5 Spinor Calculus

5.1 From triads and Euler angles to spinors. A heuristic introduction.

As mentioned already in Section 3.4.3, it is an obvious idea to enrich the Pauli algebra formalism by introducing the complex vector space $\mathcal{V}(2, C)$ on which the matrices operate. The two-component complex vectors are traditionally called *spinors*[28]. We wish to show that they give rise to a wide range of applications. In fact we shall introduce the spinor concept as a natural answer to a problem that arises in the context of rotational motion.

In Section 3 we have considered rotations as operations performed on a vector space. Whereas this approach enabled us to give a group-theoretical definition of the magnetic field, a vector is not an appropriate construct to account for the rotation of an orientable object. The simplest mathematical model suitable for this purpose is a Cartesian (orthogonal) three-frame, briefly, a *triad*. The problem is to consider two triads with coinciding origins, and the rotation of the object frame is described with respect to the space frame. The triads are represented in terms of their respective unit vectors: the space frame as $\Sigma_s(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ and the object frame as $\Sigma_c(\hat{e}_1, \hat{e}_2, \hat{e}_3)$. Here c stands for "corpus," since o for "object" seems ambiguous. We choose the frames to be right-handed.

These orientable objects are not pointlike, and their parametrization offers novel problems. In this sense we may refer to triads as "higher objects," by contrast to points which are "lower objects." The thought that comes most easily to mind is to consider the nine direction cosines $\hat{e}_i \cdot \hat{x}_k$ but this is impractical, because of the six relations connecting these parameters. This difficulty is removed by the three independent Eulerian angles, a most ingenious set of constructs, which leave us nevertheless with another problem: these parameters do not have good algebraic properties; their connection with the ordinary Euclidean vector space is provided by rather cumbersome relations. This final difficulty is solved by the *spinor* concept.

The theory of the rotation of triads has been usually considered in the context of rigid body mechanics[29] According to the traditional definition a rigid body is "a collection of point particles keeping rigid distances." Such a system does not lend itself to useful relativistic generalization [30]. Nor is this definition easily reconciled with the Heisenberg principle of uncertainty.

Since the present discussion aims at applications to relativity and quantum mechanics, we hasten

[28] In the literature the spinor concept has been introduced independently and in terms of conflicting conventions in nonrelativistic quantum mechanics on the one hand and in the relativistic theory on the other. The former is developed in the familiar bra-ket formalism, whereas the latter operates in terms of the contraction technique of tensor calculus. The present discussion is unified and uses the bra-ket formalism throughout. Moreover, the latter is introduced on purely mathematical grounds without invoking a prior knowledge of quantum mechanics. Comparison with standard procedures will be given as we go along.

[29] See [Gol50], Chapter IV. For more detail: [KS65] The fundamental theory is contained in Volume I of this four volume classic.

[30] [Pau58], p. 132. Also [Fok49]

to point out that we consider a triad as a precise mathematical model to deal with objects that are orientable in space.

Although we shall briefly consider the rigid body rotation in Section 5.2, the concept of *rigidity* in the sense defined above *is not essential in our argument*.

We turn now to a heuristic argument that leads us in a natural fashion from triad rotation to the spinor concept.

According to Euler's theorem any displacement of a rigid body fixed at a point O is equivalent to a rotation around an axis through O. (See [Whi64], page 2.)

This theorem provides the justification to describe the orientational configuration of Σ_c in terms of the unitary matrix in $\mathcal{SU}(2)$ that produces the configuration in question from the standard position in which the two frames coincide. Denoting the unitary unimodular matrices corresponding to two configuration by V_1, V_2 a transition between them is conveyed by an operator U

$$V_2 = UV_1 \tag{5.1.1}$$

Let

$$\begin{aligned} V &= \cos\frac{\phi}{2}1 - i\sin\frac{\phi}{2}\hat{v}\cdot\vec{\sigma} \\ &= q_0 1 - i\vec{q}\cdot\vec{\sigma} \end{aligned} \tag{5.1.2}$$

Here $q_0, \vec{q}$ are the so-called quaternion components, since the $(-i\sigma_k)$ obey the commutation rules of the quaternion units e_k: $e_1 e_2 = -e_2 e_1 = e_3$. We have

$$|V| = q_0^2 + \vec{q}^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1 \tag{5.1.3}$$

The Equations 5.1.1 - 5.1.3 can be given an elegant geometrical interpretation: $q_0, \vec{q}$ are considered as the coordinates of a point on the three-dimensional unit hypersphere in four-dimensional space $\mathcal{V}(4, R)$. Thus the rotation of the triad is mapped on the rotation of this hypersphere. The operation leaves 5.1.3 invariant.

The formalism is that of elliptic geometry, a counterpart to the hyperbolic geometry in Minkowski space [31].

This geometry implies a "metric": the "distance" of two displacements V_1, and V_2 is defined as

$$\frac{1}{2}Tr(V_2\tilde{V}_1) = \cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} + \sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}\hat{v}_1\cdot\hat{v}_2 \tag{5.1.4}$$

$$= \cos\frac{\phi}{2} = q_{10}q_{20} + \vec{q}_1\cdot\vec{q}_2 \tag{5.1.5}$$

[31] [KS28],[Bla60]

where ϕ is the angle of rotation carrying V_1, into V_2. Note the analogy with the hyperbolic formula 3.4.67 in Section 3.4.3.

We have here an example for an interesting principle of geometry: a "higher object" in a lower space can be often represented as a "lower object," i.e., a point in a higher space. The "higher object" is a triad in ordinary space $\mathcal{V}(3, R)$. It is represented as a point in the higher space $\mathcal{V}(4, R)$.

We shall see that this principle is instrumental in the intuitive interpretation of quantum mechanics. The *points* in the abstract spaces of this theory are to be associated with complex objects in ordinary space.

Although the representation of the rotation operator U and the rotating object V in terms of the same kind of parametrization can be considered a source of mathematical elegance, it also has a shortcoming. Rotating objects may exhibit a preferred intrinsic orientation, such as a figure axis, or the electron spin, for which there is no counterpart in Equations 5.1.1 and 5.1.3.

This situation is remedied by the following artifice. Let the figure axis point along the unit vector $\hat{e}_3$ that coincides in the standard position with $\hat{x}_3$ Instead of generating the object matrix V in terms of single rotation, we consider the following standard sequence to be read from right to left, (see Figure 5.1):

$$U(\hat{x}_3, \frac{\alpha}{2})U(\hat{x}_2, \frac{\beta}{2})U(\hat{x}_3, \frac{\gamma}{2}) = V(\alpha, \beta, \gamma) \tag{5.1.6}$$

Here α, β, γ are the well known Euler angles, and the sequence of rotations is one of the variants traditionally used for their definition.

The notation calls for explanation. We shall continue to use, as we did in Section 3, $U(\hat{u}, \phi/2)$ for the 2×2 unitary matrix parametrized in terms of axis angle variables. We shall call this also a uniaxial parametrization, to be distinguished from the *biaxial* parametrization of the unitary V matrices in which both the spatial direction $\hat{x}_3$, and the figure axis $\hat{e}_3$, play a preferred role.

In Equation 5.1.6 the rotations are defined along axes specified in the space frame Σ_s. However, in the course of each operation the axis is fixed in both frames. Thus it is merely a matter of another name (an alias I) to describe the operation (4) in Σ_c. We have then for the same unitary matrix

$$V(\alpha, \beta, \gamma) = U(\hat{e}_3, \frac{\gamma}{2})U(\hat{e}_2, \frac{\beta}{2})U(\hat{e}_3, \frac{\alpha}{2}) \tag{5.1.7}$$

Note the inversion of the sequence of operations involving the rotations a and y. This relation is to be interpreted in the kinematic sense: the body frame moves from the initial orientation of coincidence with Σ_s into the final position.

The equivalence of 5.1.6 and 5.1.7 can be recognized by geometrical intuition, or also by explicit transformations between Σ_s and Σ_c. (See [Got66], p 268).

In the literature one often considers the sequence

$$U(\hat{x}''_3, \frac{\gamma}{2})U(\hat{x}'_2, \frac{\beta}{2})U(\hat{x}_3, \frac{\alpha}{2}) \tag{5.1.8}$$

where $\hat{x}'_2$, and $\hat{x}''_3$ are axis positions after the first and the second step respectively. This procedure seems to have the awkward property that the different rotations are performed in different spaces. On closer inspection, however, one notices that Equation 5.1.8 differs only in notation from Equation 5.1.7. In the usual static interpretation $\hat{e}_1, \hat{e}_2, \hat{e}_3$, is used only for the final configuration, and $\hat{x}'_2$, $\hat{x}''_3$ are introduced as auxiliary axes. If, in contrast, one looks at the object frame kinematically, one realizes that at the instant of the particular rotations the following axes coincide:

$$\hat{x}_3 = \hat{e}_3, \quad \hat{x}'_2 = \hat{e}_2, \quad \hat{x}''_3 = \hat{e}_3, \tag{5.1.9}$$

We now write Equation 5.1.6 explicitly as

$$\begin{aligned} V(\alpha, \beta, \gamma) &= U(\hat{x}_3, \frac{\alpha}{2})\, U(\hat{x}_2, \frac{\beta}{2})\, U(\hat{x}_3, \frac{\gamma}{2}) \\ &= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} \begin{pmatrix} \cos(\beta/2) & -\sin(\beta/2) \\ \sin(\beta/2) & \cos(\beta/2) \end{pmatrix} \begin{pmatrix} e^{-i\gamma/2} & 0 \\ 0 & e^{i\gamma/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{-i\alpha/2}\cos(\beta/2)e^{-i\gamma/2} & -e^{-i\alpha/2}\sin(\beta/2)e^{i\gamma/2} \\ e^{i\alpha/2}\sin(\beta/2)e^{-i\gamma/2} & e^{i\alpha/2}\cos(\beta/2)e^{i\gamma/2} \end{pmatrix} \\ &= \begin{pmatrix} \xi_0 & -\xi_1^* \\ \xi_1 & \xi_0^* \end{pmatrix} \end{aligned} \tag{5.1.10}$$

with

$$\begin{aligned} \xi_0 &= e^{-i\alpha/2}\cos(\beta/2)e^{-i\gamma/2} \\ \xi_1 &= e^{i\alpha/2}\sin(\beta/2)e^{-i\gamma/2} \end{aligned} \tag{5.1.11}$$

The four matrix elements appearing in this relation are the so-called Cayley-Klein parameters. (See Equation 3.4.43 in Section 3.4.2.)

It is a general property of the matrices of the algebra $\mathcal{A}_2$, that they can be represented either in terms of components or in terms of matrix elements. We have arrived at the conclusion that the representation of a unitary matrix in terms of elements is suitable for the parametrization of orientational configuration, while the rotation operator is represented in terms of components (axis-angle variables).

