CHAPTER 11

Vectors and Matrices

This chapter opens up a new part of calculus. It is *multidimensional calculus*, because the subject moves into more dimensions. In the first ten chapters, all functions depended on time t or position x—but not both. We had $f(t)$ or $y(x)$. The graphs were curves in a plane. There was one independent variable (x or t) and one dependent variable (y or f). Now we meet functions $f(x, t)$ that depend on both x and t. Their graphs are ***surfaces*** instead of curves. This brings us to the ***calculus of several variables***.

Start with the surface that represents the function $f(x, t)$ or $f(x, y)$ or $f(x, y, t)$. I emphasize functions, because that is what calculus is about.

EXAMPLE 1 $f(x, t) = \cos(x - t)$ is a traveling wave (cosine curve in motion).

At $t = 0$ the curve is $f = \cos x$. At a later time, the curve moves to the right (Figure 11.1). At each t we get a cross-section of the whole x-t surface. For a wave traveling along a string, the height depends on position as well as time.

A similar function gives a wave going around a stadium. Each person stands up and sits down. Somehow the wave travels.

EXAMPLE 2 $f(x, y) = 3x + y + 1$ is a sloping roof (fixed in time).

The surface is two-dimensional—you can walk around on it. It is flat because $3x + y + 1$ is a linear function. In the y direction the surface goes up at 45°. If y increases by 1, so does f. That slope is 1. In the x direction the roof is steeper (slope 3). There is a direction in between where the roof is steepest (slope $\sqrt{10}$).

EXAMPLE 3 $f(x, y, t) = \cos(x - y - t)$ is an ocean surface with traveling waves.

This surface moves. At each time t we have a new x-y surface. There are three variables, x and y for position and t for time. I can't draw the function, it needs four dimensions! The base coordinates are x, y, t and the height is f. The alternative is a movie that shows the x-y surface changing with t.

At time $t = 0$ the ocean surface is given by $\cos(x - y)$. The waves are in straight lines. The line $x - y = 0$ follows a crest because $\cos 0 = 1$. The top of the next wave is on the parallel line $x - y = 2\pi$, because $\cos 2\pi = 1$. Figure 11.1 shows the ocean surface at a fixed time.

The line $x - y = t$ gives the crest at time t. The water goes up and down (like people in a stadium). ***The wave goes to shore, but the water stays in the ocean.***

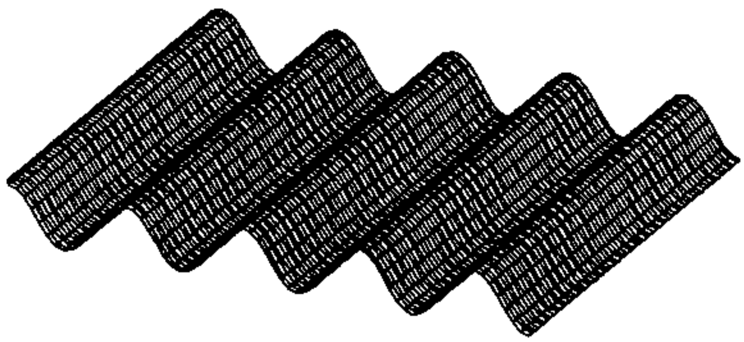


Fig. 11.1 Moving cosine with a small optical illusion—the darker bands seem to go from top to bottom as you turn.

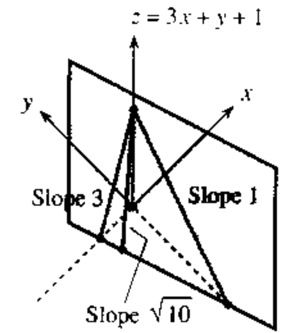


Fig. 11.2 Linear functions give planes.

Of course multidimensional calculus is not only for waves. In business, demand is a function of price and date. In engineering, the velocity and temperature depend on position x and time t. Biology deals with many variables at once (and statistics is always looking for linear relations like $z = x + 2y$). A serious job lies ahead, to carry derivatives and integrals into more dimensions.

11.1 Vectors and Dot Products

In a plane, every point is described by two numbers. We measure across by x and up by y. Starting from the origin we reach the point with coordinates (x, y). I want to describe this movement by a ***vector***—the straight line that starts at $(0, 0)$ and ends at (x, y). This vector $\mathbf{v}$ has a ***direction***, which goes from $(0, 0)$ to (x, y) and not the other way.

In a picture, the vector is shown by an arrow. In algebra, $\mathbf{v}$ is given by its two components. For a *column vector*, write x above y:

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \quad (x \text{ and } y \text{ are the components of } \mathbf{v}). \tag{1}$$

Note that $\mathbf{v}$ is printed in boldface; its components x and y are in lightface.† The vector $-\mathbf{v}$ in the opposite direction changes signs. Adding $\mathbf{v}$ to $-\mathbf{v}$ gives the ***zero vector*** (different from the zero number and also in boldface):

$$-\mathbf{v} = \begin{bmatrix} -x \\ -y \end{bmatrix} \quad \text{and} \quad \mathbf{v} - \mathbf{v} = \begin{bmatrix} x - x \\ y - y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \mathbf{0}. \tag{2}$$

Notice how vector addition or subtraction is done separately on the x's and y's:

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \tag{3}$$

†Another way to indicate a vector is $\vec{v}$. You will recognize vectors without needing arrows.

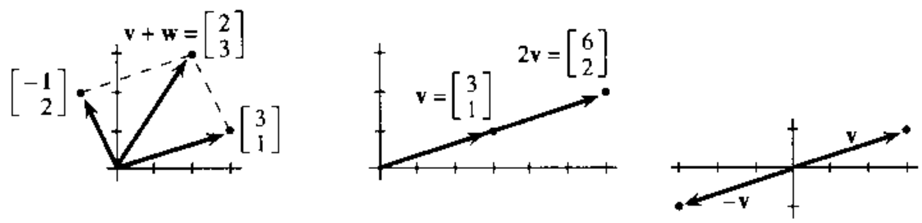


Fig. 11.3 Parallelogram for $\mathbf{v}+\mathbf{w}$, stretching for $2\mathbf{v}$, signs reversed for $-\mathbf{v}$.

The vector $\mathbf{v}$ has components $v_1 = 3$ and $v_2 = 1$. (I write v_1 for the first component and v_2 for the second component. I also write x and y, which is fine for two components.) The vector $\mathbf{w}$ has $w_1 = -1$ and $w_2 = 2$. To add the vectors, add the components. ***To draw this addition, place the start of* w *at the end of* v**. Figure 11.3 shows how $\mathbf{w}$ starts where $\mathbf{v}$ ends.

VECTORS WITHOUT COORDINATES

In that *head-to-tail addition* of $\mathbf{v} + \mathbf{w}$, we did something new. The vector $\mathbf{w}$ was moved away from the origin. Its length and direction were not changed! The new arrow is parallel to the old arrow—only the starting point is different. *The vector is the same as before.*

A vector can be defined without an origin and without x and y axes. The purpose of axes is to give the components—the separate distances x and y. Those numbers are necessary for calculations. But x and y coordinates are not necessary for head-to-tail addition $\mathbf{v}+\mathbf{w}$, or for stretching to $2\mathbf{v}$, or for linear combinations $2\mathbf{v}+3\mathbf{w}$. Some applications depend on coordinates, others don't.

Generally speaking, physics works without axes—it is "coordinate-free." A ***velocity*** has direction and magnitude, but it is not tied to a point. A ***force*** also has direction and magnitude, but it can act anywhere—not only at the origin. In contrast, a vector that gives the prices of five stocks is not floating in space. Each component has a meaning—there are five axes, and we know when prices are zero. After examples from geometry and physics (no axes), we return to vectors *with* coordinates.

EXAMPLE 1 (Geometry) Take any four-sided figure in space. Connect the midpoints of the four straight sides. ***Remarkable fact: Those four midpoints lie in the same plane.*** More than that, they form a *parallelogram*.

Frankly, this is amazing. Figure 11.4a cannot do justice to the problem, because it is printed on a flat page. Imagine the vectors $\mathbf{A}$ and $\mathbf{D}$ coming upward. $\mathbf{B}$ and $\mathbf{C}$ go down at different angles. Notice how easily we indicate the four sides as vectors, not caring about axes or origin.

I will prove that $\mathbf{V} = \mathbf{W}$. That shows that the midpoints form a parallelogram.

What is $\mathbf{V}$? It starts halfway along $\mathbf{A}$ and ends halfway along $\mathbf{B}$. The small triangle at the bottom shows $\mathbf{V} = \frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}$. This is vector addition—the tail of $\frac{1}{2}\mathbf{B}$ is at the head of $\frac{1}{2}\mathbf{A}$. Together they equal the shortcut $\mathbf{V}$. For the same reason $\mathbf{W} = \frac{1}{2}\mathbf{C} + \frac{1}{2}\mathbf{D}$. The heart of the proof is to see these relationships.

One step is left. Why is $\frac{1}{2}\mathbf{A} + \frac{1}{2}\mathbf{B}$ equal to $\frac{1}{2}\mathbf{C} + \frac{1}{2}\mathbf{D}$? In other words, why is $\mathbf{A} + \mathbf{B}$ equal to $\mathbf{C} + \mathbf{D}$? (I multiplied by 2.) When the right question is asked, the answer jumps out. A head-to-tail addition $\mathbf{A} + \mathbf{B}$ brings us to the point R. ***Also* $\mathbf{C} + \mathbf{D}$ *brings us to* R.** The proof comes down to one line:

$$\mathbf{A} + \mathbf{B} = PR = \mathbf{C} + \mathbf{D}. \text{ Then } \mathbf{V} = \tfrac{1}{2}\mathbf{A} + \tfrac{1}{2}\mathbf{B} \text{ equals } \mathbf{W} = \tfrac{1}{2}\mathbf{C} + \tfrac{1}{2}\mathbf{D}.$$

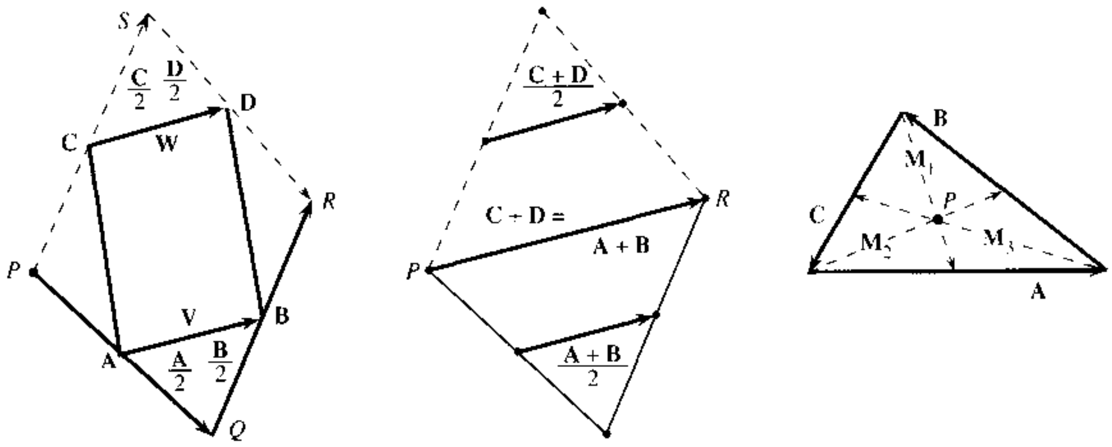


Fig. 11.4 Four midpoints form a parallelogram ($\mathbf{V} = \mathbf{W}$). Three medians meet at P.

EXAMPLE 2 (Also geometry) In any triangle, draw lines from the corners to the midpoints of the opposite sides. To prove by vectors: ***Those three lines meet at a point.*** Problem 38 finds the meeting point in Figure 11.4c. Problem 37 says that ***the three vectors add to zero.***

EXAMPLE 3 (Medicine) An electrocardiogram shows the sum of many small vectors, the voltages in the wall of the heart. What happens to this sum—the ***heart vector*** **V**—in two cases that a cardiologist is watching for?

