V \,|\, S_1 = s_1, \ldots, S_n = s_n\}$$

as was intended in the original design of the Vickrey auction. Thus, the sale price is the expectation of the value of the shares. Nevertheless, the seller's net revenue is

$$p^0 - \sum_i B_i(p^0) = \frac{1}{n}\sum_i p_i = \frac{1}{n}\sum_i \left[\frac{m+n}{k - \sum_{j\neq i}s_j + 1/nc}\right]$$
$$= \frac{1}{n}\sum_i E\left\{V|S_i = \frac{-1}{nc}, (j \neq i)S_j = s_j\right\}.$$

In particular, observe that if the parameter c is chosen to be sufficiently small, then the seller's net revenue, after payment of the bidders' rebates, is nearly zero. This feature indicates that the Vickrey auction is subject to manipulation by the bidders, at least in this example.

Example 2. This example is the same as the previous Example 2 except that a Vickrey auction is used; also, I assume that there is no risk aversion so that $r = 0$. Omitting the derivation here, I claim that an optimal strategy is

$$x(p;s_i) = (1/n)[1 + c(hm + nh's_i - p[h + nh'])],$$

where again the parameter c can be any positive constant. It is true here as well that the sale price is the conditional expectation of the value, but the seller's net revenue may be made arbitrarily small by making the parameter c sufficiently small.

CONCLUSION

I conclude from this study of examples of share auctions that, compared to unit auctions, the seller may experience a considerable reduction in revenue. The loss in revenue stems from two features: as in Example 1 it may be that the seller obtains no advantages from increased competition as the number of bidders increases, or as in Example 2 the multiplicity of optimal strategies enables the bidders to choose an optimal strategy that is severely disadvantageous to the seller. Altering the procedure to enforce discriminatory pricing, or to offer incentives to the bidders as in a Vickrey auction, does not improve matters for the seller.

This conclusion does not necessarily imply that a share auction will actually prove to be disadvantageous to the seller in practice. It may be that the bidders do not adopt optimal strategies, or do not adopt the same optimal strategy, or that other considerations lead to the choice of an optimal strategy that is favorable for the seller.

Also, the generality of the conclusions derived from the few examples studied here is open to question.

STANFORD UNIVERSITY

REFERENCES

Vickrey, W., "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance,* XVII (May 1961), 8–37.

Wilson, R., "Competitive Bidding with Disparate Information," *Management Science,* XV (March 1969), 446–48.

——, "A Bidding Model of Perfect Competition," *Review of Economic Studies,* XLIV (Oct. 1977), 511–18.