There is one more step left to express this result most efficiently. We introduce the two-component complex vectors (spinors) of $\mathcal{V}(2, C)$ already mentioned at the beginning of the chapter. In particular, we define two conjugate column vectors, or ket spinors:

$$|\xi\rangle = \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix}, \quad |\bar{\xi}\rangle = \begin{pmatrix} -\xi_1^* \\ \xi_0^* \end{pmatrix} \tag{5.1.12}$$

and write the unitary V matrix symbolically as

$$V = \big(\langle\xi|\,|\,|\bar{\xi}\rangle\big) \tag{5.1.13}$$

We define the corresponding bra vectors by splitting the Hermitian conjugate V horizontally into row vectors:

$$V^\dagger = \begin{pmatrix} \xi_0^* & \xi_1^* \\ -\xi_1 & \xi_0 \end{pmatrix} = \begin{pmatrix} \langle\xi| \\ \langle\bar{\xi}| \end{pmatrix} \tag{5.1.14}$$

or

$$\langle\xi| = (\xi_0^*, \xi_1^*)\,; \quad \langle\bar{\xi}| = (-\xi_1, \xi_0) \tag{5.1.15}$$

The condition of unitarity of V can be expressed as

$$\begin{aligned} V^\dagger V &= \begin{pmatrix} \langle\xi| \\ \langle\bar{\xi}| \end{pmatrix} \left(|\xi\rangle, |\bar{\xi}\rangle\right) & (5.1.16) \\ &= \begin{pmatrix} \langle\xi|\xi\rangle & \langle\xi|\bar{\xi}\rangle \\ \langle\bar{\xi}|\xi\rangle & \langle\bar{\xi}|\bar{\xi}\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & (5.1.17) \end{aligned}$$

yielding at once the conditions of orthonormality

$$\begin{aligned} \langle\xi|\xi\rangle &= \langle\bar{\xi}|\bar{\xi}\rangle = 1 \\ \langle\xi|\bar{\xi}\rangle &= \langle\bar{\xi}|\xi\rangle = 0 \end{aligned} \tag{5.1.18}$$

These can be, of course, verified by direct calculation. The orthogonal spinors are also called conjugate spinors.

We see from these relations that our definition of spin conjugation is, indeed, a sensible one. However, the meaning of this concept is richer than the analogy with the ortho-normality relation in the real domain might suggest.

First of all we express spin conjugation in terms of a matrix operation. The relation is nonlinear, as it involves the operation of complex conjugation $\mathcal{K}$

We have

$$|\bar{\xi}\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mathcal{K}|\xi\rangle = -i\sigma_2\mathcal{K}|\xi\rangle \tag{5.1.19}$$

and

$$\langle\bar{\xi}| = \mathcal{K}\langle\xi| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathcal{K}\langle\xi|i\sigma_2 \tag{5.1.20}$$

We obtain from here

$$|\bar{\bar{\xi}}\rangle = -|\xi\rangle, \quad \langle\bar{\bar{\xi}}| = -\langle\xi| \tag{5.1.21}$$

The bar notation for spin conjugation suggests a connection with the complex reflection of the Pauli algebra. We shall see that such a connection indeed exists. However, we have to remember that, in contrast to Equation 5.1.21, complex reflection is involutive, i.e., its iteration is the identit $\bar{\bar{A}} = A$.

The emergence of the negative sign in Equation 5.1.21 is a well known property of the spin function, however we have to defer the discussion of this intriguing fact for later.

We shall occasionally refer to spinors normalized according to Equation 5.1.18 as *unitary spinors*, in order to distinguish them from *relativistic spinors* normalized as $\langle\xi|\xi\rangle = k_0$ where k_0 is the 0-th component of a four-vector.

Let us take a closer look at the connection between spinors and triads. In our heuristic procedure we started with an object triad specified by three orthonormal unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and arrived at an equivalent specification in terms of an associated spinor $|\xi\rangle$. Our task is now to start from the spinor and establish the corresponding triad in terms of its unit vectors. This is achieved by means of quadratic expressions.

We consider the so-called outer products

$$\begin{aligned} |\xi\rangle\langle\xi| &= \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} (\xi_0^*, \xi_1^*) \\ &= \begin{pmatrix} \xi_0\xi_0^* & \xi_0\xi_1^* \\ \xi_1\xi_0^* & \xi_1\xi_1^* \end{pmatrix} \end{aligned} \tag{5.1.22}$$

and

$$\begin{aligned} |\xi\rangle\langle\bar{\xi}| &= \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} (-\xi_1, \xi_0) \\ &= \begin{pmatrix} -\xi_0\xi_1 & \xi_0^2 \\ -\xi_1^2 & \xi_0\xi_1 \end{pmatrix} \end{aligned} \tag{5.1.23}$$

which can be considered as products of a 2×1 and 1×2 matrix.

In order to establish the connection with the unit vectors $\hat{e}_k$, we consider first the unit configuration in which the triads coincide: $\alpha = \beta = \gamma = 0$, i.e.,

$$\xi_0 = 1,\ \xi_1 = 0 \quad \text{or} \quad |\xi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{5.1.24}$$

with

$$|\bar{\xi}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.1.25}$$

Denoting these spinors briefly as $|1\rangle$ and $|\bar{1}\rangle$ respectively, we obtain from 5.1.22 and 5.1.23

$$|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(1 + \sigma_3) = \frac{1}{2}(1 + \hat{x}_3 \cdot \vec{\sigma}) \tag{5.1.26}$$

$$|1\rangle\langle\bar{1}| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}(\sigma_1 + i\sigma_2) = \frac{1}{2}(\hat{x}_1 + i\hat{x}_2) \cdot \vec{\sigma} \tag{5.1.27}$$

Let V be the unitary matrix that carries the object frame from the unit position into $\Sigma_c(\hat{e}_1, \hat{e}_2, \hat{e}_3)$. Since $V^\dagger = V^{-1}$ and $\bar{V} = V$, we have

$$V|1\rangle = |\xi\rangle \qquad V|\bar{1}\rangle = |\bar{\xi}\rangle \tag{5.1.28}$$
$$\langle 1|V^{-1} = \langle\xi| \qquad \langle\bar{1}|V^{-1} = \langle\bar{\xi}| \tag{5.1.29}$$

By operating on 5.1.26 and 5.1.27 from left and right by V and V^{-1} respectively, we obtain

$$|\xi\rangle\langle\xi| = \frac{1}{2}\left(1 + \hat{e}_3 \cdot \vec{\sigma}\right) \tag{5.1.30}$$
$$|\xi\rangle\langle\bar{\xi}| = \frac{1}{2}\left(\hat{e}_1 + i\hat{e}_2\right) \cdot \vec{\sigma} \tag{5.1.31}$$

and hence, by using Equation 3.4.13 of Section 3.4.2,

$$\hat{e}_1 = Tr\left(|\xi\rangle\langle\xi|\vec{\sigma}\right) = \langle\xi|\vec{\sigma}|\xi\rangle \tag{5.1.32}$$
$$\hat{e}_1 + i\hat{e}_2 \equiv \hat{e}_+ = Tr\left(|\xi\rangle\langle\bar{\xi}|\vec{\sigma}\right) = \langle\bar{\xi}|\vec{\sigma}|\xi\rangle \tag{5.1.33}$$

We have used here the rule:

$$Tr(|\xi\rangle\langle\eta|) = \langle\eta|\xi\rangle \tag{5.1.34}$$

Equations 5.1.32 and 5.1.33 constitute a most compact expression for the relation between a spinor and its associated triad. One can extract from here the values of the direction cosines

$$\hat{e}_j \cdot \hat{x}_k \equiv e_{jk} \quad j, k = 1, 2, 3 \tag{5.1.35}$$

$$\begin{aligned} \hat{e}_{31} &= \langle\xi|\sigma_1|\xi\rangle = \xi_0\xi_1^* + \xi_0^*\xi_1 = \Re(\xi_0^*\xi_1) \qquad (a) \\ \hat{e}_{32} &= \langle\xi|\sigma_2|\xi\rangle = i(\xi_0\xi_1^* - \xi_0^*\xi_1) = \Im(\xi_0^*\xi_1) \qquad (b) \\ \hat{e}_{33} &= \langle\xi|\sigma_3|\xi\rangle = \xi_0\xi_0^* + \xi_1^*\xi_1 \qquad (c) \end{aligned} \tag{5.1.36}$$

$$\begin{aligned} \hat{e}_{11} + i\hat{e}_{21} &= \langle\bar{\xi}|\sigma_1|\xi\rangle = \xi_0^2 - \xi_1^2 \quad (a) \\ \hat{e}_{12} + i\hat{e}_{22} &= \langle\bar{\xi}|\sigma_2|\xi\rangle = i(\xi_0^2 + \xi_1^2) \quad (b) \\ \hat{e}_{13} + i\hat{e}_{23} &= \langle\bar{\xi}|\sigma_3|\xi\rangle = -2\xi_0\xi_1 \quad (c) \end{aligned} \tag{5.1.37}$$

By using Equation 5.1.11 we obtain these quantities in terms of Euler angles [32]:

$$\begin{aligned} e_{31} &= \sin\beta\cos\alpha \\ e_{32} &= \sin\beta\sin\alpha \\ e_{33} &= \cos\beta \end{aligned} \tag{5.1.38}$$

[32]Equations 5.1.38 and 5.1.39 are the standard formulas connecting the Cartesian and the spherical coordinates of a unit vector. This was our rationale for choosing in Equation 5.1.6 $U(\hat{x}_2, \beta/2)$ rather than $U(\hat{x}_1, \beta/2)$ which would have interchanged the role of $\cos\alpha$ and $\sin\alpha$. (See [Gol50], op. cit.)

$$
\begin{aligned}
e_{11} &= \cos\gamma\cos\beta\cos\alpha - \sin\gamma\sin\alpha & e_{21} &= -\sin\gamma\cos\beta\cos\alpha - \cos\gamma\sin\alpha \\
e_{12} &= \cos\gamma\cos\beta\sin\alpha + \sin\gamma\cos\alpha & e_{22} &= -\sin\gamma\cos\beta\sin\alpha + \cos\gamma\cos\alpha \\
e_{13} &= -\cos\gamma\sin\beta & e_{23} &= \sin\gamma\sin\beta
\end{aligned}
\tag{5.1.39}
$$

The relation between vectors and spinors displayed in Equations 5.1.36 can be established also by means of a stereographic projection[33]. This method yields quicker results than the present lengthy build-up which in turn has a wider scope. Instead of rotating vector spaces, we operate on triads and thus obtain also Equation 5.1.37. To my knowledge, this relation has not appeared in the literature.

The Equations 5.1.36 and 5.1.37 solve the parametrization problem stated at the beginning of this chapter. The nine interrelated direction cosines e_{jk}. are expressed by the three independent spinor parameters.

This is the counterpart of the parametrization problem concerning the nine parameters of the $\mathcal{SO}(3)$ matrices (see page 13), a problem that has been solved by the $\mathcal{SU}(2)$ representation of $\mathcal{SO}(3)$ with the unitary matrices $U(\hat{u}, \phi/2)$.

It is noteworthy that the decisive step was taken in both cases by Euler who introduced the “Euler angles” α, β, γ and also the axis-angle parameters $\hat{u}, \phi$ for the rotation operators.

Euler’s results come to fruition in the version of spinor calculus in which spinors representing orientational states are parametrized in terms of Euler angles and the unitary operators in terms of $\hat{u}, \phi$.

We propose to demonstrate the ease by which this formalism lends itself to algebraic operations. This stems particularly from the constructs 5.1.30 and 5.1.31 in which we recognize the singular matrices of Table 3.2 (page 46).