Case **1**. Part of the heart is dead (***infarction***).
Case **2**. Part of the heart is abnormally thick (***hypertrophy***).

A heart attack kills part of the muscle. A defective valve, or hypertension, overworks it. In **case 1** the cells die from the cutoff of blood (loss of oxygen). In **case 2** the heart wall can triple in size, from excess pressure. The causes can be chemical or mechanical. The effect we see is electrical.

The machine is adding small vectors and "projecting" them in twelve directions. The leads on the arms, left leg, and chest give twelve directions in the body. Each graph shows the component of **V** in one of those directions. Three of the projections two in the vertical plane, plus lead 2 for front-back—produce the "mean QRS vector" in Figure 11.5. That is the sum **V** when the ventricles start to contract. The left ventricle is larger, so the heart vector normally points down and to the left.

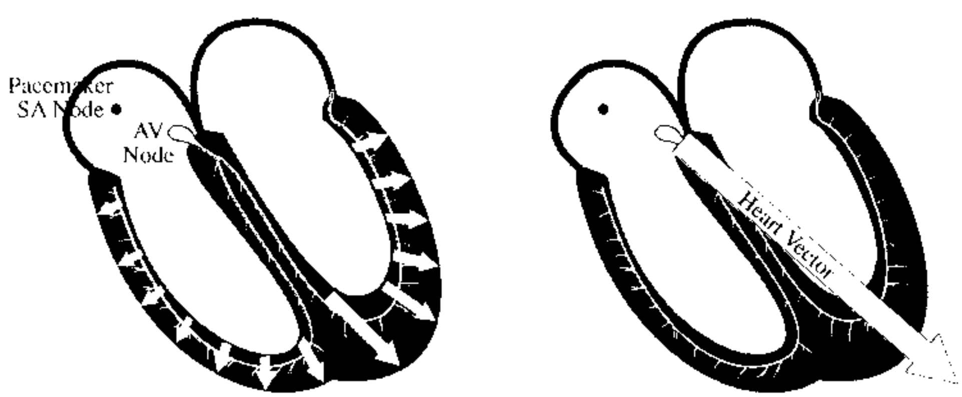


Fig. 11.5 **V** is a sum of small voltage vectors, at the moment of depolarization.

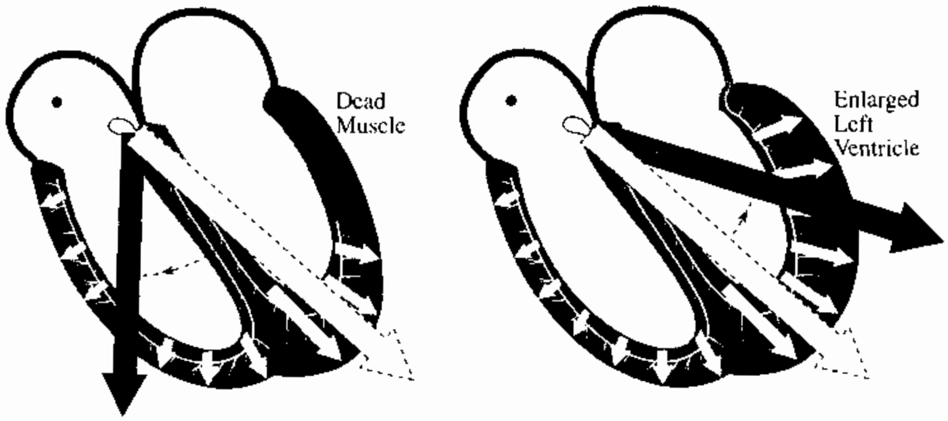


Fig. 11.6 Changes in **V** show dead muscle and overworked muscle.

We come soon to projections, but here the question is about **V** itself. How does the ECG identify the problem?

> ***Case* 1**: ***Heart attack*** The dead cells make no contribution to the electrical potential. Some small vectors are missing. Therefore the sum **V** turns ***away*** from the infarcted part.
> ***Case* 2**: ***Hypertrophy*** The overwork increases the contribution to the potential. Some vectors are larger than normal. Therefore **V** turns ***toward*** the thickened part.

When **V** points in an abnormal direction, the ECG graphs locate the problem. The P, Q, R, S, T waves on separate graphs can all indicate hypertrophy, in different regions of the heart. Infarctions generally occur in the left ventricle, which needs the greatest blood supply. When the supply of oxygen is cut back, that ventricle feels it first. The result can be a heart attack (= myocardial infarction = coronary occlusion). Section 11.2 shows how the projections on the ECG point to the location.

First come the basic facts about vectors—components, lengths, and dot products.

COORDINATE VECTORS AND LENGTH

To compute with vectors we need axes and coordinates. The picture of the heart is "coordinate-free," but calculations require numbers. A vector is known by its components. ***The unit vectors along the axes are* i *and* j *in the plane and* i, j, k *in space***:

$$\textbf{in 2D:}\quad \mathbf{i}=\begin{bmatrix}1\\0\end{bmatrix},\ \mathbf{j}=\begin{bmatrix}0\\1\end{bmatrix}\qquad \textbf{in 3D:}\quad \mathbf{i}=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{j}=\begin{bmatrix}0\\1\\0\end{bmatrix},\ \mathbf{k}=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Notice how easily we moved into three dimensions! The only change is that vectors have three components. The combinations of **i** and **j** (or **i**, **j**, **k**) produce all vectors **v** in the plane (and all vectors **V** in space):

$$\mathbf{v}=3\mathbf{i}+\mathbf{j}=\begin{bmatrix}3\\1\end{bmatrix}\qquad \mathbf{V}=\mathbf{i}+2\mathbf{j}-2\mathbf{k}=\begin{bmatrix}1\\2\\-2\end{bmatrix}.$$

Those vectors are also written $\mathbf{v} = (3, 1)$ *and* $\mathbf{V} = (1, 2, -2)$. The components of the vector are also the coordinates of a point. (The vector goes from the origin to the point.) This relation between point and vector is so close that we allow them the same notation: $P = (x, y, z)$ and $\mathbf{v} = (x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

The sum $\mathbf{v} + \mathbf{V}$ is totally meaningless. Those vectors live in different dimensions.

From the components we find the ***length***. The length of $(3, 1)$ is $\sqrt{3^2 + 1^2} = \sqrt{10}$. This comes directly from a right triangle. In three dimensions, $\mathbf{V}$ has a third component to be squared and added. The length of $\mathbf{V} = (x, y, z)$ is $|\mathbf{V}| = \sqrt{x^2 + y^2 + z^2}$.

Vertical bars indicate length, which takes the place of absolute value. The length of $\mathbf{v} = 3\mathbf{i} + \mathbf{j}$ is the distance from the point $(0, 0)$ to the point $(3, 1)$:

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2} = \sqrt{10} \qquad |\mathbf{V}| = \sqrt{1^2 + 2^2 + (-2)^2} = 3.$$

A unit vector is a vector of length one. Dividing $\mathbf{v}$ and $\mathbf{V}$ by their lengths produces unit vectors in the same directions:

$$\frac{\mathbf{v}}{|\mathbf{v}|} = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \quad \text{and} \quad \frac{\mathbf{V}}{|\mathbf{V}|} = \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix} \quad \text{are unit vectors.}$$

11A Each nonzero vector has a positive length $|\mathbf{v}|$. The direction of $\mathbf{v}$ is given by a unit vector $\mathbf{u} = \mathbf{v}/|\mathbf{v}|$. Then length times direction equals $\mathbf{v}$.

A unit vector in the plane is determined by its angle θ with the x axis:

$$\mathbf{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \text{ is a unit vector: } |\mathbf{u}|^2 = \cos^2\theta + \sin^2\theta = 1.$$

In 3-space the components of a unit vector are its "direction cosines":

$$\mathbf{U} = (\cos\alpha)\mathbf{i} + (\cos\beta)\mathbf{j} + (\cos\gamma)\mathbf{k}: \quad \alpha, \beta, \gamma = \text{angles with } x, y, z \text{ axes.}$$

Then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. We are doing algebra with numbers while we are doing geometry with vectors. It was the great contribution of Descartes to see how to study algebra and geometry at the same time.

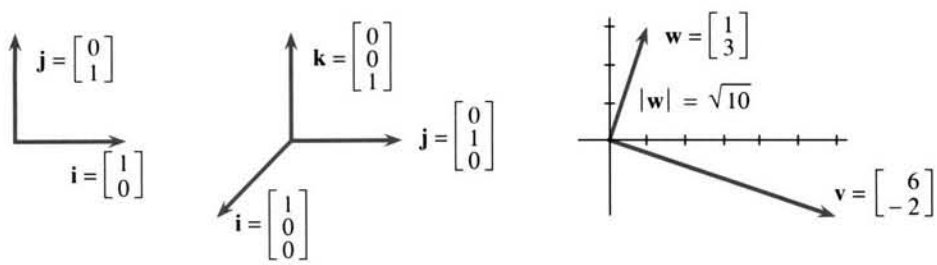


Fig. 11.7 Coordinate vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$. Perpendicular vectors $\mathbf{v} \cdot \mathbf{w} = (6)(1) + (-2)(3) = 0$.

THE DOT PRODUCT OF TWO VECTORS

There are two basic operations on vectors. First, vectors are added $(\mathbf{v} + \mathbf{w})$. Second, a vector is multiplied by a scalar ($7\mathbf{v}$ or $-2\mathbf{w}$). That leaves a natural question—how do you multiply two vectors? The main part of the answer is—you don't. But there is an extremely important operation that begins with two vectors and produces a number. It is usually indicated by a dot between the vectors, as in $\mathbf{v} \cdot \mathbf{w}$, so it is called the ***dot product***.

DEFINITION 1 *The dot product multiplies the lengths* $|\mathbf{v}|$ *times* $|\mathbf{w}|$ *times a cosine*:

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta, \quad \theta = \text{angle between } \mathbf{v} \text{ and } \mathbf{w}.$$

EXAMPLE $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ has length 3, $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ has length $\sqrt{8}$, the angle is 45°.

The dot product is $|\mathbf{v}|\,|\mathbf{w}| \cos\theta = (3)(\sqrt{8})(1/\sqrt{2})$, which simplifies to 6. The square roots in the lengths are "canceled" by square roots in the cosine. For computing $\mathbf{v} \cdot \mathbf{w}$, a second and much simpler way involves no square roots in the first place.

DEFINITION 2 *The dot product* $\mathbf{v} \cdot \mathbf{w}$ *multiplies component by component and adds*:

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 \qquad \begin{bmatrix} 3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = (3)(2) + (0)(2) = 6.$$

The first form $|\mathbf{v}|\,|\mathbf{w}| \cos\theta$ is coordinate-free. The second form $v_1 w_1 + v_2 w_2$ computes with coordinates. Remark 4 explains why these two forms are equal.

> **11B** The ***dot product*** or ***scalar product*** or ***inner product*** of three-dimensional vectors is
>
> $$\mathbf{V} \cdot \mathbf{W} = |\mathbf{V}|\,|\mathbf{W}| \cos\theta = V_1 W_1 + V_2 W_2 + V_3 W_3. \tag{4}$$
>
> If the vectors are perpendicular then $\theta = 90^\circ$ and $\cos\theta = 0$ and $\mathbf{V} \cdot \mathbf{W} = 0$.

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 32\ (\textit{not perpendicular}) \qquad \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 0\ (\textit{perpendicular}).$$

These dot products 32 and 0 equal $|\mathbf{V}|\,|\mathbf{W}| \cos\theta$. In the second one, $\cos\theta$ must be zero. The angle is $\pi/2$ or $-\pi/2$—in either case a right angle. Fortunately the cosine is the same for θ and $-\theta$, so we need not decide the sign of θ.

