We now continue our discussion of infinite series.

Sometimes we have to deal with series where the terms being added are indexed by multiple indices, as in this example here.

We're given numbers, aij, and i ranges over all the positive integers.

j also ranges over all the positive integers.

So what does this sum represent?

We can think of it as follows.

We have here a two-dimensional grid that corresponds to all the pairs (i,j).

And in essence, each one of those points corresponds to one of the terms that we want to add.

So we can sum the different terms in some arbitrary order.

Let's say we start from here.

Take that term, add this term, then add this term here, then add this term, then the next term, next term, and so on.

And we can keep going that way, adding the different terms according to some sequence.

So essentially, what we're doing here is we're taking this two-dimensional grid and arranging the terms associated with that grid, in some particular linear order.

And we're summing those terms in sequence.

As long as this sum converges to something as we keep adding more and more terms, then this double series will be well defined.

Notice, however, we can add those terms in many different orders.

And in principle, those different orders might give us different kinds of results.

On the other hand, as long as the sum of the absolute values of all the terms turns out to be finite, then the particular order in which we're adding the different terms will turn out that it doesn't matter.

There's another way that we can add the terms together, and this is the following.

Let us consider fixing a particular choice of i, and adding all of the terms that are associated with this particular choice of i, as j ranges from 1 to infinity.

So what we're doing is we're taking the summation from j equal to 1 to infinity, while keeping the value of i fixed.

We do this for every possible i.

So for every possible i, we're going to get a particular number.

And then we take the numbers that we obtain for the different choices if i, so i ranges from 1 to infinity.

And we add all those terms together.

So this is one particular order, one particular way of doing the infinite summation.

Now, why start with the summation over j's while keeping i fixed?

There's no reason for that.

We could also carry out the summation by fixing a particular choice of j and summing over all i's.

So now it is i that ranges from 1 to infinity.

And we look at this infinite sum.

This is the infinite sum of those terms.

We obtain one such infinite sum for every choice of j.

And then we take that sum that we obtain for any particular choice of j, and add over the different possible values of j.

So j goes from 1 to infinity.

This is a different way of carrying out the summation.

And these are going to give us the same result, and the same result that we would also obtain if we were to add the terms in this particular order, as long as the double series is well-defined, in the following sense.

If we have a guarantee that the sum of the absolute values of those numbers is finite, no matter which way we add them, then it turns out that we can use any particular order to add the terms in the series.

We're going to get the same result.

And we can also carry out the double summation by doing-- by adding over one index at a time.

A word of caution-- this condition is not always satisfied.

And in those cases, strange things can happen.

Suppose that the sequences we're dealing with, the aij's, take those particular values indicated in this picture.

And all the remaining terms, the aij's associated with the other dots, are all 0's.

So all these terms out there will be 0's.

If we carry out the summation by fixing a j and adding over all i's, what we get here is 0, and a 0, and a 0, and a 0.

That's because in each row we have a 1 and a minus 1, which cancel out and give us 0's.

So if we carry out the summation in this manner, we get a sum of 0's, which is 0.

But if we carry out the summation in this order, fix an i, and then add over all j's, the first term that we get here is going to be 1, because in this column, this is the only non-zero number.

And then in the remaining columns, as we add the terms, we're going to get 0's, and 0's, and so on.

And so if we carry out the summation in this way, we obtain a 1.

So this is an example that shows you that the order of summation actually may matter.

In this example, the sum of the absolute values of all of the terms that are involved is infinity, because we have infinitely many plus or minus 1's, so this condition here is not satisfied in this example.

Let us now consider the case where we want to add the terms of a double sequence, but over a limited range of indices as in this example, where we have coefficients aij, which we want to add, but only for those i's and j's for which j is less than or equal to i.

Graphically, this means that we only want to consider the pairs shown in this picture.

So these points here correspond to i,j pairs for which i is equal to j.

Terms on the right, or points to the right, correspond to i,j pairs for which i is at least as large as j.

We can carry out this summation in two ways.

One way is the following.

We fix a value of i, and we consider all of the corresponding terms, that correspond to different choices of j.

But we only go up to the point where i is equal to j.

This is the largest term.

So what are we doing here?

We're taking the coefficients aij, and we are adding over all j's, starting from 1, which corresponds to this term.

And j goes up to the point where it becomes equal to i.

We do this for every value of i.

And so we get a number for the sum of each one of the columns, and then we add those numbers together.

So we're adding over all i's, and i ranges from 1 up to infinity.

This is one way of carrying out the summation.

Alternatively, we could fix a value of j, and consider doing the summation over all choices of i.

So this corresponds to the sum over all choices of i, from where?

The smallest term, the first term, happens when i is equal to the value of j.

And then we have larger choices of i, numbers for which i is bigger than the corresponding value of j.

And so i ranges from j all the way to infinity.

And this is the sum over one of the rows in this diagram.

We do this for every j.

We get a result, and then we need to add all those results together.

So we're summing for all j's from 1 up to infinity.

So these are two different ways that we can evaluate this series associated with a double sequence.

We can either add over all j's first and then over i's, or we can sum over all i's first, and then over all j's.

The two ways of approaching this problem, this summation, should give us the same answer.

And this is going to be, again, subject to the usual qualification.

As long as the sum of the absolute values of the terms that we're trying to add is less than infinity-- if this condition is true, then the two ways of carrying out the summation are equal, and they're just two different alternatives.