We define more fully

$$
\begin{aligned}
|\xi\rangle\langle\xi| &= \frac{1}{2}(1 + \hat{e}_3\cdot\vec{\sigma}) \equiv E_3 && (a) \\
|\bar{\xi}\rangle\langle\bar{\xi}| &= \frac{1}{2}(1 - \hat{e}_3\cdot\vec{\sigma}) \equiv \bar{E}_3 && (b) \\
|\xi\rangle\langle\bar{\xi}| &= \frac{1}{2}(\hat{e}_1 + i\hat{e}_2)\cdot\vec{\sigma} \equiv E_+ && (c) \\
|\bar{\xi}\rangle\langle\xi| &= \frac{1}{2}(\hat{e}_1 - i\hat{e}_2)\cdot\vec{\sigma} \equiv E_- = -\bar{E}_+ && (d)
\end{aligned}
\tag{5.1.40}
$$

Here $E_3, \bar{E}_3$ are *idempotent projection operators* and E_+, E_- *nilpotent step operators*. Since $E_3 + \bar{E}_3 = 1$, we have

$$
|\eta\rangle = |\xi\rangle\langle\xi|\eta\rangle + |\bar{\xi}\rangle\langle\bar{\xi}|\eta\rangle \tag{5.1.41}
$$

$$
= |\xi\rangle a_0 + |\bar{\xi}\rangle a_1 \tag{5.1.42}
$$

[33]The surface of the unit sphere is projected from the south pole on to the equatorial plane; the latter is considered the complex plane in terms of homogeneous coordinates $z = \xi_1/\xi_0$. See [Wey50], page 144.

with

$$a_0 = \langle \xi | \eta \rangle, \quad a_1 = \langle \bar{\xi} | \eta \rangle \tag{5.1.43}$$

Also

$$E_+ |\bar{\xi}\rangle = |\xi\rangle \qquad E_- |\xi\rangle = |\bar{\xi}\rangle \tag{5.1.44}$$

$$E_+ |\xi\rangle = 0 \qquad E_- |\bar{\xi}\rangle = 0 \tag{5.1.45}$$

We see from Equations 5.1.40 that the transition $|\xi\rangle \to |\bar{\xi}\rangle$ corresponds to an inversion of the figure axis with a simultaneous inversion of the γ-rotation around the axis. Therefore the transformation corresponds to a transition from a right to a left frame with a simultaneous change from counter-clockwise to clockwise as the positive sense of rotation. Thus we should look at the transition from 5.1.40c to 5.1.40d as $E_+ \to \bar{E}_+$, or

$$\frac{1}{2}\left(\hat{e}_1 + i\hat{e}_2\right) \cdot \vec{\sigma} \to \frac{1}{2}\left[-\hat{e}_1 - i(-\hat{e}_2)\right] \cdot \vec{\sigma} \tag{5.1.46}$$

All this is apparent also if we represent the transition $|\xi\rangle \to |\bar{\xi}\rangle$ in terms of Euler angles as

$$\alpha \to \pi + \alpha \tag{5.1.47}$$

$$\beta \to \pi - \beta \tag{5.1.48}$$

$$\gamma \to \pi - \gamma \tag{5.1.49}$$

We note also the following relations for later use:

$$E_- E_+ = \bar{E}_3, \quad E_+ E_- = E_3 \tag{5.1.50}$$

In addition to the short symbols $|\xi\rangle$, $|\bar{\xi}\rangle$ for spinors and their conjugates, we shall use also more explicit notations depending on the context:

$$\overline{|\alpha, \beta, \gamma\rangle} = |\pi + \alpha, \pi - \beta, \pi - \gamma\rangle \tag{5.1.51}$$

$$|\hat{k}, \gamma\rangle, \; |\overline{\hat{k}, \gamma}\rangle = | - \hat{k}, \pi - \gamma\rangle \tag{5.1.52}$$

$$|\hat{k}\rangle \; |\bar{\hat{k}}\rangle = | - \hat{k}\rangle \tag{5.1.53}$$

Here $\hat{k}$ is the unit vector denoted by $\hat{e}_3$, in Equation 5.1.30. Its association with the spinor is evident from the following eigenvalue problem.

By using Equations 5.1.40 and 5.1.18 we obtain

$$\frac{1}{2}\left(1 + \hat{k} \cdot \vec{\sigma}\right) |\hat{k}\rangle = |\hat{k}\rangle\langle \hat{k}|\hat{k}\rangle = |\hat{k}\rangle \tag{5.1.54}$$

$$\frac{1}{2}\left(1 - \hat{k} \cdot \vec{\sigma}\right) |\bar{\hat{k}}\rangle = |\bar{\hat{k}}\rangle\langle \bar{\hat{k}}|\bar{\hat{k}}\rangle = |\bar{\hat{k}}\rangle \tag{5.1.55}$$

Hence

$$\hat{k}\cdot\vec{\sigma}|\hat{\rangle}k = |\hat{k}\rangle \tag{5.1.56}$$

$$\hat{k}\cdot\vec{\sigma}|\hat{\rangle}k = |\hat{k}\rangle \tag{5.1.57}$$

Thus $|k\rangle$ and $|\bar{\hat{k}}\rangle$ are eigenvectors of the Hermitian operator $\hat{k}\cdot\vec{\sigma}$, with the eigenvalues $+1$ and -1 respectively. This is a well known result, although usually obtained by a somewhat longer computation[34].

By using the explicit expression for $U(\hat{k}, \phi/2)$ we obtain from 5.1.56 and 5.1.57:

$$\begin{aligned} U(\hat{k}, \phi/2)|\hat{k}, \gamma\rangle &= \exp\left(-i\phi/2\right)|\hat{k}, \gamma\rangle = |\hat{k}, \gamma + \phi\rangle && (5.1.58) \\ U(\hat{k}, \phi/2)\overline{|\hat{k}, \gamma\rangle} &= \exp\left(i\phi/2\right)\overline{|\hat{k}, \gamma\rangle} = \overline{|\hat{k}, \gamma + \phi\rangle} && (5.1.59) \end{aligned}$$

There is also the unitary diagonal matrix

$$U(\hat{x}_3, \phi/2) = \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \tag{5.1.60}$$

the effect of which is easily described:

$$U(\hat{x}_3, \phi/2)|\alpha, \beta, \gamma\rangle = |\alpha + \phi, \beta, \gamma\rangle \tag{5.1.61}$$

These relations bring out the "biaxial" character of spinors: both $\hat{x}_3$, and $\hat{k}$ play a distinguished role. The same is true of a unitary matrix parametrized in terms of Euler angles: $V(\alpha, \beta, \gamma)$ or Cayley-Klein parameters. This is to be contrasted with the uniaxial form $U(\hat{u}, \phi/2)$.

Our discussion in this chapter has been thus far purely geometrical although active transformations of geometrical objects can be given a kinematic interpretation. We go now one step further and introduce the conept of time. By setting $\phi = \omega t$ with a constant ω in the unitary rotation operator we obtain the description of rotation processes:

$$\begin{aligned} U(\hat{k}, \frac{\omega t}{2})|\hat{k}, \frac{\gamma}{2}\rangle &= \exp\left(-i\omega t/2\right)|\hat{k}, \frac{\gamma}{2}\rangle = |\hat{k}, \frac{\gamma + \omega t}{2}\rangle && (5.1.62) \\ U(\hat{k}, \frac{\omega t}{2})\overline{|\hat{k}, \gamma\rangle} &= \exp\left(i\omega t/2\right)\overline{|\hat{k}, \gamma\rangle} = \overline{|\hat{k}, \frac{\gamma + \omega t}{2}\rangle} \end{aligned}$$

These rotations are stationary, because U operates on its eigenspinors. There are various ways to represent the evolution of arbitrary spinors as well. We have

$$\begin{aligned} U(\hat{k}, \frac{\omega t}{2})|\eta\rangle &= \exp\left(-i\frac{\omega t}{2}\hat{k}\cdot\vec{\sigma}\right)|\eta\rangle && (5.1.63) \\ \langle\eta|U^{-1}(\hat{k}, \frac{\omega t}{2}) &= \langle\eta|\exp\left(i\frac{\omega t}{2}\hat{k}\cdot\vec{\sigma}\right) \end{aligned}$$

[34] See [Kae65], pp. 11-16. This is a highly recommended supplementary reading.

Or, in differential form

$$\begin{aligned} i|\dot{\eta}\rangle &= \frac{\omega}{2}\hat{k}\cdot\vec{\sigma}|\eta\rangle \\ -i\langle\dot{\eta}| &= \langle\eta|\frac{\omega}{2}\hat{k}\cdot\vec{\sigma} \end{aligned} \tag{5.1.64}$$

The state functions solving these differential equations are obtained explicitly by using Equations 5.1.42, 5.1.43, 5.1.62 and 5.1.63:

$$|\eta(t)\rangle = U\left(\hat{k}, \frac{\omega t}{2}\right)|\eta(0)\rangle = \exp\left(-i\omega t/2\right)|\hat{k}\rangle a_0 + \exp\left(i\omega t/2\right)|\bar{\hat{k}}\rangle a_1 \tag{5.1.65}$$

and similarly for $\langle\eta(t)|$.

By introducing the symbol H for the Hermitian operator $H = (\omega t/2)\hat{k}\cdot\vec{\sigma}$ in 5.1.64 we obtain

$$i|\dot{\eta}\rangle = H|\eta\rangle \tag{5.1.66}$$

$$-i\langle\dot{\eta}| = \langle\eta|H \tag{5.1.67}$$

These equations are reminiscent of the Schrödinger equation. Also it would be easy to derive from here a Heisenberg type operator equation.

It must be apparent to those familiar with quantum mechanics, that our entire spinor formalism has a markedly quantum mechanical flavor. What all this means is that the orientability of objects is of prime importance in quantum mechanics and the concept of the triad provides us with a more direct path to quantization, or to some aspects of it, than the traditional point mass approach.

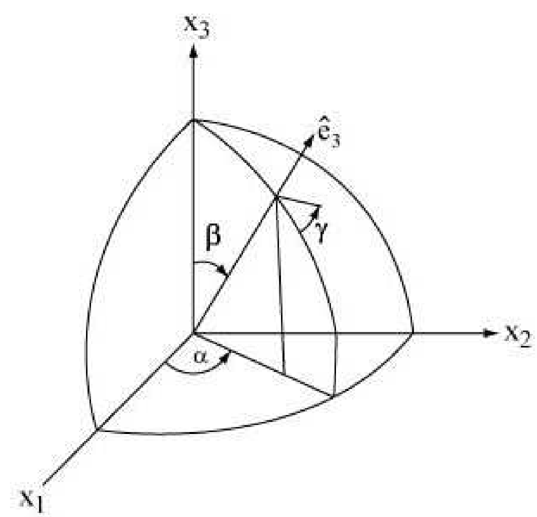
In order to make use of this opportunity, we have to apply our spinor formalism to physical systems.

Our use of the concept of time in Equations 5.1.62–5.1.66 is quite formal. We merely selected a one-parameter subgroup of the rotation group to describe possible types of stationary rotation.