Remark* 1** When $\mathbf{V} = \mathbf{W}$ the angle is zero but not the cosine! In this case $\cos\theta = 1$ and $\mathbf{V} \cdot \mathbf{V} = |\mathbf{V}|^2$. ***The dot product of* V *with itself is the length squared:

$$\mathbf{V} \cdot \mathbf{V} = (V_1, V_2, V_3) \cdot (V_1, V_2, V_3) = V_1^2 + V_2^2 + V_3^2 = |\mathbf{V}|^2. \tag{5}$$

***Remark* 2** The dot product of $\mathbf{i} = (1, 0, 0)$ with $\mathbf{j} = (0, 1, 0)$ is $\mathbf{i} \cdot \mathbf{j} = 0$. The axes are perpendicular. Similarly $\mathbf{i} \cdot \mathbf{k} = 0$ and $\mathbf{j} \cdot \mathbf{k} = 0$. Those are unit vectors: $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$.

***Remark* 3** The dot product has three properties that keep the algebra simple:

1. $\mathbf{V} \cdot \mathbf{W} = \mathbf{W} \cdot \mathbf{V}$ 2. $(c\mathbf{V}) \cdot \mathbf{W} = c(\mathbf{V} \cdot \mathbf{W})$ 3. $(\mathbf{U} + \mathbf{V}) \cdot \mathbf{W} = \mathbf{U} \cdot \mathbf{W} + \mathbf{V} \cdot \mathbf{W}$

When $\mathbf{V}$ is doubled ($c = 2$) the dot product is doubled. When $\mathbf{V}$ is split into $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components, the dot product splits in three pieces. The same applies to $\mathbf{W}$, since

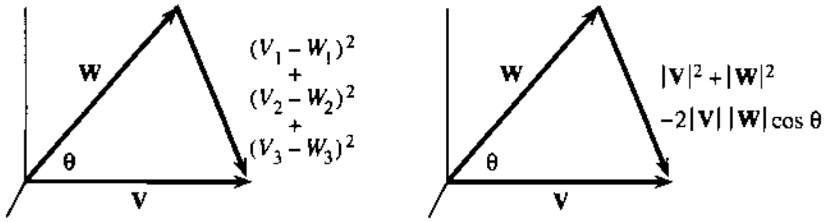


Fig. 11.8 Length squared $= (\mathbf{V}-\mathbf{W})\cdot(\mathbf{V}-\mathbf{W})$, from coordinates and the cosine law.

$\mathbf{V}\cdot\mathbf{W}=\mathbf{W}\cdot\mathbf{V}$. The nine dot products of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are zeros and ones, and a giant splitting of both $\mathbf{V}$ and $\mathbf{W}$ gives back the correct $\mathbf{V}\cdot\mathbf{W}$:

$$\mathbf{V}\cdot\mathbf{W}=V_1\mathbf{i}\cdot W_1\mathbf{i}+V_2\mathbf{j}\cdot W_2\mathbf{j}+V_3\mathbf{k}\cdot W_3\mathbf{k}+\text{six zeros}=V_1W_1+V_2W_2+V_3W_3.$$

Remark 4 ***The two forms of the dot product are equal.*** This comes from computing $|\mathbf{V}-\mathbf{W}|^2$ by coordinates and also by the "law of cosines":

$$\text{with coordinates: } |\mathbf{V}-\mathbf{W}|^2=(V_1-W_1)^2+(V_2-W_2)^2+(V_3-W_3)^2$$
$$\text{from cosine law: } |\mathbf{V}-\mathbf{W}|^2=|\mathbf{V}|^2+|\mathbf{W}|^2-2|\mathbf{V}|\,|\mathbf{W}|\cos\theta.$$

Compare those two lines. Line 1 contains V_1^2 and V_2^2 and V_3^2. Their sum matches $|\mathbf{V}|^2$ in the cosine law. Also $W_1^2+W_2^2+W_3^2$ matches $|\mathbf{W}|^2$. Therefore the terms containing -2 are the same (you can mentally cancel the -2). *The definitions agree*:

$$-2(V_1W_1+V_2W_2+V_3W_3)\ \textit{equals}\ -2|\mathbf{V}|\,|\mathbf{W}|\cos\theta\ \textit{equals}\ -2\mathbf{V}\cdot\mathbf{W}.$$

The cosine law is coordinate-free. It applies to all triangles (even in n dimensions). Its vector form in Figure 11.8 is $|\mathbf{V}-\mathbf{W}|^2=|\mathbf{V}|^2-2\mathbf{V}\cdot\mathbf{W}+|\mathbf{W}|^2$. This application to $\mathbf{V}\cdot\mathbf{W}$ is its brief moment of glory.

Remark 5 The dot product is the best way to compute the cosine of θ:

$$\cos\theta=\frac{\mathbf{V}\cdot\mathbf{W}}{|\mathbf{V}|\,|\mathbf{W}|}. \tag{6}$$

Here are examples of $\mathbf{V}$ and $\mathbf{W}$ with a range of angles from 0 to π:

$\mathbf{i}$ and $3\mathbf{i}$ have the same direction	$\cos\theta=1$	$\theta=0$
$\mathbf{i}\cdot(\mathbf{i}+\mathbf{j})=1$ is positive	$\cos\theta=1/\sqrt{2}$	$\theta=\pi/4$
$\mathbf{i}$ and $\mathbf{j}$ are perpendicular: $\mathbf{i}\cdot\mathbf{j}=0$	$\cos\theta=0$	$\theta=\pi/2$
$\mathbf{i}\cdot(-\mathbf{i}+\mathbf{j})=-1$ is negative	$\cos\theta=-1/\sqrt{2}$	$\theta=3\pi/4$
$\mathbf{i}$ and $-3\mathbf{i}$ have opposite directions	$\cos\theta=-1$	$\theta=\pi$

Remark 6 ***The Cauchy-Schwarz inequality*** $|\mathbf{V}\cdot\mathbf{W}|\leqslant|\mathbf{V}|\,|\mathbf{W}|$ ***comes from*** $|\cos\theta|\leqslant 1$.

The left side is $|\mathbf{V}|\,|\mathbf{W}|\,|\cos\theta|$. It never exceeds the right side $|\mathbf{V}|\,|\mathbf{W}|$. This is a key inequality in mathematics, from which so many others follow:

Geometric mean $\sqrt{xy}\leqslant$ ***arithmetic mean*** $\frac{1}{2}(x+y)$ (true for any $x\geqslant 0$ and $y\geqslant 0$).

Triangle inequality $|\mathbf{V}+\mathbf{W}|\leqslant|\mathbf{V}|+|\mathbf{W}|$ ($|\mathbf{V}|$, $|\mathbf{W}|$, $|\mathbf{V}+\mathbf{W}|$ are lengths of sides).

These and other examples are in Problems 39 to 44. The Schwarz inequality $|\mathbf{V}\cdot\mathbf{W}|\leqslant|\mathbf{V}|\,|\mathbf{W}|$ becomes an equality when $|\cos\theta|=1$ and the vectors are _______.

11.1 EXERCISES

Read-through questions

A vector has length and __a__. If $\mathbf{v}$ has components 6 and -8, its length is $|\mathbf{v}| =$ __b__ and its direction vector is $\mathbf{u} =$ __c__. The product of $|\mathbf{v}|$ with $\mathbf{u}$ is __d__. This vector goes from $(0, 0)$ to the point $x =$ __e__, $y =$ __f__. A combination of the coordinate vectors $\mathbf{i} =$ __g__ and $\mathbf{j} =$ __h__ produces $\mathbf{v} =$ __i__ $\mathbf{i}$ + __j__ $\mathbf{j}$.

To add vectors we add their __k__. The sum of $(6, -8)$ and $(1, 0)$ is __l__. To see $\mathbf{v} + \mathbf{i}$ geometrically, put the __m__ of $\mathbf{i}$ at the __n__ of $\mathbf{v}$. The vectors form a __o__ with diagonal $\mathbf{v} + \mathbf{i}$. (The other diagonal is __p__.) The vectors $2\mathbf{v}$ and $-\mathbf{v}$ are __q__ and __r__. Their lengths are __s__ and __t__.

In a space without axes and coordinates, the tail of $\mathbf{V}$ can be placed __u__. Two vectors with the same __v__ are the same. If a triangle starts with $\mathbf{V}$ and continues with $\mathbf{W}$, the third side is __w__. The vector connecting the midpoint of $\mathbf{V}$ to the midpoint of $\mathbf{W}$ is __x__. That vector is __y__ the third side. In this coordinate-free form the dot product is $\mathbf{V} \cdot \mathbf{W} =$ __z__.

Using components, $\mathbf{V} \cdot \mathbf{W} =$ __A__ and $(1, 2, 1) \cdot (2, -3, 7) =$ __B__. The vectors are perpendicular if __C__. The vectors are parallel if __D__. $\mathbf{V} \cdot \mathbf{V}$ is the same as __E__. The dot product of $\mathbf{U} + \mathbf{V}$ with $\mathbf{W}$ equals __F__. The angle between $\mathbf{V}$ and $\mathbf{W}$ has $\cos\theta =$ __G__. When $\mathbf{V} \cdot \mathbf{W}$ is negative then θ is __H__. The angle between $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$ is __I__. The Cauchy-Schwarz inequality is __J__, and for $\mathbf{V} = \mathbf{i} + \mathbf{j}$ and $\mathbf{W} = \mathbf{i} + \mathbf{k}$ it becomes $1 \leqslant$ __K__.

In 1–4 compute $\mathbf{V} + \mathbf{W}$ and $2\mathbf{V} - 3\mathbf{W}$ and $|\mathbf{V}|^2$ and $\mathbf{V} \cdot \mathbf{W}$ and $\cos\theta$.

1 $\mathbf{V} = (1, 1, 1)$, $\mathbf{W} = (-1, -1, -1)$

2 $\mathbf{V} = \mathbf{i} + \mathbf{j}$, $\mathbf{W} = \mathbf{j} - \mathbf{k}$

3 $\mathbf{V} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{W} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$

4 $\mathbf{V} = (1, 1, 1, 1)$, $\mathbf{W} = (1, 2, 3, 4)$

5 (a) Find a vector that is perpendicular to (v_1, v_2).
(b) Find two vectors that are perpendicular to (v_1, v_2, v_3).

6 Find two vectors that are perpendicular to $(1, 1, 0)$ and to each other.

7 What vector is perpendicular to all 2-dimensional vectors? What vector is parallel to all 3-dimensional vectors?

8 In Problems 1–4 construct unit vectors in the same direction as $\mathbf{V}$.

9 If $\mathbf{v}$ and $\mathbf{w}$ are unit vectors, what is the geometrical meaning of $\mathbf{v} \cdot \mathbf{w}$? What is the geometrical meaning of $(\mathbf{v} \cdot \mathbf{w})\mathbf{v}$? Draw a figure with $\mathbf{v} = \mathbf{i}$ and $\mathbf{w} = (3/5)\mathbf{i} + (4/5)\mathbf{j}$.

10 Write down all unit vectors that make an angle θ with the vector $(1, 0)$. Write down *all* vectors at that angle.

11 ***True or false*** in three dimensions:
1. If both $\mathbf{U}$ and $\mathbf{V}$ make a 30° angle with $\mathbf{W}$, so does $\mathbf{U} + \mathbf{V}$.
2. If they make a 90° angle with $\mathbf{W}$, so does $\mathbf{U} + \mathbf{V}$.
3. If they make a 90° angle with $\mathbf{W}$ they are perpendicular: $\mathbf{U} \cdot \mathbf{V} = 0$.

12 From $\mathbf{W} = (1, 2, 3)$ subtract a multiple of $\mathbf{V} = (1, 1, 1)$ so that $\mathbf{W} - c\mathbf{V}$ is perpendicular to $\mathbf{V}$. Draw $\mathbf{V}$ and $\mathbf{W}$ and $\mathbf{W} - c\mathbf{V}$.

13 (a) What is the sum $\mathbf{V}$ of the twelve vectors from the center of a clock to the hours?
(b) If the 4 o'clock vector is removed, find $\mathbf{V}$ for the other eleven vectors.
(c) If the vectors to 1, 2, 3 are cut in half, find $\mathbf{V}$ for the twelve vectors.

14 (a) By removing one or more of the twelve clock vectors, make the length $|\mathbf{V}|$ as large as possible.
(b) Suppose the vectors start from the top instead of the center (the origin is moved to 12 o'clock, so $\mathbf{v}_{12} = \mathbf{0}$). What is the new sum $\mathbf{V}^*$?