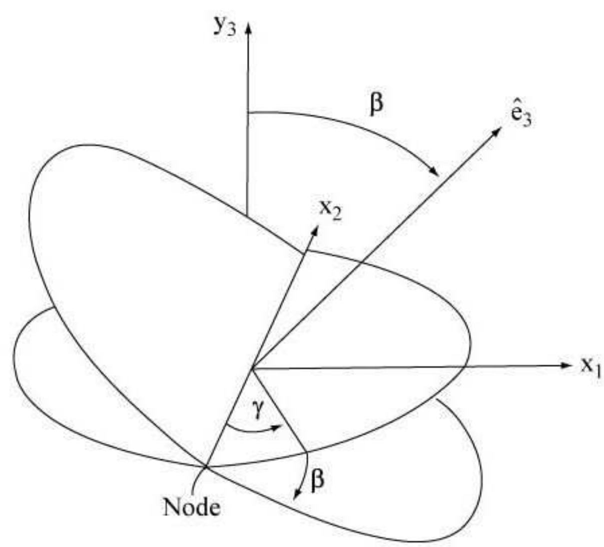
We have to turn to experiment, or to an experimentally established dynamical theory, to decide whether such motions actually occur in nature. We shall examine this question in connection with the rigid body rotation in the next section.

However, our main objective is the discussion of polarized light. Here the connection between classical and quantum theories is very close and the quantization procedure is particularly clear in terms of the spinor formalism.

The fact that the same formalism can be adjusted both to rigid body motion and to a wave phenomenon is interesting by itself. We know that the particle-wave duality is among the central themes in quantum mechanics. The contrast between these objects is very pronounced if we confine ourselves to point particles and to scalar waves. It is remarkable how this contrast is toned down within the context of rotational problems.



(a)



(b)

Figure 5.1: Euler Angles: (a) Static Picture. (b) Kinematic Display.

5.2 Rigid Body Rotation [35]

In Equations 5.1.62–5.1.65 we have introduced the concept of time formally as a parameter to specify some simple types of motion which have a stationary character.

We examine now the usefulness of these results by considering the inertial motions of a rigid body fixed at one of its points, the so-called gyroscope.

We may sum up the relevant experimental facts as follows: there are objects of a sufficiently high symmetry (the spherical top) that indeed display a stationary, inertial rotation around any of their axes. In the general case (asymmetric top) such a stationary rotation is possible only around three principal directions marked out in the body triad.

The point of greatest interest for us, however, is the fact that there are also modes of motion that can be considered stationary in a weaker sense of the word.

We mean the so-called precession. We shall consider here only the regular precession of the symmetric top, or gyroscope, [36] that can be visualized in terms of the well known geometrical construction developed by Poinsot in 1853. The motion is produced by letting a circular cone fixed in the body triad Σ_c roll over a circular cone fixed in the space triad Σ_s (see Figure 5.2) [37].

The noteworthy point is that the biaxial nature of spinors renders them well suited to provide an algebraic counterpart to this geometric picture.

In order to prove this point we have to make use of the theorem that angular velocities around different axes can be added according to the rules of vectorial addition. This theorem is a simple corollary of our formalism.

Let us consider the composition of infinitesimal rotations with $\delta\phi = \omega\delta t << 1$:

$$U_2(\hat{u}_2, \frac{\omega_2\delta t}{2})U_1(\hat{u}_1, \frac{\omega_1\delta t}{2}) \simeq \left(1 - \frac{\omega_2\delta t}{2}\hat{u}_2\cdot\vec{\sigma}\right)\left(1 - \frac{\omega_1\delta t}{2}\hat{u}_1\cdot\vec{\sigma}\right) \tag{5.2.1}$$

$$\simeq 1 - \frac{\delta t}{2}\left(\omega_2\hat{u}_2 + \omega_1\hat{u}_1\right)\cdot\vec{\sigma} \tag{5.2.2}$$

We define the angular velocity vectors

$$\vec{\omega} = \omega\hat{u} \tag{5.2.3}$$

and notice from Equation 5.2.1 that

$$\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega} \tag{5.2.4}$$

[35] Section 5.2 is not required for the continuity of the argument.

[36] The principal moments of inertia are $I_1 = I_2 \neq I_3$, and the external torque $N = 0$.

[37] See for a detailed geometrical discussion of the various possibilities in Klein-Sommerfeld [KS65]. Although many aspects of the spinor concept are found at least implicitly in this work, the formalism was not carried far enough to include the spinorial representation of the Poinsot precession.

Consequently we obtain for the situation presented in Figure 5.2:

$$\begin{aligned}\vec{\omega} &= \dot{\gamma}\hat{e}_3 + \dot{\alpha}\hat{x}_3 && (5.2.5)\\ \omega^2 &= \dot{\alpha}^2 + \dot{\gamma}^2 + 2\dot{\alpha}\dot{\gamma}\cos\beta && (5.2.6)\\ \beta &= \hat{x}_3 \cdot \hat{e}_3 && (5.2.7)\end{aligned}$$

We can describe the precession in spinorial terms as follows. We describe the gyroscope configuration in terms of the unitary matrix 5.1.10 and operate on it from right and left with two unitary operators:

$$\begin{aligned} V(t) &= U(\hat{x}_3, \frac{\dot{\alpha}t}{2})V(0)U(\hat{e}_3, \frac{\dot{\gamma}t}{2}) \qquad (5.2.8)\\ &= \begin{pmatrix} e^{-i\dot{\alpha}t/2} & 0 \\ 0 & e^{i\dot{\alpha}t/2} \end{pmatrix} \begin{pmatrix} e^{-i\alpha(0)/2}\cos(\beta/2)e^{-i\gamma(0)/2} & -e^{-i\alpha(0)/2}\sin(\beta/2)e^{i\gamma(0)/2} \\ e^{i\alpha(0)/2}\sin(\beta/2)e^{-i\gamma(0)/2} & e^{i\alpha(0)/2}\cos(\beta/2)e^{i\gamma(0)/2} \end{pmatrix} \\ &\quad \times \begin{pmatrix} e^{-i\dot{\gamma}t/2} & 0 \\ 0 & e^{i\dot{\gamma}t/2} \end{pmatrix}\end{aligned}$$

Thus

$$\alpha = \alpha(0) + \dot{\alpha}t, \quad \gamma = \gamma(0) + \dot{\gamma}t \qquad (5.2.9)$$

This relation displays graphically the biaxial character of the V matrix. Thus premultiplication corresponds to rotation in Σ_s and postmultiplication to that in Σ_c.

Note that the situations represented in Figure 5.2 (a) and (b) are called progressive and retrograde precessions respectively.

The rotational axis $\vec{\omega}$ is instantaneously at rest in both frames. The vector components can be expressed as follows:

$$\begin{array}{c|c|c} & \Im\Sigma_s & \Im\Sigma_c \\ \hline (a) & \omega_1 = \dot{\gamma}\sin\beta\cos\alpha & \omega_1 = \dot{\alpha}\sin\beta\cos\gamma \\ (b) & \omega_2 = \dot{\gamma}\sin\beta\sin\alpha & \omega_2 = \dot{\alpha}\sin\beta\sin\gamma \\ (a) & \omega_3 = \dot{\gamma}\cos\beta + \dot{\alpha} & \omega_3 = \dot{\alpha}\cos\beta + \dot{\gamma} \end{array} \qquad (5.2.10)$$

These expressions can be derived formally from Equation 5.2.8. The left column of Equation 5.2.10 follows from the application of the left-operator on a ket spinor and the right column of Equation 5.2.10 from a right operation on a bra spinor.

Another way of arriving at these results is as follows: Expressions in the left column of Equation 5.2.10 are evident from the vector addition rule given in 5.2.4. Expressions in the right column of Equation 5.2.10 do not follow so easily from geometrical intuition. However, we can invoke the kinematic relativity between the two triads. A rotation of Σ_c with respect to Σ_s can be thought of also as the reverse rotation of Σ_s in Σ_c. Thus V_c is equivalent to

$$V_s^{-1} = V_s^\dagger = V_s(-\alpha, -\beta, -\gamma) \qquad (5.2.11)$$

and we arrive from left to the right columns in Equation 5.2.10 by the following substitution:

$$\alpha \rightarrow -\gamma \tag{5.2.12}$$
$$\beta \rightarrow -\beta \tag{5.2.13}$$
$$\gamma \rightarrow -\alpha \tag{5.2.14}$$
$$t \rightarrow -t \tag{5.2.15}$$
$$\tag{5.2.16}$$

Up to this point the discussion has been only descriptive, kinematic. We have to turn to dynamics to answer the deeper questions as to the factors that determine the nature of the precession in any particular instance.

We invoke the kinematic relation Equation 5.1.62:

$$\exp\left(-i\frac{\omega t}{2}\hat{k}\cdot\vec{\sigma}\right)|\hat{k},\frac{\gamma}{2}\rangle = \exp\left(-i\frac{\omega t}{2}\right)|\hat{k},\frac{\gamma}{2}\rangle \tag{5.2.17}$$

This expression offers a way of generalization to dynamics. If this relation indeed describes a stationary process, then the generators $\frac{i}{2}\sigma_j$ of the unitary operator are constants of that motion. Later we shall pursue this idea to establish the concept of angular momentum and its quantization. However, at this preliminary stage we are merely looking for an elementary illustration of the formalism, and we draw on the standard results of rigid body dynamics.

The dynamic law consists of three propositions. First, we have in Σ_s

$$\frac{d\vec{L}_s}{dt} = \vec{N} \tag{5.2.18}$$

where N is the external torque.

In Σ_c we have a *constitutive relation* connecting *angular velocity* and *angular momentum*. We assume that the object triad is along the principal axes of inertia:

$$\begin{aligned} L_{c1} &= I_1\omega_1 \\ L_{c2} &= I_2\omega_2 \\ L_{c3} &= I_3\omega_3 \end{aligned} \tag{5.2.19}$$

Finally, the angular momentum components in Σ_s and Σ_c are connected by the relation

$$\frac{d\vec{L}_s}{dt} = \frac{d\vec{L}_c}{dt} + \vec{\omega}\times\vec{L}_c \tag{5.2.20}$$

Equations 5.2.18–5.2.20 imply the Euler equations. Dynamically the precession may stem either from an external torque, or from the anisotropy of the moment of inertia (or both).

We shall assume $\vec{N} = 0$ and $I_1 = I_2 \neq I_3$. The Euler equations simplified accordingly yield for the precession as viewed in Σ_c :

$$\begin{aligned} \dot{\omega}_{c1} + i\dot{\omega}_{c2} &= -i\left(\omega_{c1} + i\omega_{c2}\right)\omega_3\delta \\ I_3\dot{\omega}_3 &= 0 \end{aligned} \quad (5.2.21)$$

with

$$\delta = 1 - \frac{I_3}{I_1} \quad (5.2.22)$$

From Equation 5.2.10, right hand column, row (a) and (b), we obtain

$$\dot{\omega}_1 + i\dot{\omega}_2 = -i\dot{\gamma}\left(\omega_1 + i\omega_2\right) \quad (5.2.23)$$

and by comparison with 5.2.21 we have

$$\dot{\gamma} = \omega_3\delta = \omega_3\left(1 - \frac{I_3}{I_1}\right) \quad (5.2.24)$$

We obtain from 5.2.24, 5.2.22 and 5.2.10 (the right hand column, row c):

$$I_3\omega_3 = I_1\dot{\alpha}\cos\beta \quad (5.2.25)$$

and

$$\frac{\dot{\gamma}}{\dot{\alpha}\cos\beta} = \frac{I_1}{I_3} - 1 \quad (5.2.26)$$

Thus the nature of inertial precession is determined by the inertial anisotropy 5.2.22. In particular, let $\cos\beta > 0$, then

$$I_1 > I_3 \rightarrow \frac{\dot{\gamma}}{\dot{\alpha}} > 0 \quad \text{see Figure 5.2-a} \quad (5.2.27)$$

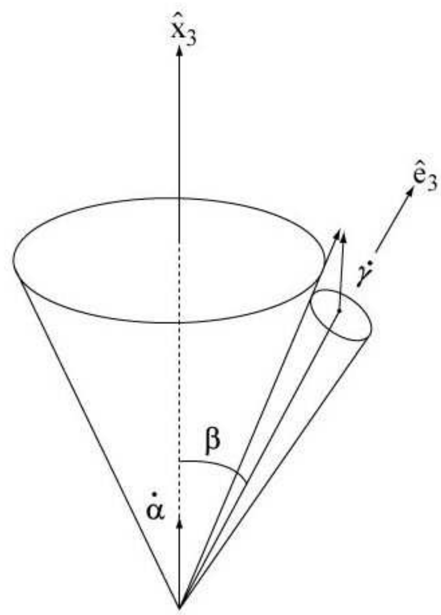
$$I_1 < I_3 \rightarrow \frac{\dot{\gamma}}{\dot{\alpha}} < 0 \quad \text{see Figure 5.2-b} \quad (5.2.28)$$

Finally, from Equation 5.2.25 $L_{c3} = I_3\omega_3 = I_1\dot{\alpha}\cos\beta$ and the total angular momentum squared is

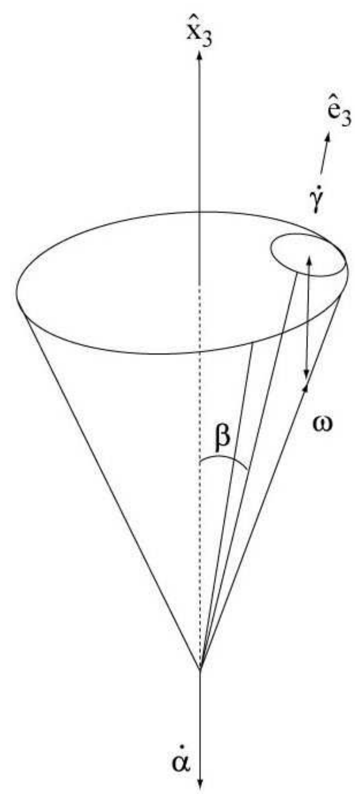
$$L^2 = I_1^2\dot{\alpha}^2 \quad (5.2.29)$$

Note that $L_{c3} = I_1\dot{\alpha}\cos\beta$ is the projection of the total angular momentum on to the figure axis. The precession γ in Σ_c comes about if $I_3 \neq I_1$. For further detail we refer to [KS65].

We note also that Euler's theory has been translated into the modern language of Lie Groups by V. Arnold ([Arn66] pp 319-361). However in this work stationary motions are required to have fixed rotational axes.



(a)



(b)

Figure 5.2: Progressive (a) and Retrograde (b) Precession.

5.3 Polarized light

Polarization optics provides a most appropriate field of application for the Pauli algebra and the spinor formalism. Historically, of course, it went the other way around, and various aspects of the formalism had been advanced by many authors, often through independent discovery in response to a practical need[38].

In the present discussion we forego the historical approach and by using the mathematical formalism already developed, we arrive at the consolidation and streamlining of much disconnected material.

Another factor which simplifies our argument is that we do not attempt to describe polarization in all the complexity of a real situation, but concentrate first on a simple mathematical model, the two-dimensional isotropic, harmonic oscillator. This is, of course, the standard method of the elementary theory, however, by translating this description into the spinorial formalism, we set the stage for generalizations. A potential generalization would be to establish the connection with the statistical theory of coherence[39]. However, at the present stage we shall be more concerned with applications to quantum mechanics[40].

Let us consider a monochromatic, polarized plane wave propagating in the z direction and write for the x and y components of the electric field

$$\begin{aligned} E_x &= p_1 \cos(\omega t + \phi_1) = p_1 \cos\tau \\ E_y &= p_2 \cos(\omega t + \phi_2) = p_2 \cos(\tau - \phi) \end{aligned} \tag{5.3.1}$$

where

$$\phi = \phi_1 - \phi_2, \qquad p_1,\, p_2 \geq 0 \tag{5.3.2}$$

Let us define new parameters:

$$\begin{aligned} p_1 &= p\cos\frac{\theta}{2} \\ & \quad 0 \leq \theta \leq \pi \\ p_2 &= p\sin\frac{\theta}{2} \end{aligned} \tag{5.3.3}$$

It is convenient to express the information contained in Equations 5.3.1–5.3.3 in terms of the spinor

$$|\hat{k}\rangle = p \begin{pmatrix} e^{-i\phi/2} & \cos(\theta/2) \\ e^{i\phi/2} & \sin(\theta/2) \end{pmatrix} e^{-i\psi/2} \tag{5.3.4}$$

[38] [Sto52]; [Poi92]; See [Shu62], with an extensive bibliography. Closest in line with the present approach is [Whi71]. This paper contains also a selection of references to theoretical papers.

[39] [Wie30, Bar63]

[40] Such connections may have been first suggested by [FM51, Fan51, Fan54]

Here $\psi = \omega t + \phi_1$ represents the common phase of the two components which does not affect the state of polarization. However, the presence of this third angle is in line with our definition of spinor in Equations 5.1.10 and 5.1.11 in Section 5.1. It will prove to be of significance in the problem of beam splitting and composition. By normalizing the intensity and setting $p = 1$, the spinor 5.3.4 conforms to our unitary normalization of Section 5.1 [41].

By using Equations 5.1.30, 5.1.36 and 5.1.38 of Section 5.1 we obtain

$$|\hat{k}\rangle\langle\hat{k}| = \frac{1}{2}(1 + \hat{k} \cdot \vec{\sigma}) \tag{5.3.5}$$

with

$$\begin{aligned} k_1 = \langle\hat{k}|\sigma_1|\hat{k}\rangle &= \sin\theta\cos\phi = 2p_1p_2\cos\phi \\ k_2 = \langle\hat{k}|\sigma_2|\hat{k}\rangle &= \sin\theta\sin\phi = 2p_1p_2\sin\phi \\ k_3 = \langle\hat{k}|\sigma_3|\hat{k}\rangle &= \cos\theta = p_2^2 - p_1^2 \end{aligned} \tag{5.3.6}$$

In such a fashion the spinor 5.3.5, and hence each state of polarization is mapped on the surface of the unit sphere, the so-called Poincaré sphere.

We see that the unit vector $(1, 0, 0)(\theta = \pi/2, \phi = 0)$ corresponds to linear polarization along

$$\frac{1}{\sqrt{2}}(\hat{x} + \hat{y}), \quad \text{or} \quad |45^\circ\rangle$$

$(0, 1, 0)(\theta = \pi/2, \phi = \pi/2)$ corresponds to right circularly polarized light $|R\rangle$[42], and $(0, 0, 1)$ or $\theta = 0$ to linear polarization in the $\hat{x}$ direction: $|\hat{x}\rangle$. (See Figures 5.3 and 5.3.)

There is an alternative, and even more favored method of parametrizing the Poincaré sphere, in which the preferred axis for the definition of spherical coordinates corresponds to light of positive helicity $|R\rangle$. This choice implies a new set of spherical angles, say α, β to replace ϕ, θ. Their relation is displayed geometrically in Figures 5.3 and 5.4. The corresponding algebraic treatment is summed up as follows.

We relabel the Cartesian axes in the "Poincaré space" as

$$\begin{aligned} k_3 &= s_1 = \sin\beta\cos\alpha \\ k_1 &= s_2 = \sin\beta\sin\alpha \\ k_2 &= s_3 = \cos\beta \end{aligned} \tag{5.3.7}$$

[41] This is a good opportunity to amend an unnecessary narrowness in the definition of the spinor in Section 5.1. "Spinor" should denote any vector in $\mathcal{V}(2, C)$. The unitary normalization is required only for the basis spinors or unit spinors.

[42] Contrary to the traditional convention in classical optics we call right polarization light of positive helicity: linear and angular momentum are parallel. Note that left polarization is the conjugate of right polarization: $|L\rangle = |\bar{R}\rangle$; likewise $|\bar{\hat{y}}\rangle = |\bar{\hat{x}}\rangle$, and $|135^\circ\rangle = |\overline{45^\circ}\rangle$.

The vector $\hat{s}$ is associated with the unitary spinor

$$|\hat{s}\rangle = \begin{pmatrix} \exp(-i\alpha/2) & \cos(\beta/2) \\ \exp(i\alpha/2) & \sin(\beta/2) \end{pmatrix} \tag{5.3.8}$$

and

$$|\hat{s}\rangle\langle\hat{s}| = \frac{1}{2}(1 + \hat{s}\cdot\sigma) \tag{5.3.9}$$

The advantage of this choice is that the angles α, β have a simple meaning. We claim that

$$\begin{aligned} a_1 &= a\cos\left(\frac{1}{2}(\frac{\pi}{2} - \beta)\right) \\ a_2 &= a\sin\left(\frac{1}{2}(\frac{\pi}{2} - \beta)\right) \end{aligned} \tag{5.3.10}$$

where a_1, a_2 are the half major and minor axes of the ellipse traced by the $\vec{E}$ vector; we associate a positive and negative a_2 with an ellipse circled in the positive and negative sense respectively. Moreover, the angle α is twice the angle of inclination of the major axis against the x axis (Figure 5.4-d). The angle γ refers to the overall phase in complete analogy to ψ.

The proof of these statements are found in Born and Wolf (see pp. 24-32 of [BW64], the later editions are almost unchanged). A somewhat simplified derivation follows.

First we prove that Equations 5.3.1 and 5.3.2 provide indeed a parametric representation of an ellipse. The elimination of τ from the two equations 5.3.1 yields

$$\left(\frac{E_1}{p_1\sin\phi}\right)^2 - \left(\frac{2E_1E_2\cos\phi}{p_1p_2\sin^2\phi}\right) + \left(\frac{E_2}{p_2\sin\phi}\right)^2 = 1 \tag{5.3.11}$$

This is an equation of the form

$$\sum_{i=1}^{2} a_{ik}x_ix_k = 1 \tag{5.3.12}$$

with the a_{ik}, real, symmetric, and $a_{11} > 0$,

$$a_{11}a_{22} - a_{12}^2 > 0 \tag{5.3.13}$$

The axes of the ellipse are derived from the eigenvalue problem:

$$\begin{aligned} (a_{11} - \lambda)x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 &= 0 \end{aligned} \tag{5.3.14}$$

Hence

$$\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}^2 = 0 \tag{5.3.15}$$

with

$$\lambda_1 = \frac{1}{a_1^2}, \quad \lambda_2 = \frac{1}{a_2^2} \tag{5.3.16}$$

where a_1, a_2 are the half major and half minor axes respectively.

We have, by inserting for the a_{ik}, from Equation 5.3.11

$$\lambda_1 + \lambda_1 = a_{11} + a_{22} = \left(\frac{1}{p_1^2} + \frac{1}{p_2^2}\right)\frac{1}{\sin^2\phi} \tag{5.3.17}$$

$$\lambda_1 + \lambda_1 = a_{11}a_{22} - a_{12}^2 = \frac{1}{p_1^2 p_2^2 \sin^2\phi} \tag{5.3.18}$$

From these equations we have

$$a_1^2 a_2^2 = p_1^2 p_2^2 \sin^2\phi \tag{5.3.19}$$

$$a_1^2 + a_2^2 = p_1^2 + p_2^2 \tag{5.3.20}$$

From Equation 5.3.19 we have $a_1 a_2 = \pm p_1 p_2 \sin\phi$. We require

$$a_1 a_2 = p_1 p_2 \sin\phi \tag{5.3.21}$$

and let $a_2 < 0$ for $sin\phi < 0$.