15 Find the angle POQ by vector methods if $P = (1, 1, 0)$, $O = (0, 0, 0)$, $Q = (1, 2, -2)$.

16 (a) Draw the unit vectors $\mathbf{u}_1 = (\cos\theta, \sin\theta)$ and $\mathbf{u}_2 = (\cos\phi, \sin\phi)$. By dot products find the formula for $\cos(\theta - \phi)$.
(b) Draw the unit vector $\mathbf{u}_3$ from a 90° rotation of $\mathbf{u}_2$. By dot products find the formula for $\sin(\theta + \phi)$.

17 Describe all points (x, y) such that $\mathbf{v} = x\mathbf{i} + y\mathbf{j}$ satisfies
(a) $|\mathbf{v}| = 2$ (b) $|\mathbf{v} - \mathbf{i}| = 2$
(c) $\mathbf{v} \cdot \mathbf{i} = 2$ (d) $\mathbf{v} \cdot \mathbf{i} = |\mathbf{v}|$

18 (Important) If $\mathbf{A}$ and $\mathbf{B}$ are non-parallel vectors from the origin, describe
(a) the endpoints of $t\mathbf{B}$ for all numbers t
(b) the endpoints of $\mathbf{A} + t\mathbf{B}$ for all t
(c) the endpoints of $s\mathbf{A} + t\mathbf{B}$ for all s and t
(d) the vectors $\mathbf{v}$ that satisfy $\mathbf{v} \cdot \mathbf{A} = \mathbf{v} \cdot \mathbf{B}$

19 (a) If $\mathbf{v} + 2\mathbf{w} = \mathbf{i}$ and $2\mathbf{v} + 3\mathbf{w} = \mathbf{j}$ find $\mathbf{v}$ and $\mathbf{w}$.
(b) If $\mathbf{v} = \mathbf{i} + \mathbf{j}$ and $\mathbf{w} = 3\mathbf{i} + 4\mathbf{j}$ then $\mathbf{i} =$ ________ $\mathbf{v}$ + ________ $\mathbf{w}$.

20 If $P = (0, 0)$ and $R = (0, 1)$ choose Q so the angle PQR is 90°. All possible Q's lie in a ________.

21 (a) Choose d so that $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j}$ is perpendicular to $\mathbf{B} = 9\mathbf{i} + d\mathbf{j}$.
(b) Find a vector $\mathbf{C}$ perpendicular to $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{B} = \mathbf{i} - \mathbf{k}$.

22 If a boat has velocity **V** with respect to the water and the water has velocity **W** with respect to the land, then ________. The speed of the boat is not $|\mathbf{V}|+|\mathbf{W}|$ but ________.

23 Find the angle between the diagonal of cube and (a) an edge (b) the diagonal of a face (c) another diagonal of the cube. Choose lines that meet.

24 Draw the triangle PQR in Example 1 (the four-sided figure in space). By geometry not vectors, show that PR is twice as long as **V**. Similarly $|PR| = 2|\mathbf{W}|$. Also **V** is parallel to **W** because both are parallel to ________. So $\mathbf{V}=\mathbf{W}$ as before.

25 (a) If **A** and **B** are unit vectors, show that they make equal angles with $\mathbf{A}+\mathbf{B}$.

(b) If **A**, **B**, **C** are unit vectors with $\mathbf{A}+\mathbf{B}+\mathbf{C}=0$, they form a ________ triangle and the angle between any two is ________.

26 (a) Find perpendicular unit vectors **I** and **J** in the plane that are different from **i** and **j**.

(b) Find perpendicular unit vectors **I**, **J**, **K** different from **i**, **j**, **k**.

27 If **I** and **J** are perpendicular, take their dot products with $\mathbf{A}=a\mathbf{I}+b\mathbf{J}$ to find a and b.

28 Suppose $\mathbf{I}=(\mathbf{i}+\mathbf{j})/\sqrt{2}$ and $\mathbf{J}=(\mathbf{i}-\mathbf{j})/\sqrt{2}$. Check $\mathbf{I}\cdot\mathbf{J}=0$ and write $\mathbf{A}=2\mathbf{i}+3\mathbf{j}$ as a combination $a\mathbf{I}+b\mathbf{J}$. (Best method: use a and b from Problem 27. Alternative: Find **i** and **j** from **I** and **J** and substitute into **A**.)

29 (a) Find the position vector OP and the velocity vector PQ when the point P moves around the unit circle (see figure) with speed 1. (b) Change to speed 2.

30 The sum $(\mathbf{A}\cdot\mathbf{i})^2+(\mathbf{A}\cdot\mathbf{j})^2+(\mathbf{A}\cdot\mathbf{k})^2$ equals ________.

31 In the semicircle find **C** and **D** in terms of **A** and **B**. Prove that $\mathbf{C}\cdot\mathbf{D}=0$ (they meet at right angles).

35 The vector from the earth's center to Seattle is $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$.

(a) Along the circle at the latitude of Seattle, what two functions of a, b, c stay constant? **k** goes to the North Pole.

(b) On the circle at the longitude of Seattle—the meridian—what two functions of a, b, c stay constant?

(c) Extra credit: Estimate a, b, c in your present position. The 0° meridian through Greenwich has $b=0$.

36 If $|\mathbf{A}+\mathbf{B}|^2=|\mathbf{A}|^2+|\mathbf{B}|^2$, prove that **A** is perpendicular to **B**.

37 In Figure 11.4, the medians go from the corners to the midpoints of the opposite sides. Express $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ in terms of **A**, **B**, **C**. Prove that $\mathbf{M}_1+\mathbf{M}_2+\mathbf{M}_3=0$. What relation holds between **A**, **B**, **C**?

38 The point $\frac{2}{3}$ of the way along is the same for all three medians. This means that $\mathbf{A}+\frac{2}{3}\mathbf{M}_3=\frac{2}{3}\mathbf{M}_2=$ ________. Prove that those three vectors are equal.

39 (a) Verify the Schwarz inequality $|\mathbf{V}\cdot\mathbf{W}|\leqslant|\mathbf{V}||\mathbf{W}|$ for $\mathbf{V}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $\mathbf{W}=2\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

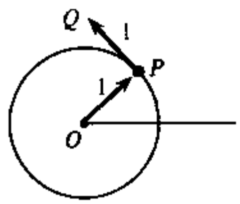
(b) What does the inequality become when $\mathbf{V}=(\sqrt{x}, \sqrt{y})$ and $\mathbf{W}=(\sqrt{y}, \sqrt{x})$?

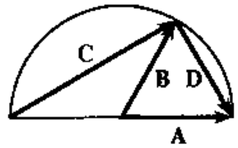
40 By choosing the right vector **W** in the Schwarz inequality, show that $(V_1+V_2+V_3)^2\leqslant 3(V_1^2+V_2^2+V_3^2)$. What is **W**?

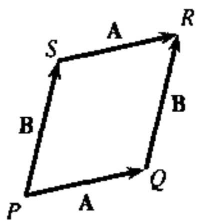
41 The Schwarz inequality for $a\mathbf{i}+b\mathbf{j}$ and $c\mathbf{i}+d\mathbf{j}$ says that $(a^2+b^2)(c^2+d^2)\geqslant(ac+bd)^2$. Multiply out to show that the difference is $\geqslant 0$.

42 The vectors **A**, **B**, **C** form a triangle if $\mathbf{A}+\mathbf{B}+\mathbf{C}=\mathbf{0}$. The **triangle inequality** $|\mathbf{A}+\mathbf{B}|\leqslant|\mathbf{A}|+|\mathbf{B}|$ says that any one side length is less than ________. The proof comes from Schwarz:

$$|\mathbf{A}+\mathbf{B}|^2=\mathbf{A}\cdot\mathbf{A}+2\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{B}$$
$$\leqslant|\mathbf{A}|^2+\underline{\qquad\qquad}+|\mathbf{B}|^2=(|\mathbf{A}|+|\mathbf{B}|)^2.$$







32 The diagonal PR has $|PR|^2=(\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})=\mathbf{A}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}+\mathbf{B}\cdot\mathbf{B}$. Add $|QS|^2$ from the other diagonal to prove the parallelogram law: $|PR|^2+|QS|^2=$ sum of squares of the four side lengths.

33 If (1, 2, 3), (3, 4, 7), and (2, 1, 2) are corners of a parallelogram, find all possible fourth corners.

34 The diagonals of the parallelogram are $\mathbf{A}+\mathbf{B}$ and ________. If they have the same length, prove that $\mathbf{A}\cdot\mathbf{B}=0$ and the region is a ________.

43 **True or false**, with reason or example:

(a) $|\mathbf{V}+\mathbf{W}|^2$ is never larger than $|\mathbf{V}|^2+|\mathbf{W}|^2$

(b) In a real triangle $|\mathbf{V}+\mathbf{W}|$ never equals $|\mathbf{V}|+|\mathbf{W}|$

(c) $\mathbf{V}\cdot\mathbf{W}$ equals $\mathbf{W}\cdot\mathbf{V}$

(d) The vectors perpendicular to $\mathbf{i}+\mathbf{j}+\mathbf{k}$ lie along a line.

44 If $\mathbf{V}=\mathbf{i}+2\mathbf{k}$ choose **W** so that $\mathbf{V}\cdot\mathbf{W}=|\mathbf{V}||\mathbf{W}|$ and $|\mathbf{V}+\mathbf{W}|=|\mathbf{V}|+|\mathbf{W}|$.

45 A methane molecule has a carbon atom at $(0, 0, 0)$ and hydrogen atoms at $(1, 1, -1)$, $(1, -1, 1)$, $(-1, 1, 1)$, and $(-1, -1, -1)$. Find

(a) the distance between hydrogen atoms

(b) the angle between vectors going out from the carbon atom to the hydrogen atoms.

46 (a) Find a vector **V** at a 45° angle with **i** and **j**.

(b) Find **W** that makes a 60° angle with **i** and **j**.

(c) Explain why no vector makes a 30° angle with **i** and **j**.

11.2 Planes and Projections

The most important "curves" are straight lines. The most important functions are linear. Those sentences take us back to the beginning of the book—the graph of $mx + b$ is a line. The goal now is to move into three dimensions, where ***graphs are surfaces***. Eventually the surfaces will be curved. But calculus starts with the flat surfaces that correspond to straight lines:

What are the most important surfaces? ***Planes***.
What are the most important functions? ***Still linear***.

The geometrical idea of a plane is turned into algebra, by finding ***the equation of a plane***. Not just a general formula, but the particular equation of a particular plane.

A line is determined by one point (x_0, y_0) and the slope m. The point-slope equation is $y - y_0 = m(x - x_0)$. That is a linear equation, it is satisfied when $y = y_0$ and $x = x_0$, and dy/dx is m. For a plane, we start again with a particular point—which is now (x_0, y_0, z_0). But the slope of a plane is not so simple. Many planes climb at a 45° angle—with "slope 1"—and more information is needed.

The direction of a plane is described by a vector **N**. The vector is not *in* the plane, but *perpendicular* to the plane. In the plane, there are many directions. Perpendicular to the plane, there is only one direction. A vector in that perpendicular direction is a ***normal vector***.

The normal vector **N** can point "up" or "down". The length of **N** is not crucial (we often make it a unit vector and call it **n**). Knowing **N** and the point $P_0 = (x_0, y_0, z_0)$, we know the plane (Figure 11.9). For its equation we switch to algebra and use the dot product—which is the key to perpendicularity.