We introduce now the auxiliary angle 3 as defined in Equation 5.3.10. With such an assignment $\beta = 0, \pi$ correspond indeed to right and left circularly polarized light $|R\rangle$, $|\bar{R}\rangle$ respectively. Moreover $a_1 \geq |a_2|$. Hence a_1 is the half major axis.

From Equations 5.3.3, 5.3.19 and 5.3.10 we obtain

$$\cos\beta = \sin\theta\sin\phi \tag{5.3.22}$$

We complete the parametrization of ellipticity by introducing $\alpha/2$ for the angle between the major axis[43] and the $\hat{x}$ direction (Figure 5.4-c).

From Equation 5.3.14 we have

$$\tan\frac{\alpha}{2} = \frac{x_2}{x_1} = \frac{\lambda - a_{11}}{a_{12}} = \frac{a_{12}}{\lambda - a_{22}} \tag{5.3.23}$$

and

$$\begin{aligned} \tan\alpha &= \frac{\tan\frac{\alpha}{2} + \tan\frac{\alpha}{2}}{1 - \tan^2\frac{\alpha}{2}} = \frac{\frac{\lambda - a_{11}}{a_{12}} + \frac{a_{12}}{\lambda - a_{22}}}{\frac{\lambda - a_{11}}{\lambda - a_{22}}} \\ &= \frac{(\lambda - a_{11})(\lambda - a_{22})\frac{1}{a_{12}} + a_{12}}{a_{11} - a_{22}} = \frac{2a_{12}}{a_{11} - a_{22}} \\ &= \frac{2p_1 p_2 \cos\phi}{p_1^2 - p_2^2} = \frac{\sin\theta\cos\phi}{\cos\theta} = \tan\theta\cos\phi \end{aligned} \tag{5.3.24}$$

[43]That $\alpha/2$ refers to the major rather than the minor axis will be evident from the resulting parametrization of the Poincaré sphere.

It is apparent from Equation 5.3.22 that the axis s_3 can be indeed identified with k_2. Moreover Equation 5.3.24 yields

$$\frac{s_2}{s_1} = \frac{k_1}{k_3} \tag{5.3.25}$$

Since $k_1^2 + k_2^2 + k_3^2 = 1 = s_1^2 = s_2^2 = s_3^2$, we arrive at the rest of the identification suggested in Equation 5.3.7.

We shall refer to the formalism based on the parametrizations $\hat{k}(\phi, \theta, \psi)$ and $\hat{s}(\alpha, \beta, \gamma)$ as the $\hat{k}$-scheme and the $\hat{s}$-scheme respectively. Since either of the two pairs of angles ϕ, θ and α, β provide a satisfactory description of the polarization state, it is worthwhile to deal with both schemes.

The role of the "third angle" ψ or γ, respectively, is more subtle. It is well known that a spinor can be visualized as a vector and an angle, a "flagpole" and a "flag" in Penrose's terminology [44]. However, the angle represents a phase, and as such has notoriously ambivalent properties. While a single phase is usually unimportant, *phase relations* are often most significant. Although one can solve particular problems in polarization optics in terms of the Poincaré sphere without an explicit use of the third angle, for us these problems are merely stepping stones for deeper problems and we prefer to present them as instances of the general formalism. No matter if this seems to be a somewhat heavy gun for the purpose.

In proceeding this way we have to ignore some fine distinctions; thus we assign $|\xi\rangle$ and $-|\xi\rangle$ to the same state of polarization. We consider it an advantage that the formalism has the reserve capacity to be used later to such problems as the electron spin[45].

We demonstrate the usefulness of the spinor formalism by translating one of its simple propositions into, what might be called the fundamental theorem of polarization optics.

Consider two pairs of conjugate spinors $|\xi\rangle$, $|\bar{\xi}\rangle$ and $|\xi'\rangle$, $|\bar{\xi}'\rangle$.

Theorem 3. There is a uniquely determined unimodular unitary matrix U such that

$$\begin{aligned} |\xi'\rangle &= U|\xi\rangle \\ |\bar{\xi}'\rangle &= U|\bar{\xi}\rangle \end{aligned} \tag{5.3.26}$$

Proof. By using Equation 5.1.27 of Section 5.1 we consider the unitary matrices associated with the spinor pairs:

$$V = \big(|\xi\rangle, |\bar{\xi}\rangle\big) \tag{5.3.27}$$

$$V' = \big(|\xi'\rangle, |\bar{\xi}'\rangle\big) \tag{5.3.28}$$

□

[44] See [MTW73], p 1157. The equivalent term "ax and blade" has been used by [Pay52]. A number of ideas in the present version of spinor theory have been advanced in this paper and its sequels by the same author: [Pay55, Pay59].

[45] Remember that in Section 3 we use the $\mathcal{SU}(2)$ representation of $\mathcal{SO}(3)$, not because we need its two-valuedness, but because it offers an economical and adequate parametrization of the problem.

The matrix $U = V'V^{-1}$ has the desired properties, since $UV = V'$.

Let the monoaxial parametrization of U be $U(\hat{u}, \chi/2)$. By using Equations 5.1.58 and 5.1.59 of Section 5.1 we see that U has two eigenspinors:

$$U|\hat{u}\rangle = \exp(-i\chi/2)\,|\hat{u}\rangle \tag{5.3.29}$$

$$U|\bar{\hat{u}}\rangle = \exp(i\chi/2)\,|\bar{\hat{u}}\rangle \tag{5.3.30}$$

Hence U produces a phase shift between the conjugate states $|\hat{u}\rangle$ and $|\bar{\hat{u}}\rangle$; moreover it rotates their linear combinations:

$$|\xi\rangle = a_0|\hat{u}\rangle + a_1|\bar{\hat{u}}\rangle \tag{5.3.31}$$

where

$$|a_0|^2 + |a_1|^2 = 1 \tag{5.3.32}$$

These results translate into polarization optics as follows. An arbitrary, fully polarized beam can be transformed into another beam of the same kind by a phase shifter, the axis $\hat{u}$ of which is uniquely determined in terms of the spinor representation of the given beams. Since the result is a rotation of the Poincaré sphere, the axis of the phase shifter can be determined also geometrically.

To counteract the complete generality of the Poincaré construction, let us consider the special cases

$$U = U\left(\hat{k}_3, \frac{\Delta\phi}{2}\right) \tag{5.3.33}$$

$$U = U\left(\hat{s}_3, \frac{\Delta\alpha}{2}\right) \tag{5.3.34}$$

The phase shifter, Equation 5.3.33, is called a *linear retarder*, it establishes a phase lag between one state of linear polarization and its antipodal state. For $\Delta\phi = \pi/2$ we have a quarter waveplate that transforms elliptic into linear polarization or vice versa.

The phase shifter, Equation 5.3.34, produces a phase lag between right and left circularly polarized beams. (A circularly biregringent crystal, say quartz is cut perpendicularly to the optic axis: spiral staircase effect.)

Since a linearly polarized beam is the linear composition $|R\rangle$ and $|L\rangle$ the phase lag manifests itself in a rotation of the plane of polarization, hence a rotation around $\hat{s}_3$. The device is called a *rotator*.

Thus rotations of the Poincaré sphere may produce either a change of shape, or a change of orientation in ordinary space.

We may add that by combining two quarter waveplates with one general rotator we can realize an arbitrary phase shifter $U(\hat{u}, \chi/2)$.

Our main theorem on the representation of the transformation of fully polarized beams is evidently the counterpart of Euler's theorem on the displacements of the gyroscope mentioned on page 56.

Although we have a formal identity, in the sense that we have in both cases the rotation of a triad, there is a great deal of difference in the physical interpretation. The rotation takes place now in an abstract space, we may call it the *Poincaré space*. Also it makes a great deal of difference that the angular velocities of the rotating object are now replaced by the time rates of change of the phase difference between pairs of conjugate polarizations. On going from rigid bodies to polarized waves (degenerate vibrations) we do not have to modify the formalism, but the new interpretation opens up new opportunities. The concept of phase difference evokes the idea of coherent superposition as contrasted with incoherent composition. These matters have no analog in the case of rigid rotation, and we now turn to the examination of the new features.

Let us consider a polarized beam represented in the $\hat{s}(\alpha, \beta)$ scheme by the spinor $|\hat{s}\rangle$ where

$$S = |\hat{s}\rangle\langle\hat{s}| = \frac{1}{2}(1 + \hat{s}\cdot\vec{\sigma}) \tag{5.3.35}$$

or alternatively

$$S = \begin{pmatrix} s_0 s_0^* & s_0 s_1^* \\ s_1 s_0^* & s_0 s_1^* \end{pmatrix} \tag{5.3.36}$$

S is called the density matrix or coherency matrix associated with a polarized beam (a *pure state* in quantum mechanics). As we have seen already, it is idempotent and the determinant $|S| = 0$.

We analyze this beam with an instrument $U(u, \Delta\psi/2)$ where $\hat{u} \neq \hat{s}$, and obtain

$$|\hat{s}\rangle = a_0|\hat{u}\rangle + a_1|\bar{\hat{u}}\rangle \tag{5.3.37}$$

with

$$a_0 = \langle\hat{u}|\hat{s}\rangle \qquad a_1 = \langle\bar{\hat{u}}|\hat{s}\rangle \tag{5.3.38}$$

$$|a_0|^2 + |a_1|^2 \;=\; 1 \tag{5.3.39}$$

From Equations 5.1.58 and 5.1.58 we have

$$\begin{aligned} |\xi'\rangle &= U\left(\hat{u}, \frac{\Delta\psi}{2}\right)|\hat{s}\rangle + \\ &= a_0 \exp\left(-i\frac{\Delta\psi}{2}\right)|\hat{u}\rangle a_1 \exp\left(i\frac{\Delta\psi}{2}\right)|\bar{\hat{u}}\rangle \\ &= a_0|\hat{u}, \frac{\psi+\Delta\psi}{2}\rangle + a_1\overline{|\hat{u}, \frac{\psi+\Delta\psi}{2}}\rangle \end{aligned} \tag{5.3.40}$$

Let us now assume that the instrument U is doubled up with a reverse instrument that reunites the two beams that have been separated in the first step. This reunification may happen after certain manipulations have been performed on the separated beams. Such a device, the so-called analyzing loop has been used more for the conceptual analysis of the quantum mechanical formalism than for the practical purposes of polarization optics.

Depending on the nature of the manipulations we have a number of distinct situations which we proceed to disentangle on hand of the following formulas.

We obtain from Equation 5.3.40

$$\frac{1}{2}S' = |\hat{s'}\rangle\langle\hat{s'}| = |a_0|^2|\hat{u}\rangle\langle\hat{u}| + |a_1|^2|\bar{\hat{u}}\rangle\langle\bar{\hat{u}}| + a_0 a_1^* \exp\left(-i\Delta\psi\right)|\hat{u}\rangle\langle\bar{\hat{u}}| \tag{5.3.41}$$

Here S' is idempotent and of determinant zero just as S is, since $|\hat{s'}\rangle$ arises out of $|\hat{s}\rangle$ by means of a unitary operation.

Let us consider now a different case in which the phase difference between the two partial beams has been randomized. In fact, take first the extreme case in which the interference terms vanish:

$$\langle a_0 a_1^* \exp\left(-i\Delta\psi\right)\rangle_{av} = \langle a_1 a_0^* \exp\left(i\Delta\psi\right)\rangle_{av} = 0 \tag{5.3.42}$$

We obtain from Equations 5.3.41, 5.3.42 and 5.3.39

$$S' = 1 + \left(|a_0|^2 - |a_1|^2\right)\hat{u}\cdot\vec{\sigma} \tag{5.3.43}$$

We write S' as

$$S' = 1 + s'\hat{u}\cdot\vec{\sigma} \tag{5.3.44}$$

where $0 \leq s' < 1$ and

$$0 < |S| = 1 + s'^2 \leq 1 \tag{5.3.45}$$

We have now a generazized form of the density matrix associated with a partially polarized or even natural light (if $s' = 0$). In quantum mechanics we speak of a mixture of states.

It is usual in optics to change the normalization and set for a partially polarized beam

$$S = s_0 + s\hat{s}\cdot\vec{\sigma} \tag{5.3.46}$$

where s_0 is the total intensity and s the intensity of the polarized component. We have for the determinant

$$0 \leq |S| = s_0^2 - s^2 \leq s_0^2(31) \tag{5.3.47}$$

which is zero for polarized light and positive otherwise.

In addition to conserving or destroying phase relations, one may operate directly on the intensity as well. If one of the components of the analyser, say $|\hat{u}\rangle$ or $|\bar{\hat{u}}\rangle$ is blocked off, the instrument acts as a perfect polarizer.

Formally, we can let the projection operator

$$\frac{1}{2}\left(1 \pm \hat{u}\cdot\vec{\sigma}\right)$$

act on the density matrix of the beam, which may be polarized fully, partially, or not at all. Non-polarized or natural light can be considered as a statistical ensemble of polarized light beams uniformly distributed over the Poincaré sphere. (See problem #15.)

An imperfect polarizer (such as a sheet of polaroid) exhibits an unequal absorption of two conjugate linear polarizations. It can be represented as a Hermitian operator acting on S.

We have seen above that the inchoherent composition of the two beams of an analyzer is accounted, for by the addition of the density matrix.

Conversely, every partially polarized beam can be constructed in such a fashion. (See problem #13.)

Yet we may wish to add incoherently an arbitrary set of partially polarized beams, and this is always accomplished by adding their density matrices.

The question arises then: Could we not operate phenomenologically in terms of the density matrices alone?

The matter was considered already by Stokes (1852) who introduced a column vector with the four components I, M, C, S corresponding to our $s_0, \vec{s}$. A general instrument is represented by a real 4×4 matrix. Note that the "instrument" might be also a molecule producing a change of polarization on scattering.

The 4x 4 matrices are commonly called Mueller matrices[46]. This formalism is usually mentioned along with the Jones calculus of 2×2 complex matrices. This was developed by R. Clark Jones of the Polaroid Co. and his collaborators in a long series of papers in Journal of the American Optical Society in the 1940's (quoted e.g., by Shurcliff, and C. Whitney). This is basically a two-component spinor theory to deal with instruments which modify the polarization without depolarization or loss of intensity. It was developed in close contact with experiment without reliance on an existing mathematical formalism.

Mueller liked to emphasize the purely phenomenological character of his formalism. The four Stokes parameters of a beam can be determined from measurements by four filters [47]. However a difficulty of this phenomenological approach is that not every 4×4 matrix corresponds to a physically realizable instrument or scattering object. This means that a so-called passive instrument must neither increase total intensity nor create phase correlations. The situation is simpler in the 2×2 matrix formulation in which the redundant parameters have been eliminated.

However, we do not enter into such details, since polarization optics is not our primary concern. In fact, the two-valuedness of the full spinor formalism brings about a certain complication which

[46] The formalism was developed by the late Hans Mueller of M.I.T. Unfortunately these results were not adequately published, but Mueller's influence has been considerable through his lectures and his students. The discussion of this section may never have come about without the conversations I had with Hans Mueller in the course of years. For an independent approach see also [Per42].

[47] See [GR60]. Also [Whi71]

is justified by the fact that our main interest is in the applications to quantum mechanics. We shall compare the different types of applications which are available at this juncture in Section 5.5.

Meanwhile in the next section we show that the concept of unitary spinor can be generalized to relativistic situations. This is indispensable if the formalism is to be applied also to the propagation rather than only the polarization of light.

5.4 Relativistic triads and spinors. A preliminary discussion.

We have arrived at the concept of unitary spinors by searching for the proper parametrization of a Euclidean triad. We shall arrive at relativistic spinors by parametrising the *relativistic triad*. This is not a standard term, but it seems appropriate to so designate the configuration $\vec{E}$, $\vec{B}$, $\vec{k}$ (electric and magnetic fields, and the wave vector) in a monochromatic electromagnetic plane wave in vacuum[48].

The propagation of light is a dynamic problem and we are not ready to discuss it within the geometric-kinematic context of this chapter.

The purpose of this section is only to show that the formalism of unitary spinors developed thus far can be extended to relativistic situations with only a few indispensable adjustments[49].

It is a remarkable fact that the mutual orthogonality of the above mentioned vectors is a Lorentz invariant property. However, we have to abandon the unitary normalization since the length of the vectors is affected by inertial transformations.

Accordingly, we set up the relativistic analog of the Equations 5.1.40. We consider first

$$|\xi\rangle\langle\xi| = \frac{1}{2}\left(k_0 + \vec{k}\cdot\vec{\sigma}\right) = \frac{1}{2}K \tag{5.4.1}$$

$$|\bar{\xi}\rangle\langle\bar{\xi}| = \frac{1}{2}\left(k_0 - \vec{k}\cdot\vec{\sigma}\right) = \frac{1}{2}\bar{K} \tag{5.4.2}$$

with the unitary normalization changed to

$$\langle\xi|\xi\rangle = \langle\bar{\xi}|\bar{\xi}\rangle = k_0 \tag{5.4.3}$$

The Lorentz transformation properties of the spinors follow from that of K:

$$|\xi'\rangle = V|\xi\rangle \tag{5.4.4}$$

$$|\bar{\xi}'\rangle = \bar{V}|\bar{\xi}\rangle \tag{5.4.5}$$

$$\langle\xi'| = \langle\xi|V^\dagger \tag{5.4.6}$$

$$\langle\bar{\xi}'| = \langle\bar{\xi}|V^{-1} \tag{5.4.7}$$

[48]One might think of using the Poynting vector $\vec{E}\times\vec{B}$ instead of the wave vector $\vec{k}$. This turns out to be quite impractical.

[49]This is in contrast with the standard method in which the unitary inner product of nonrelativistic quantum mechanics is replaced by an analog to the contraction method of tensor calculus.

If $V = U$ is unitary, we have $\bar{U} = U$, $U^{\dagger} = U^{-1}$.

Let us define a second spinor by

$$|\eta\rangle\langle\eta| = \frac{1}{2}\left(r_0 + \vec{r}\cdot\vec{\sigma}\right) = \frac{1}{2}R \tag{5.4.8}$$

The relativistic invariant, (2.2.3a)[50] appears now as

$$\begin{aligned}\frac{1}{2}Tr\left(R\bar{K}\right) &= \frac{1}{2}Tr\left(|\eta\rangle\langle\eta|\bar{\xi}\rangle\langle\bar{\xi}|\right) \\ &= \langle\bar{\xi}|\eta\rangle\langle\eta|\bar{\xi}\rangle = |\langle\bar{\xi}|\eta\rangle|^2\end{aligned} \tag{5.4.9}$$

It follows from Equations 5.4.4 and 5.4.7 that even the amplitude is invariant

$$\langle\bar{\xi}|\eta\rangle = \text{invariant} \tag{5.4.10}$$

Explicitly it is equal to

$$(-\xi_1, \xi_0)\begin{pmatrix}\eta_0 \\ \eta_1\end{pmatrix} = \xi_0\eta_1 - \xi_1\eta_0 = \begin{vmatrix}\xi_0 & \eta_0 \\ \xi_1 & \eta_1\end{vmatrix} \tag{5.4.11}$$

We turn now to the last two of the Equations 5.1.40 and write by analogy

$$|\xi\rangle\langle\bar{\xi}| \sim \left(\vec{E} + i\vec{B}\right)\cdot\vec{\sigma} = F \tag{5.4.12}$$

$$|\bar{\xi}\rangle\langle\xi| \sim \left(\vec{E} - i\vec{B}\right)\cdot\vec{\sigma} = -\bar{F} = F^{\dagger} \tag{5.4.13}$$

We see that the field quantities have, in view of Equations 5.4.4–5.4.7 the correct transformation properties.

The occurrence of the same spinor in Equations 5.4.1, 5.4.2, 5.4.12 and 5.4.13 ensures the expected orthogonality properties of the triad.

However, in Equations 5.4.12 and 5.4.13 we write proportionality instead of equality, because we have to admit a different normalization for the four-vector and the six-vector respectively. We are not ready to discuss the matter at this point.

If in Equation 5.4.10 we choose the two spinors to be identical, the invariant vanishes:

$$\langle\bar{\xi}|\xi\rangle = 0 \tag{5.4.14}$$

[50] need to determine the correct equation reference – Editor.

The same is true of the invariant of the electromagnetic field:

$$\begin{aligned}\frac{1}{2}Tr(F\tilde{F}) &= -\frac{1}{2}TrF^2 \simeq -\frac{1}{2}Tr\left(|\xi\rangle\langle\bar{\xi}|\xi\rangle\langle\bar{\xi}|\right) \\ &= -\left(\langle\bar{\xi}|\xi\rangle\right)^2 = 0 \end{aligned} \tag{5.4.15}$$

Thus from a single spinor we can build up only constructs corresponding toa plane wave. We do not enter here into the discussion of more complicated situations and note only that we cannot use the device of taking linear combination of conjugate spinors in the usual form $a_0|\xi\rangle + a_1|\bar{\xi}\rangle$, because the two terms have contragradient Lorentz transformation properties. We write them, displaying their Lorentz transformations as

$$\begin{pmatrix} |\xi'\rangle \\ |\bar{\xi}'\rangle \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} |\xi\rangle \\ |\bar{\xi}\rangle \end{pmatrix} \tag{5.4.16}$$

We have arrived at spinors of the Dirac type. We return to their discussion later.

Let us conclude this section by considering the relation of the formalism to the standard formalism of van der Waerden. (See e.g., [MTW73])

The point of departure is Equation 5.4.4 applied to two spinors yielding the determinantal invariant of 5.4.11. The first characteristic aspect of the theory is the rule for raising the indices:

$$\eta_1 = \eta^0, \quad \eta_0 = -\eta^1 \tag{5.4.17}$$

Hence the invariant appears as

$$\xi_0\eta^0 + \xi_1\eta^1 \tag{5.4.18}$$

The motivation for writing the invariant in this form is to harmonize the presentation with the standard tensor formalism. In contrast, our expression 5.4.10, is an extension of the bra-ket formalism of nonrelativistic quantum mechanics, that is also quite natural for the linear algebra of complex vector spaces.