N is described by its components (a, b, c). In other words **N** is $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$. ***This vector is perpendicular to every direction in the plane***. A typical direction goes from

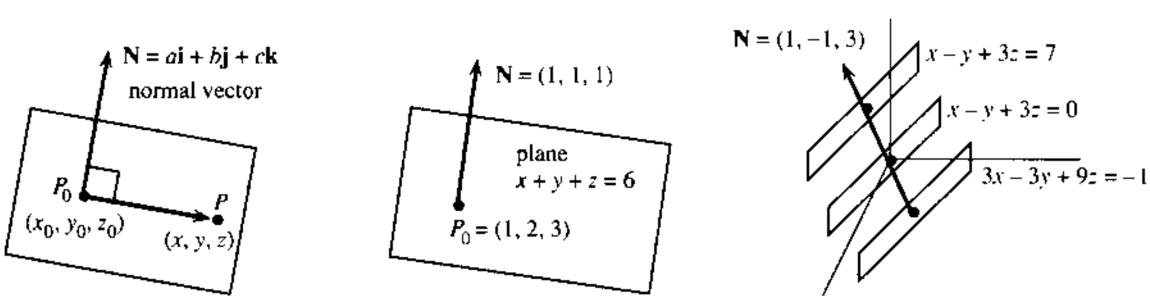


Fig. 11.9 The normal vector to a plane. Parallel planes have the same **N**.

P_0 to another point $P = (x, y, z)$ in the plane. The vector from P_0 to P has components $(x - x_0, y - y_0, z - z_0)$. This vector lies in the plane, so *its dot product with* **N** *is zero:*

> **11C** The plane through P_0 perpendicular to $\mathbf{N} = (a, b, c)$ has the equation
> $$(a, b, c) \cdot (x - x_0, y - y_0, z - z_0) = 0 \qquad \text{or}$$
> $$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{1}$$
> The point P lies on the plane when its coordinates x, y, z satisfy this equation.

EXAMPLE 1 The plane through $P_0 = (1, 2, 3)$ perpendicular to $\mathbf{N} = (1, 1, 1)$ has the equation $(x - 1) + (y - 2) + (z - 3) = 0$. That can be rewritten as $x + y + z = 6$.

Notice three things. First, P_0 lies on the plane because $1 + 2 + 3 = 6$. Second, $\mathbf{N} = (1, 1, 1)$ can be recognized from the x, y, z coefficients in $x + y + z = 6$. Third, we could change $\mathbf{N}$ to $(2, 2, 2)$ and we could change P_0 to $(8, 2, -4)$—because $\mathbf{N}$ is still perpendicular and P_0 is still in the plane: $8 + 2 - 4 = 6$.

The new normal vector $\mathbf{N} = (2, 2, 2)$ produces $2(x - 1) + 2(y - 2) + 2(z - 3) = 0$. That can be rewritten as $2x + 2y + 2z = 12$. Same normal direction, same plane.

The new point $P_0 = (8, 2, -4)$ produces $(x - 8) + (y - 2) + (z + 4) = 0$. That is another form of $x + y + z = 6$. All we require is a perpendicular $\mathbf{N}$ and a point P_0 in the plane.

EXAMPLE 2 The plane through $(1, 2, 4)$ with the same $\mathbf{N} = (1, 1, 1)$ has a different equation: $(x - 1) + (y - 2) + (z - 4) = 0$. This is $x + y + z = 7$ (instead of 6). ***These planes with 7 and 6 are parallel.***

Starting from $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, we often move $ax_0 + by_0 + cz_0$ to the right hand side—and call this constant d:

> **11D** With the P_0 terms on the right side, the equation of the plane is $\mathbf{N} \cdot \mathbf{P} = d$:
> $$ax + by + cz = ax_0 + by_0 + cz_0 = d. \tag{2}$$
> A different d gives a ***parallel plane***; $d = 0$ gives a ***plane through the origin***.

EXAMPLE 3 The plane $x - y + 3z = 0$ goes through the origin $(0, 0, 0)$. The normal vector is read directly from the equation: $\mathbf{N} = (1, -1, 3)$. The equation is satisfied by $P_0 = (1, 1, 0)$ and $P = (1, 4, 1)$. Subtraction gives a vector $\mathbf{V} = (0, 3, 1)$ that is in the plane, and $\mathbf{N} \cdot \mathbf{V} = 0$.

The parallel planes $x - y + 3z = d$ have the same $\mathbf{N}$ but different d's. These planes miss the origin because d is not zero ($x = 0$, $y = 0$, $z = 0$ on the left side needs $d = 0$ on the right side). Note that $3x - 3y + 9z = -15$ is parallel to both planes. $\mathbf{N}$ is changed to $3\mathbf{N}$ in Figure 11.9, but its direction is not changed.

EXAMPLE 4 ***The angle between two planes is the angle between their normal vectors.***

The planes $x - y + 3z = 0$ and $3y + z = 0$ are perpendicular, because $(1, -1, 3) \cdot (0, 3, 1) = 0$. The planes $z = 0$ and $y = 0$ are also perpendicular, because $(0, 0, 1) \cdot (0, 1, 0) = 0$. (Those are the xy plane and the xz plane.) The planes $x + y = 0$ and $x + z = 0$ make a $60°$ angle, because $\cos 60° = (1, 1, 0) \cdot (1, 0, 1)/\sqrt{2}\sqrt{2} = \frac{1}{2}$.

The cosine of the angle between two planes is $|\mathbf{N}_1 \cdot \mathbf{N}_2|/|\mathbf{N}_1||\mathbf{N}_2|$. See Figure 11.10.

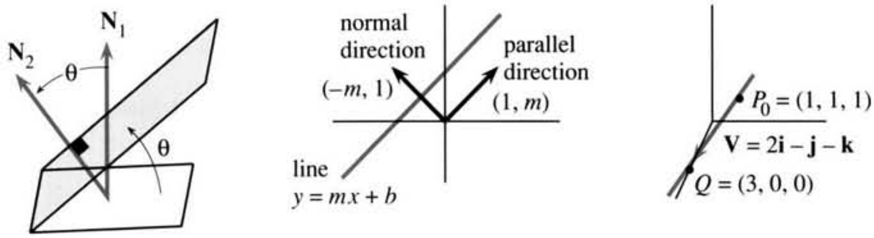


Fig. 11.10 Angle between planes = angle between normals. Parallel and perpendicular to a line. *A line in space through P_0 and Q.*

***Remark* 1** We gave the "point-slope" equation of a line (using m), and the "point-normal" equation of a plane (using $\mathbf{N}$). What is the normal vector $\mathbf{N}$ to a line?

The vector $\mathbf{V} = (1, m)$ is parallel to the line $y = mx + b$. The line goes across by 1 and up by m. ***The perpendicular vector is*** $\mathbf{N} = (-m, 1)$. The dot product $\mathbf{N} \cdot \mathbf{V}$ is $-m + m = 0$. Then the point-normal equation matches the point-slope equation:

$$-m(x - x_0) + 1(y - y_0) = 0 \text{ is the same as } y - y_0 = m(x - x_0). \tag{3}$$

Remark* 2** What is the point-slope equation for a plane? The difficulty is that a plane has different slopes in the x and y directions. The function $f(x, y) = m(x - x_0) + M(y - y_0)$ has ***two derivatives m and M.

This remark has to stop. In Chapter 13, "slopes" become "***partial derivatives***."

A LINE IN SPACE

In three dimensions, a line is not as simple as a plane. ***A line in space needs two equations***. Each equation gives a plane, and the line is the ***intersection of two planes***.

The equations $x + y + z = 3$ *and* $2x + 3y + z = 6$ *determine a line.*

Two points on that line are $P_0 = (1, 1, 1)$ and $Q = (3, 0, 0)$. They satisfy both equations so they lie on both planes. Therefore they are on the line of intersection. The direction of that line, subtracting coordinates of P_0 from Q, is along the vector $\mathbf{V} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

The line goes through $P_0 = (1, 1, 1)$ *in the direction of* $\mathbf{V} = 2\mathbf{i} - \mathbf{j} - \mathbf{k}$.

Starting from $(x_0, y_0, z_0) = (1, 1, 1)$, ***add on any multiple*** $t\mathbf{V}$. Then $x = 1 + 2t$ and $y = 1 - t$ and $z = 1 - t$. Those are the components of the vector equation $\mathbf{P} = \mathbf{P}_0 + t\mathbf{V}$—which produces the line.

Here is the problem. The line needs two equations—or a vector equation with a *parameter* t. Neither form is as simple as $ax + by + cz = d$. Some books push ahead anyway, to give full details about both forms. After trying this approach, I believe that those details should wait. Equations with parameters are the subject of Chapter 12, and a line in space is the first example. Vectors and planes give plenty to do here—especially when a vector is projected onto another vector or a plane.

PROJECTION OF A VECTOR

What is the projection of a vector $\mathbf{B}$ onto another vector $\mathbf{A}$? One part of $\mathbf{B}$ goes ***along*** $\mathbf{A}$—that is the projection. The other part of $\mathbf{B}$ is ***perpendicular*** to $\mathbf{A}$. We now compute these two parts, which are $\mathbf{P}$ and $\mathbf{B} - \mathbf{P}$.

In geometry, projections involve $\cos\theta$. In algebra, we use the dot product (which is closely tied to $\cos\theta$). In applications, the vector **B** might be a *velocity* **V** or a *force* **F**:

> An airplane flies northeast, and a 100-mile per hour wind blows due east. What is the projection of $\mathbf{V} = (100, 0)$ in the flight direction **A**?
>
> Gravity makes a ball roll down the surface $2x + 2y + z = 0$. What are the projections of $\mathbf{F} = (0, 0, -mg)$ in the plane and perpendicular to the plane?

The component of **V** along **A** is the push from the wind (tail wind). The other component of **V** pushes sideways (crosswind). Similarly the force parallel to the surface makes the ball move. Adding the two components brings back **V** or **F**.

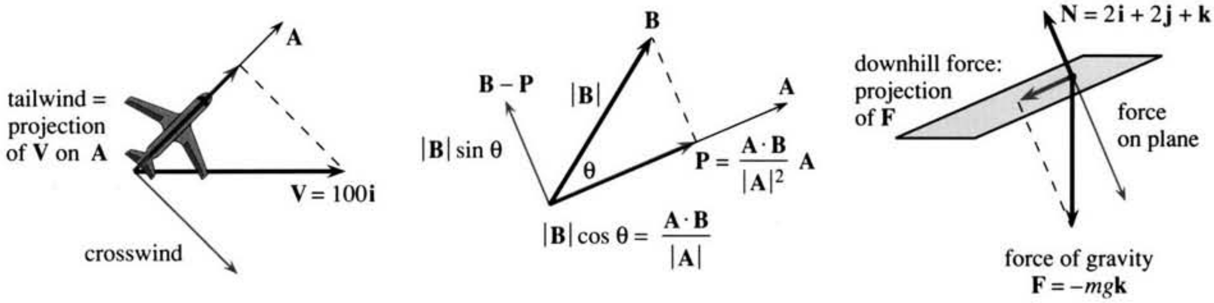


Fig. 11.11 Projections along **A** of wind velocity **V** and force **F** and vector **B**.

***We now compute the projection of* B *onto* A**. Call this projection **P**. Since its direction is known—**P** is along **A**—we can describe **P** in two ways:

1) *Give the* **length** *of* P *along* A
2) *Give the* **vector** P *as a multiple of* A.

Figure 11.11b shows the projection **P** and its length. The hypotenuse is $|\mathbf{B}|$. The length is $|\mathbf{P}| = |\mathbf{B}|\cos\theta$. The perpendicular component $\mathbf{B} - \mathbf{P}$ has length $|\mathbf{B}|\sin\theta$. The cosine is positive for angles less than 90°. The cosine (and **P**!) are zero when **A** and **B** are perpendicular. $|\mathbf{B}|\cos\theta$ is negative for angles greater than 90°, and the projection points along $-\mathbf{A}$ (the length is $|\mathbf{B}|\,|\cos\theta|$). Unless the angle is 0° or 30° or 45° or 60° or 90°, we don't want to compute cosines—and we don't have to. The dot product does it automatically:

$$|\mathbf{A}|\,|\mathbf{B}|\cos\theta = \mathbf{A}\cdot\mathbf{B} \textit{ so the length of } \mathrm{P} \textit{ along } \mathrm{A} \textit{ is } |\mathbf{B}|\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|}. \tag{4}$$

Notice that the length of **A** cancels out at the end of (4). If **A** is doubled, **P** is unchanged. But if **B** is doubled, the projection is doubled.

What is the vector **P**? Its length along **A** is $\mathbf{A}\cdot\mathbf{B}/|\mathbf{A}|$. If **A** is a unit vector, then $|\mathbf{A}| = 1$ and the projection is $\mathbf{A}\cdot\mathbf{B}$ times **A**. Generally **A** is not a unit vector, until we divide by $|\mathbf{A}|$. ***Here is the projection* P *of* B *along* A**:

$$\mathbf{P} = (\textit{length of } \mathrm{P})(\textit{unit vector}) = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|}\right)\left(\frac{\mathbf{A}}{|\mathbf{A}|}\right) = \frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|^2}\mathbf{A}. \tag{5}$$

EXAMPLE 5 For the wind velocity $\mathbf{V} = (100, 0)$ and flying direction $\mathbf{A} = (1, 1)$, find $\mathbf{P}$. Here $\mathbf{V}$ points east, $\mathbf{A}$ points northeast. The projection of $\mathbf{V}$ onto $\mathbf{A}$ is $\mathbf{P}$:

$$\text{length } |\mathbf{P}| = \frac{\mathbf{A} \cdot \mathbf{V}}{|\mathbf{A}|} = \frac{100}{\sqrt{2}} \qquad \text{vector } \mathbf{P} = \frac{\mathbf{A} \cdot \mathbf{V}}{|\mathbf{A}|^2}\mathbf{A} = \frac{100}{2}(1, 1) = (50, 50).$$

EXAMPLE 6 Project $\mathbf{F} = (0, 0, -mg)$ onto the plane with normal $\mathbf{N} = (2, 2, 1)$.

The projection of $\mathbf{F}$ along $\mathbf{N}$ is *not* the answer. But compute that first:

$$\frac{\mathbf{F} \cdot \mathbf{N}}{|\mathbf{N}|} = -\frac{mg}{3} \qquad \mathbf{P} = \frac{\mathbf{F} \cdot \mathbf{N}}{|\mathbf{N}|^2}\mathbf{N} = -\frac{mg}{9}(2, 2, 1).$$

$\mathbf{P}$ is the component of $\mathbf{F}$ *perpendicular* to the plane. It does *not* move the ball. The in-plane component is the difference $\mathbf{F} - \mathbf{P}$. Any vector $\mathbf{B}$ has two projections, along $\mathbf{A}$ and perpendicular:

The projection $\mathbf{P} = \dfrac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|^2}\mathbf{A}$ ***is perpendicular to the remaining component*** $\mathbf{B} - \mathbf{P}$.

EXAMPLE 7 Express $\mathbf{B} = \mathbf{i} - \mathbf{j}$ as the sum of a vector $\mathbf{P}$ parallel to $\mathbf{A} = 3\mathbf{i} + \mathbf{j}$ and a vector $\mathbf{B} - \mathbf{P}$ perpendicular to $\mathbf{A}$. Note $\mathbf{A} \cdot \mathbf{B} = 2$.

Solution $\quad \mathbf{P} = \dfrac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}|^2}\mathbf{A} = \dfrac{2}{10}\mathbf{A} = \dfrac{6}{10}\mathbf{i} + \dfrac{2}{10}\mathbf{j}. \qquad$ Then $\mathbf{B} - \mathbf{P} = \dfrac{4}{10}\mathbf{i} - \dfrac{12}{10}\mathbf{j}.$

Check: $\mathbf{P} \cdot (\mathbf{B} - \mathbf{P}) = (\frac{6}{10})(\frac{4}{10}) - (\frac{2}{10})(\frac{12}{10}) = 0$. These projections of $\mathbf{B}$ are perpendicular.
Pythagoras: $|\mathbf{P}|^2 + |\mathbf{B} - \mathbf{P}|^2$ equals $|\mathbf{B}|^2$. Check that too: $0.4 + 1.6 = 2.0$.

Question When is $\mathbf{P} = \mathbf{0}$? Answer When $\mathbf{A}$ and $\mathbf{B}$ are perpendicular.

EXAMPLE 8 ***Find the nearest point to the origin on the plane*** $x + 2y + 2z = 5$.

The shortest distance from the origin is along the normal vector $\mathbf{N}$. The vector $\mathbf{P}$ to the nearest point (Figure 11.12) is t times $\mathbf{N}$, for some unknown number t. We find t by requiring $\mathbf{P} = t\mathbf{N}$ to lie on the plane.

The plane $x + 2y + 2z = 5$ has normal vector $\mathbf{N} = (1, 2, 2)$. Therefore $\mathbf{P} = t\mathbf{N} = (t, 2t, 2t)$. To lie on the plane, this must satisfy $x + 2y + 2z = 5$:

$$t + 2(2t) + 2(2t) = 5 \quad \text{or} \quad 9t = 5 \quad \text{or} \quad t = \tfrac{5}{9}. \tag{6}$$

Then $\mathbf{P} = \frac{5}{9}\mathbf{N} = (\frac{5}{9}, \frac{10}{9}, \frac{10}{9})$. That locates the nearest point. The distance is $\frac{5}{9}|\mathbf{N}| = \frac{5}{3}$. This example is important enough to memorize, with letters not numbers:

> **11E** On the plane $ax + by + cz = d$, the nearest point to $(0, 0, 0)$ is
>
> $$P = \frac{(da, db, dc)}{a^2 + b^2 + c^2}. \qquad \text{The distance is} \quad \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}. \tag{7}$$

The steps are the same. $\mathbf{N}$ has components a, b, c. The nearest point on the plane is a multiple (ta, tb, tc). It lies on the plane if $a(ta) + b(tb) + c(tc) = d$.

Thus $t = d/(a^2 + b^2 + c^2)$. The point $(ta, tb, tc) = t\mathbf{N}$ is in equation (7). The distance to the plane is $|t\mathbf{N}| = |d|/|\mathbf{N}|$.

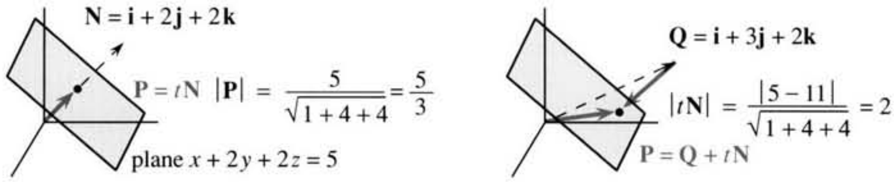


Fig. 11.12 Vector to the nearest point P is a multiple $t\mathbf{N}$. The distance is in (7) and (9).

Question How far is the plane from an arbitrary point $Q = (x_1, y_1, z_1)$?
Answer *The vector from Q to P is our multiple $t\mathbf{N}$*. In vector form $\mathbf{P} = \mathbf{Q} + t\mathbf{N}$. This reaches the plane if $\mathbf{P} \cdot \mathbf{N} = d$, and again we find t:

$$(\mathbf{Q} + t\mathbf{N}) \cdot \mathbf{N} = d \qquad \text{yields} \qquad t = (d - \mathbf{Q} \cdot \mathbf{N})/|\mathbf{N}|^2. \tag{8}$$

This new term $\mathbf{Q} \cdot \mathbf{N}$ enters the distance from Q to the plane:

$$\textit{distance} = |t\mathbf{N}| = |d - \mathbf{Q} \cdot \mathbf{N}|/|\mathbf{N}| = |d - ax_1 - by_1 - cz_1|/\sqrt{a^2 + b^2 + c^2}. \tag{9}$$

When the point is on the plane, that distance is zero—because $ax_1 + by_1 + cz_1 = d$. When $\mathbf{Q}$ is $\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$, the figure shows $\mathbf{Q} \cdot \mathbf{N} = 11$ and distance $= 2$.

PROJECTIONS OF THE HEART VECTOR

An electrocardiogram has leads to your right arm–left arm–left leg. *You produce the voltage*. The machine amplifies and records the readings. There are also six chest leads, to add a front-back dimension that is monitored across the heart. We will concentrate on the big "Einthoven triangle," named after the inventor of the ECG.

The graphs show voltage variations plotted against time. The first graph plots the voltage difference between the arms. Lead II connects the left leg to the right arm. Lead III completes the triangle, which has roughly equal sides (especially if you are a little lopsided). So the projections are based on 60° and 120° angles.

The heart vector $\mathbf{V}$ is the sum of many small vectors—all moved to the same origin. $\mathbf{V}$ is the net effect of action potentials from the cells—small dipoles adding to a single dipole. The pacemaker (*S–A* node) starts the impulse. The atria depolarize to give the P wave on the graphs. This is actually a P *loop* of the heart vector—the

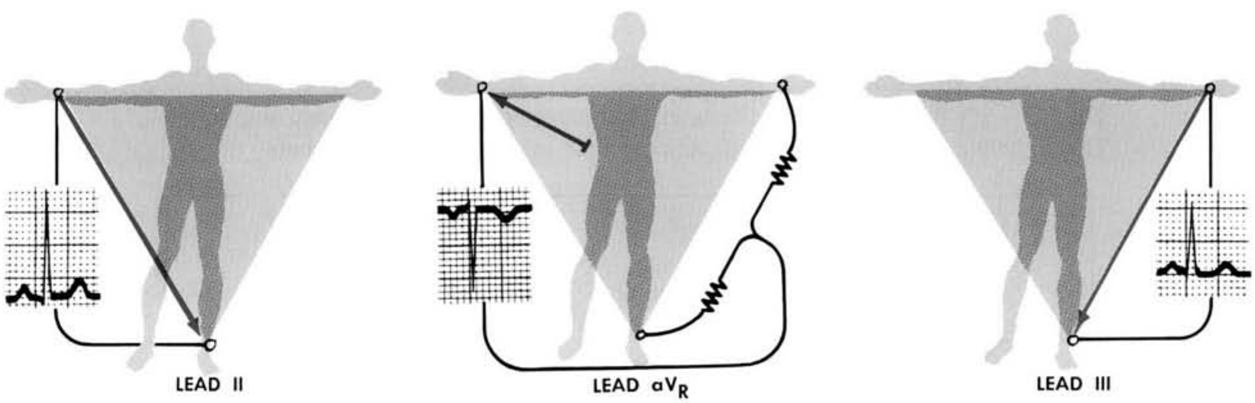


Fig. A The graphs show the component of the moving heart vector along each lead. These figures are reproduced with permission from the CIBA Collection of Medical Illustrations by Frank H. Netter, M.D. Copyright 1978 CIBA-GEIGY, all rights reserved.

graphs only show its projections. The impulse reaches the *AV* node, pauses, and moves quickly through the ventricles. This produces the QRS complex—the large sharp movement on the graph.

The total* QRS *interval should not exceed* 1/10 *second ($2\frac{1}{2}$ spaces on the printout). V points first toward the right shoulder. This direction is opposite to the leads, so the tracings go slightly down. That is the Q wave, small and negative. Then the heart vector sweeps toward the left leg. In positions 3 and 4, its projection on lead I (between the arms) is strongly positive. The R wave is this first upward deflection in each lead. Closing the loop, the S wave is negative (best seen in leads I and aVR).

Question 1 How many graphs from the arms and leg are really independent?
Answer Only two! In a plane, the heart vector V has two components. If we know two projections, we can compute the others. (The ECG does that for us.) Different vectors show better in different projections. A mathematician would use 90° angles, with an electrode at your throat.

Question 2 How are the voltages related? What is the aVR lead?
Answer Project the heart vector V onto the sides of the triangle:

The lead vectors have $\mathbf{L}_{\mathrm{I}} - \mathbf{L}_{\mathrm{II}} + \mathbf{L}_{\mathrm{III}} = \mathbf{0}$—they form a triangle.

The projections have $\mathbf{V}_{\mathrm{I}} - \mathbf{V}_{\mathrm{II}} + \mathbf{V}_{\mathrm{III}} = \mathbf{V} \cdot \mathbf{L}_{\mathrm{I}} - \mathbf{V} \cdot \mathbf{L}_{\mathrm{II}} + \mathbf{V} \cdot \mathbf{L}_{\mathrm{III}} = 0$.

The aVR lead is $-\frac{1}{2}\mathbf{L}_{\mathrm{I}} - \frac{1}{2}\mathbf{L}_{\mathrm{II}}$. It is pure algebra (no wire). By vector addition it points toward the electrode on the right arm. Its length is $\sqrt{3}$ if the other lengths are 2.