A second distinctive feature is connected with the method of complexification. Van der Waerden takes the complex conjugate of the matrix V by taking the complex conjugate of its elements, whereas we deal with the Hermitian conjugate $V^\dagger$ and the complex reflection $\bar{V}$.

From the practical point of view we tend to develop unitary and relativistic spinors in as united a form as objectively possible.

5.5 Review of SU(2) and preview of quantization.

Our introduction of the spinor concept at the beginning of Section 5.1 can be rationalized on the basis of the following guidelines. First, we require a certain economy and wish to avoid dealing with redundant parameters in specifying a rotating triad, just as we have previously solved the analogous problem for the rotation operator[51].

Second, we wish to have an efficient formalism to represent the rotational problem.

We have seen that the matrices of $\mathcal{SU}(2)$ satisfy all these requirements, but we find ourselves saddled with a two-valuedness of the representation: $|\xi\rangle$ and $|\xi\rangle$ correspond to the same triad configuration. This is not a serious trouble, since our key relations 5.1.36 and 5.1.37 are quadratic in $|\xi\rangle$ [52].

Thus the two-valuedness apperas here only as a computational aid that disappears in the final result.

The situation is different if we look at the formulas 5.1.42–5.1.66 of the same section (Section 5.1). These equations are linear, they have a quantum mechanical character and we know that they are indeed applicable in the proper context. It is a pragmatic fact the two-valuedness is not just a necessary nuisance, but has physical meaning. But to understand this meaning is a challenge which we can meet only in carefully chosen steps.

We wish to give a more physical interpretation to the triad, but avoid the impasse of the rigid body. First we associate the abstract Poincaré space with the physical system of the two-dimensional degenerate oscillator. The rotation in the Poincaré space is associated with a phase shift between conjugate states, which is translated into a rotation of the Poincaré sphere, interpreted in turn as a *change in orientation*, or *change of shape* of the vibrational patterns in ordinary space.

It is only a mild exaggeration to say that our transition from the triad in Euclidean space to that in Poincaré space is something like a "quantization," in the sense as Schrödinger's wave equation associates a wave with a particle. (Planck's h is to enter shortly!)

In this theory we have a good use for the conjugate spinors $|\xi\rangle$ and $|\bar{\xi}\rangle$ representing opposite polarizations, but we must identify $|\bar{\bar{\xi}}\rangle = -|\xi\rangle$ with $|\xi\rangle$[53].

The foregoing is still nothing but a perfectly well-defined kinematic model. The next step is different. As a "second quantization" we introduce Planck's h to define single photons[54]. A beam splitting represented by a projection operator can be expressed in probabilistic terms.

Formally all this is easy and we would at once have a great deal of the quantum mechanical formalism involving the theory of measurement.

Next, we could take the two-valuedness of the spinor seriously and obtain the formalism of isospin and of the neutrino, say as in Section 17, Fermion States, in [Kae65].

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[51] These two problems have been posed and solved by Euler in 1776! See [Whi64], page 8-12.

[52] Some caution is needed in the practical application of Equation 5.2.8 in Section 5.2; however, this is only a sideline in the present context.

[53] I failed to observe this rule in formulating the instructions for #11 in the Fourth Assignment. The result is a spurious phase in $\langle x|R\rangle$! However all this trouble is easily avoided.

[54] If you want to call it the "first quantization," I will not argue the point.

Finally, instead of doubly degenerate vibrations, we could consider the triply degenerate vibrator and handle it by $SU(3)$ [Lip02]. We shall not consider these generalizations at this point, however. Before further expanding the formalism we should hope to understand better what we already have.

First, a formal remark. Our results thus far developed are uniquely determined by the spinor formalism of Section 5.1 and by the program of considering the Poincaré sphere as the basic configuration space to be described by conventional spherical coordinates α, β or ϕ, θ [55].

It is noteworthy that this modest conceptual equipment carries us so far. We have obtained spinors, density matrices and have discussed at least fleetingly coherence, incoherence, quantum theory of measurement and transformation theory.

What we do not get out of the theory is a specific interpretation of the underlying vibrational process since the formalism is thus far entirely independent of it. This fact gives us some understanding of the scope and limit of quantum mechanics. We can apply the formalism to phenomena we understand very little. However, since the same $SU(2)$ formalism applies to polarized light, spin, isospin, strangeness and other phenomena, we learn little about their distinctive aspects.

In order to overcome this limitation we need a deeper understanding of what a quantized angular momentum is in the framework of a dynamical problem.

The next chapter is devoted to a phenomenological discussion of the concepts of particle and wave. We shall attempt to obtain sufficient hints for developing a dynamic theory in the form of a phase space geometry in Chapter VI[56].

[55]These angles should be replaced by $2\alpha, \pi/2-2\beta$, ϕ and 2θ respectively in order to arrive at the angles conventional in polarization optics.

[56]This chapter was not included in the Spring 1976 notes - Editor.

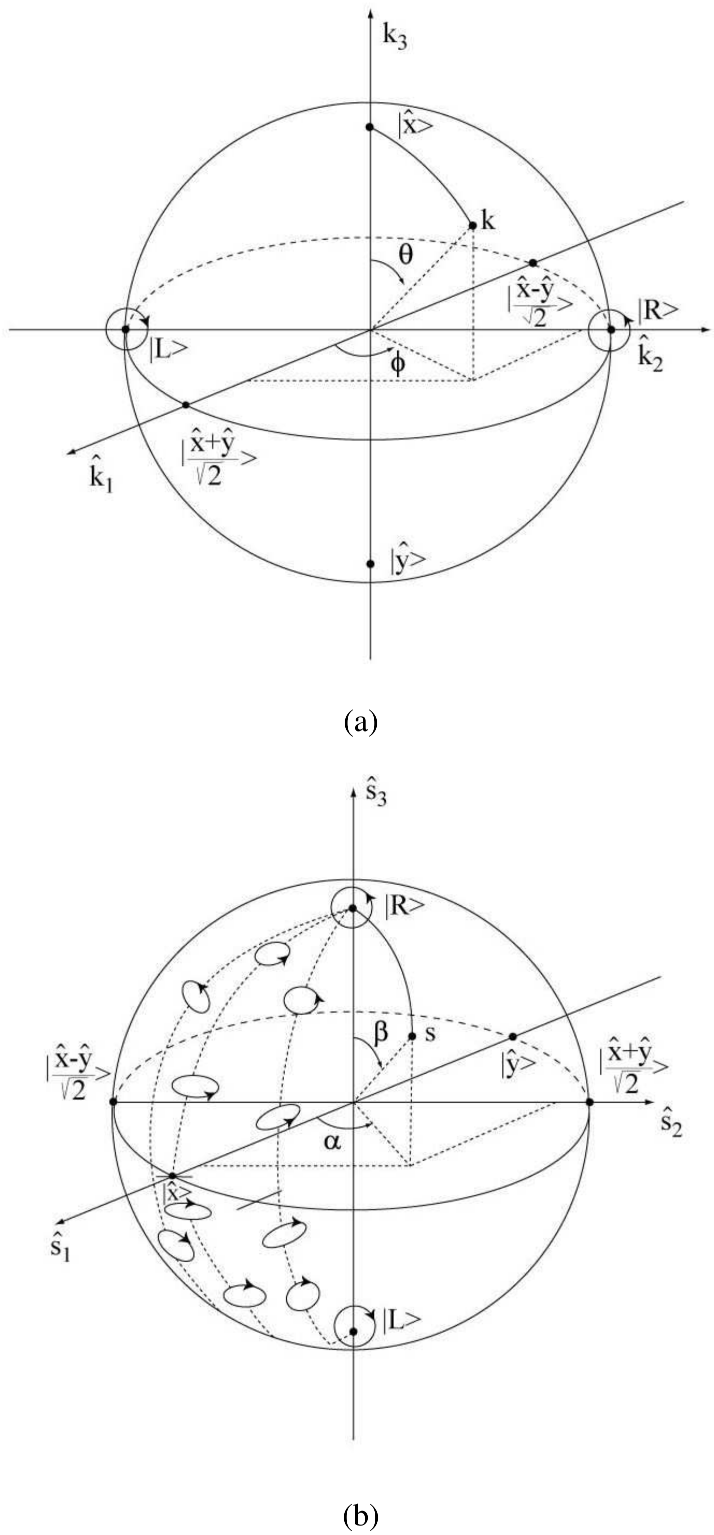
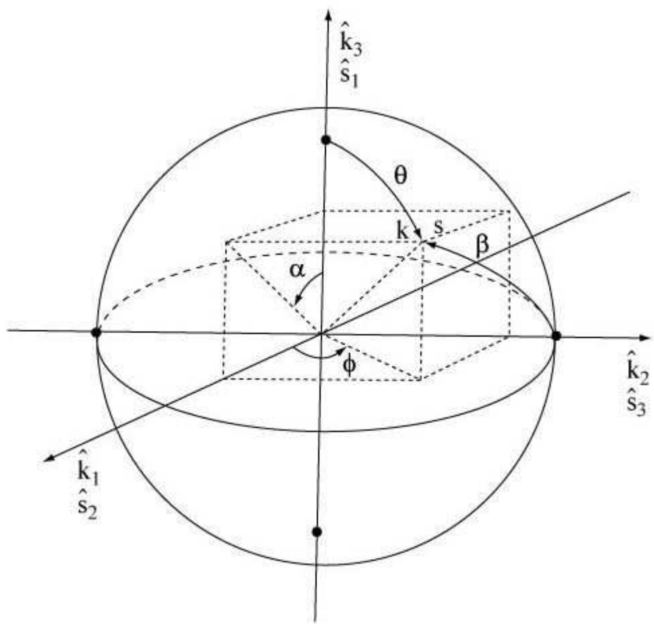
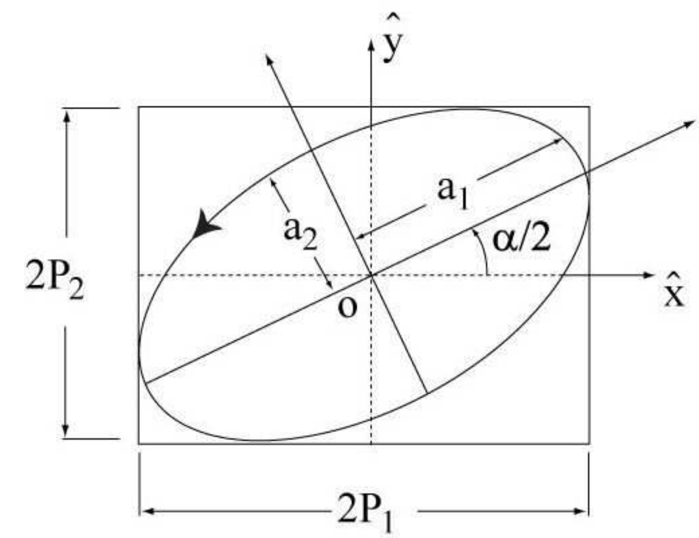


Figure 5.3: Representation of Polarization in the Poincaré Sphere. Connection between the schemes: (a) $\hat{k}(\phi, \theta)$ scheme and (b) $\hat{s}(\alpha, \beta)$ scheme.



(c)



(d)

Figure 5.4: Representation of Polarization in the Poincaré Sphere. Connection between the schemes (c) and (d).