Including aVL and aVF to the left arm and foot, there are ***six leads intersecting at equal angles***. Visualize them going out from a single point (the origin in the chest).

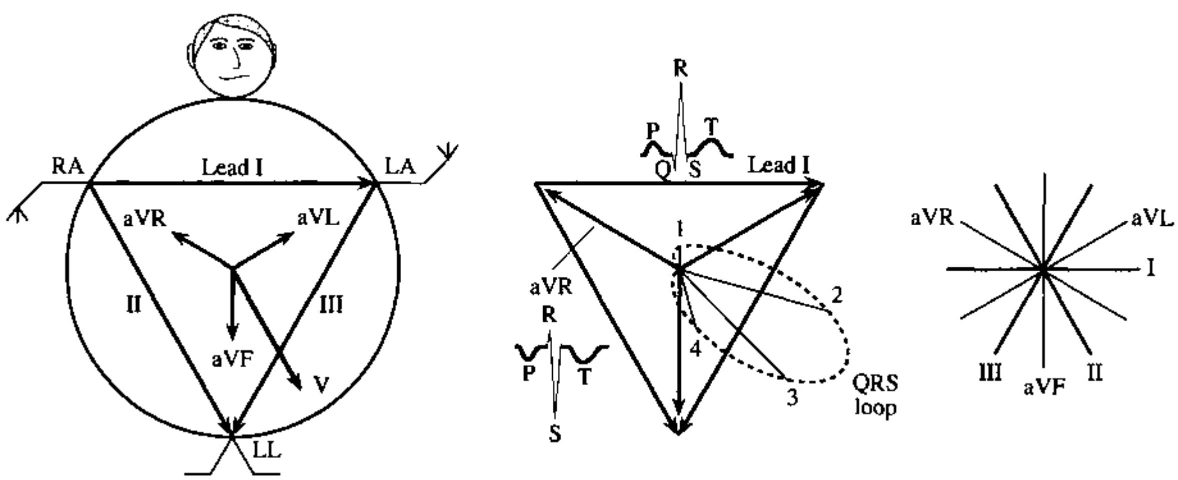


Fig. B Heart vector goes around the QRS loop. Projections are spikes on the ECG.

Question 3 If the heart vector is $\mathbf{V} = 2\mathbf{i} - \mathbf{j}$, what voltage differences are recorded?
Answer The leads around the triangle have length 2. The machine projects V:

Lead I is the horizontal vector $2\mathbf{i}$. So $\mathbf{V} \cdot \mathbf{L}_{\mathrm{I}} = 4$.
Lead II is the $-60°$ vector $\mathbf{i} - \sqrt{3}\mathbf{j}$. So $\mathbf{V} \cdot \mathbf{L}_{\mathrm{II}} = 2 + \sqrt{3}$.
Lead III is the $-120°$ vector $-\mathbf{i} - \sqrt{3}\mathbf{j}$. So $\mathbf{V} \cdot \mathbf{L}_{\mathrm{III}} = -2 + \sqrt{3}$.

The first and third add to the second. The largest R waves are in leads I and II. In aVR the projection of V will be negative (Problem 46), and will be labeled an S wave.

Question 4 What about the *potential* (not just its differences). Is it zero at the center?
Answer ***It is zero if we say so.*** The potential contains an arbitrary constant C. (It is like an indefinite integral. Its differences are like definite integrals.) Cardiologists define a "central terminal" where the potential is zero.

The average of $\mathbf{V}$ over a loop is the *mean heart vector* $\mathbf{H}$. This average requires $\int \mathbf{V}\,dt$, by Chapter 5. With no time to integrate, the doctor looks for a lead where the area under the QRS complex is zero. Then the direction of $\mathbf{H}$ (the *axis*) is perpendicular to that lead. There is so much to say about calculus in medicine.

11.2 EXERCISES

Read-through questions

A plane in space is determined by a point $P_0 = (x_0, y_0, z_0)$ and a __a__ vector $\mathbf{N}$ with components (a, b, c). The point $P = (x, y, z)$ is on the plane if the dot product of $\mathbf{N}$ with __b__ is zero. (*That answer was not* P!) The equation of this plane is $a($__c__$) + b($__d__$) + c($__e__$) = 0$. The equation is also written as $ax + by + cz = d$, where d equals __f__. A parallel plane has the same __g__ and a different __h__. A plane through the origin has $d =$ __i__.

The equation of the plane through $P_0 = (2, 1, 0)$ perpendicular to $\mathbf{N} = (3, 4, 5)$ is __j__. A second point in the plane is $P = (0, 0,$ __k__$)$. The vector from P_0 to P is __l__, and it is __m__ to $\mathbf{N}$. (Check by dot product.) The plane through $P_0 = (2, 1, 0)$ perpendicular to the z axis has $\mathbf{N} =$ __n__ and equation __o__.

The component of $\mathbf{B}$ in the direction of $\mathbf{A}$ is __p__, where θ is the angle between the vectors. This is $\mathbf{A} \cdot \mathbf{B}$ divided by __q__. The projection vector $\mathbf{P}$ is $|\mathbf{B}| \cos \theta$ times a __r__ vector in the direction of $\mathbf{A}$. Then $\mathbf{P} = (|\mathbf{B}| \cos \theta)(\mathbf{A}/|\mathbf{A}|)$ simplifies to __s__. When $\mathbf{B}$ is doubled, $\mathbf{P}$ is __t__. When $\mathbf{A}$ is doubled, $\mathbf{P}$ is __u__. If $\mathbf{B}$ reverses direction then $\mathbf{P}$ __v__. If $\mathbf{A}$ reverses direction then $\mathbf{P}$ __w__.

When $\mathbf{B}$ is a velocity vector, $\mathbf{P}$ represents the __x__. When $\mathbf{B}$ is a force vector, $\mathbf{P}$ is __y__. The component of $\mathbf{B}$ perpendicular to $\mathbf{A}$ equals __z__. The shortest distance from $(0, 0, 0)$ to the plane $ax + by + cz = d$ is along the __A__ vector. The distance is __B__ and the closest point on the plane is $P =$ __C__. The distance from $\mathbf{Q} = (x_1, y_1, z_1)$ to the plane is __D__.

Find two points P and P_0 on the planes 1–6 and a normal vector $\mathbf{N}$. Verify that $\mathbf{N} \cdot (P - P_0) = 0$.

1 $x + 2y + 3z = 0$ **2** $x + 2y + 3z = 6$ **3** the yz plane

4 the plane through $(0, 0, 0)$ perpendicular to $\mathbf{i} + \mathbf{j} - \mathbf{k}$

5 the plane through $(1, 1, 1)$ perpendicular to $\mathbf{i} + \mathbf{j} - \mathbf{k}$

6 the plane through $(0, 0, 0)$ and $(1, 0, 0)$ and $(0, 1, 1)$.

Find an $x - y - z$ equation for planes 7–10.

7 The plane through $P_0 = (1, 2, -1)$ perpendicular to $\mathbf{N} = \mathbf{i} + \mathbf{j}$

8 The plane through $P_0 = (1, 2, -1)$ perpendicular to $\mathbf{N} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$

9 The plane through $(1, 0, 1)$ parallel to $x + 2y + z = 0$

10 The plane through (x_0, y_0, z_0) parallel to $x + y + z = 1$.

11 When is a plane with normal vector $\mathbf{N}$ parallel to the vector $\mathbf{V}$? When is it perpendicular to $\mathbf{V}$?

12 (a) If two planes are perpendicular (front wall and side wall), is every line in one plane perpendicular to every line in the other?

(b) If a third plane is perpendicular to the first, it might be (parallel) (perpendicular) (at a 45° angle) to the second.

13 Explain why a plane cannot

(a) contain $(1, 2, 3)$ and $(2, 3, 4)$ and be perpendicular to $\mathbf{N} = \mathbf{i} + \mathbf{j}$

(b) be perpendicular to $\mathbf{N} = \mathbf{i} + \mathbf{j}$ and parallel to $\mathbf{V} = \mathbf{i} + \mathbf{k}$

(c) contain $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$

(d) contain $(1, 1, -1)$ if it has $\mathbf{N} = \mathbf{i} + \mathbf{j} - \mathbf{k}$ *(maybe it can)*

(e) go through the origin and have the equation $ax + by + cz = 1$.

14 The equation $3x + 4y + 7z - t = 0$ yields a hyperplane in four dimensions. Find its normal vector $\mathbf{N}$ and two points P, Q on the hyperplane. Check $(P - Q) \cdot \mathbf{N} = 0$.

15 The plane through (x, y, z) perpendicular to $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ goes through $(0, 0, 0)$ if ________. The plane goes through (x_0, y_0, z_0) if ________.

16 A curve in three dimensions is the intersection of ________ surfaces. A line in four dimensions is the intersection of ________ hyperplanes.

17 (angle between planes) Find the cosine of the angle between $x + 2y + 2z = 0$ and (a) $x + 2z = 0$ (b) $x + 2z = 5$ (c) $x = 0$.

18 N is perpendicular to a plane and V is along a line. Draw the angle θ between the plane and the line, and explain why $\mathbf{V}\cdot\mathbf{N}/|\mathbf{V}||\mathbf{N}|$ is $\sin\theta$ not $\cos\theta$. Find the angle between the xy plane and $\mathbf{V}=\mathbf{i}+\mathbf{j}+\sqrt{2}\mathbf{k}$.

In 19–26 find the projection P of B along A. Also find |P|.

19 $\mathbf{A}=(4, 2, 4)$, $\mathbf{B}=(1, -1, 0)$

20 $\mathbf{A}=(1, -1, 0)$, $\mathbf{B}=(4, 2, 4)$

21 $\mathbf{B}=$ unit vector at 60° angle with $\mathbf{A}$

22 $\mathbf{B}=$ vector of length 2 at 60° angle with $\mathbf{A}$

23 $\mathbf{B}=-\mathbf{A}$ **24** $\mathbf{A}=\mathbf{i}+\mathbf{j}$, $\mathbf{B}=\mathbf{i}+\mathbf{k}$

25 $\mathbf{A}$ is perpendicular to $x-y+z=0$, $\mathbf{B}=\mathbf{i}+\mathbf{j}$.

26 $\mathbf{A}$ is perpendicular to $x-y+z=5$, $\mathbf{B}=\mathbf{i}+\mathbf{j}+5\mathbf{k}$.

27 The force $\mathbf{F}=3\mathbf{i}-4\mathbf{k}$ acts at the point (1, 2, 2). How much force pulls toward the origin? How much force pulls vertically down? Which direction does a mass move under the force $\mathbf{F}$?

28 The projection of $\mathbf{B}$ along $\mathbf{A}$ is $\mathbf{P}=$ ________. The projection of $\mathbf{B}$ perpendicular to $\mathbf{A}$ is ________. Check the dot product of the two projections.

29 $\mathbf{P}=(x, y, z)$ is on the plane $ax+by+cz=5$ if $\mathbf{P}\cdot\mathbf{N}=|\mathbf{P}||\mathbf{N}|\cos\theta=5$. Since the largest value of $\cos\theta$ is 1, the smallest value of $|\mathbf{P}|$ is ________. This is the distance between ________.

30 If the air speed of a jet is 500 and the wind speed is 50, what information do you need to compute the jet's speed over land? What is that speed?

31 How far is the plane $x+y-z=1$ from (0, 0, 0) and also from (1, 1, −1)? Find the nearest points.

32 Describe all points at a distance 1 from the plane $x+2y+2z=3$.

33 The shortest distance from $Q=(2, 1, 1)$ to the plane $x+y+z=0$ is along the vector ________. The point $\mathbf{P}=\mathbf{Q}+t\mathbf{N}=(2+t, 1+t, 1+t)$ lies on the plane if $t=$ ________. Then $\mathbf{P}=$ ________ and the shortest distance is ________. (This distance is not $|\mathbf{P}|$.)

34 The plane through (1, 1, 1) perpendicular to $\mathbf{N}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is a distance ________ from (0, 0, 0).

35 (*Distance between planes*) $2x-2y+z=1$ is parallel to $2x-2y+z=3$ because ________. Choose a vector $\mathbf{Q}$ on the first plane and find t so that $\mathbf{Q}+t\mathbf{N}$ lies on the second plane. The distance is $|t\mathbf{N}|=$ ________.

36 The distance between the planes $x+y+5z=7$ and $3x+2y+z=1$ is zero because ________.

In Problems 37–41 all points and vectors are in the xy plane.

37 The line $3x+4y=10$ is perpendicular to the vector $\mathbf{N}=$ ________. On the line, the closest point to the origin is $P=t\mathbf{N}$. Find t and P and $|P|$.

38 Draw the line $x+2y=4$ and the vector $\mathbf{N}=\mathbf{i}+2\mathbf{j}$. The closest point to $Q=(3, 3)$ is $P=Q+t\mathbf{N}$. Find t. Find P.

39 A new way to find P in Problem 37: minimize $x^2+y^2=x^2+(\frac{10}{4}-\frac{3}{4}x)^2$. By calculus find the best x and y.

40 To catch a drug runner going from (0, 0) to (4, 0) at 8 meters per second, you must travel from (0, 3) to (4, 0) at ________ meters per second. The projection of your velocity vector onto his velocity vector will have length ________.

41 Show by vectors that the distance from (x_1, y_1) to the line $ax+by=d$ is $|d-ax_1-by_1|/\sqrt{a^2+b^2}$.

42 It takes three points to determine a plane. So why does $ax+by+cz=d$ contain four numbers a, b, c, d? When does $ex+fy+gz=1$ represent the same plane?

43 (projections by calculus) The dot product of $\mathbf{B}-t\mathbf{A}$ with itself is $|\mathbf{B}|^2-2t\mathbf{A}\cdot\mathbf{B}+t^2|\mathbf{A}|^2$. (a) This has a minimum at $t=$ ________. (b) Then $t\mathbf{A}$ is the projection of ________. A figure showing $\mathbf{B}$, $t\mathbf{A}$, and $\mathbf{B}-t\mathbf{A}$ is worth 1000 words.

44 From their equations, how can you tell if two planes are (a) parallel (b) perpendicular (c) at a 45° angle?

Problems 45–48 are about the ECG and heart vector.

45 The aVR lead is $-\frac{1}{2}\mathbf{L}_{\mathrm{I}}-\frac{1}{2}\mathbf{L}_{\mathrm{II}}$. Find the aVL and aVF leads toward the left arm and foot. Show that $\mathrm{aVR}+\mathrm{aVL}+\mathrm{aVF}=\mathbf{0}$. They go out from the center at 120° angles.

46 Find the projection on the aVR lead of $\mathbf{V}=2\mathbf{i}-\mathbf{j}$ in Question 3.

47 If the potentials are $\varphi_{\mathrm{RA}}=1$ (right arm) and $\varphi_{\mathrm{LA}}=2$ and $\varphi_{\mathrm{LL}}=-3$, find the heart vector $\mathbf{V}$. The *differences* in potential are the projections of $\mathbf{V}$.

48 If $\mathbf{V}$ is perpendicular to a lead $\mathbf{L}$, the reading on that lead is ________. If $\int\mathbf{V}(t)dt$ is perpendicular to lead $\mathbf{L}$, why is the *area* under the reading zero?

11.3 Cross Products and Determinants

After saying that vectors are not multiplied, we offered the dot product. Now we contradict ourselves further, by defining the cross product. Where $\mathbf{A} \cdot \mathbf{B}$ was a number, ***the cross product*** $\mathbf{A} \times \mathbf{B}$ ***is a vector***. It has length and direction:

The length is $|\mathbf{A}||\mathbf{B}||\sin\theta|$. ***The direction is perpendicular to*** $\mathbf{A}$ ***and*** $\mathbf{B}$.

The cross product (also called vector product) is defined in three dimensions only. $\mathbf{A}$ and $\mathbf{B}$ lie on a plane through the origin. $\mathbf{A} \times \mathbf{B}$ is along the normal vector $\mathbf{N}$, perpendicular to that plane. We still have to say whether it points "up" or "down" along $\mathbf{N}$.

The length of $\mathbf{A} \times \mathbf{B}$ depends on $\sin\theta$, where $\mathbf{A} \cdot \mathbf{B}$ involved $\cos\theta$. The dot product rewards vectors for being parallel ($\cos 0 = 1$). The cross product is largest when $\mathbf{A}$ is perpendicular to $\mathbf{B}$ ($\sin \pi/2 = 1$). At every angle

$$|\mathbf{A} \cdot \mathbf{B}|^2 + |\mathbf{A} \times \mathbf{B}|^2 = |\mathbf{A}|^2|\mathbf{B}|^2 \cos^2\theta + |\mathbf{A}|^2|\mathbf{B}|^2 \sin^2\theta = |\mathbf{A}|^2|\mathbf{B}|^2. \tag{1}$$

That will be a bridge from geometry to algebra. ***This section goes from definition to formula to volume to determinant.*** Equations (6) and (14) are the key formulas for $\mathbf{A} \times \mathbf{B}$.

Notice that $\mathbf{A} \times \mathbf{A} = \mathbf{0}$. (This is the zero vector, not the zero number.) When $\mathbf{B}$ is parallel to $\mathbf{A}$, the angle is zero and the sine is zero. Parallel vectors have $\mathbf{A} \times \mathbf{B} = \mathbf{0}$. Perpendicular vectors have $\sin\theta = 1$ and $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| =$ area of rectangle with sides $\mathbf{A}$ and $\mathbf{B}$.

Here are four examples that lead to the cross product $\mathbf{A} \times \mathbf{B}$.

EXAMPLE 1 (From geometry) Find the area of a parallelogram and a triangle.

Vectors $\mathbf{A}$ and $\mathbf{B}$, going out from the origin, form two sides of a triangle. They produce the parallelogram in Figure 11.13, which is twice as large as the triangle.

The area of a parallelogram is base times height (perpendicular height not sloping height). The base is $|\mathbf{A}|$. The height is $|\mathbf{B}||\sin\theta|$. We take absolute values because height and area are not negative. Then the area is the length of the cross product:

$$\textbf{\textit{area of parallelogram}} = |\mathbf{A}||\mathbf{B}||\sin\theta| = |\mathbf{A} \times \mathbf{B}|. \tag{2}$$

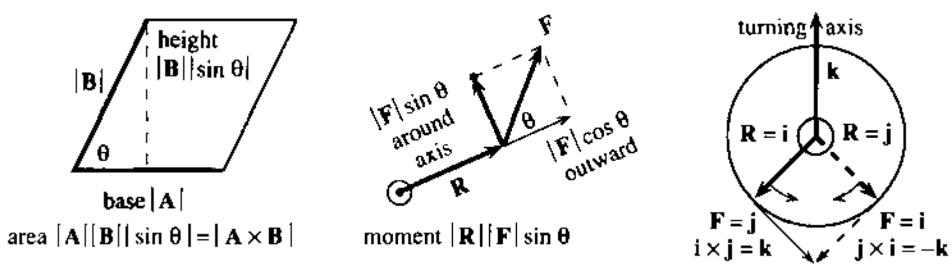


Fig. 11.13 Area $|\mathbf{A} \times \mathbf{B}|$ and moment $|\mathbf{R} \times \mathbf{F}|$. Cross products are perpendicular to the page.

EXAMPLE 2 (From physics) The torque vector $\mathbf{T} = \mathbf{R} \times \mathbf{F}$ produces rotation.

The force F acts at the point (x, y, z). When $\mathbf{F}$ is parallel to the position vector $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the force pushes outward (*no turning*). When $\mathbf{F}$ is perpendicular to $\mathbf{R}$, the force creates *rotation*. For in-between angles there is an outward force $|\mathbf{F}|\cos\theta$ and a turning force $|\mathbf{F}|\sin\theta$. The turning force times the distance $|\mathbf{R}|$ is the ***moment*** $|\mathbf{R}||\mathbf{F}|\sin\theta$.

The moment gives the magnitude and sign of the ***torque vector*** $\mathbf{T} = \mathbf{R} \times \mathbf{F}$. The direction of $\mathbf{T}$ is along the axis of rotation, at right angles to $\mathbf{R}$ and $\mathbf{F}$.

EXAMPLE 3 Does the cross product go up or down? *Use the right-hand rule.*

Forces and torques are probably just fine for physicists. Those who are not natural physicists want to see something turn.† We can visualize a record or compact disc rotating around its axis—which comes up through the center.

At a point on the disc, you give a push. When the push is outward (hard to do), nothing turns. Rotation comes from force "around" the axis. The disc can turn either way—depending on the angle between force and position. A sign convention is necessary, and it is the ***right-hand rule***:

$\mathbf{A} \times \mathbf{B}$ *points along your right thumb when the fingers curl from* $\mathbf{A}$ *toward* $\mathbf{B}$.

This rule is simplest for the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ in Figure 11.14—which is all we need.

Suppose the fingers curl from $\mathbf{i}$ to $\mathbf{j}$. The thumb points along $\mathbf{k}$. ***The*** x-y-z ***axes form a* "right-handed triple."** Since $|\mathbf{i}| = 1$ and $|\mathbf{j}| = 1$ and $\sin \pi/2 = 1$, the length of $\mathbf{i} \times \mathbf{j}$ is 1. ***The cross product is*** $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. The disc turns counterclockwise—its angular velocity is up—when the force acts at $\mathbf{i}$ in the direction $\mathbf{j}$.

Figure 11.14b reverses $\mathbf{i}$ and $\mathbf{j}$. The force acts at $\mathbf{j}$ and its direction is $\mathbf{i}$. The disc turns clockwise (the way records and compact discs actually turn). When the fingers curl from $\mathbf{j}$ to $\mathbf{i}$, the thumb points *down*. Thus $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$. This is a special case of an amazing rule:

$$\textit{The cross product is anticommutative}: \ \mathbf{B} \times \mathbf{A} = -(\mathbf{A} \times \mathbf{B}). \tag{3}$$

That is quite remarkable. Its discovery by Hamilton produced an intellectual revolution in 19th century algebra, which had been totally accustomed to $AB = BA$. This commutative law is old and boring for numbers (it is new and boring for dot products). Here we see its *opposite* for vector products $\mathbf{A} \times \mathbf{B}$. Neither law holds for matrix products.

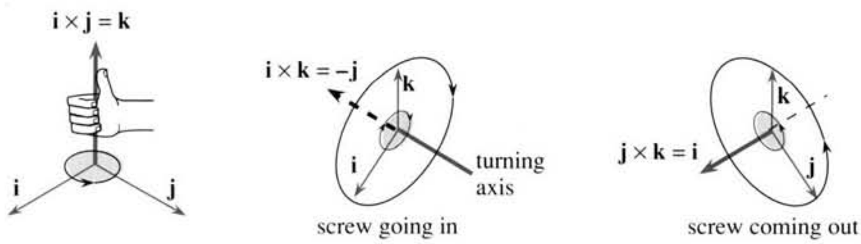


Fig. 11.14 $\mathbf{i} \times \mathbf{j} = \mathbf{k} = -(\mathbf{j} \times \mathbf{i}) \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j} = -(\mathbf{k} \times \mathbf{i}) \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} = -(\mathbf{k} \times \mathbf{j})$.

EXAMPLE 4 A screw goes into a wall or out, following the right-hand rule.

The disc was in the xy plane. So was the force. (We are not breaking records here.) The axis was up and down. To see the cross product more completely we need to turn a screw into a wall.

Figure 11.14b shows the xz plane as the wall. The screw is in the y direction. By turning from x toward z we drive the screw *into* the wall—which is the *negative* y direction. In other words $\mathbf{i} \times \mathbf{k}$ equals *minus* $\mathbf{j}$. We turn the screw clockwise to make it go in. To take out the screw, twist from $\mathbf{k}$ toward $\mathbf{i}$. Then $\mathbf{k} \times \mathbf{i}$ equals *plus* $\mathbf{j}$.

†Everybody is a natural mathematician. That is the axiom behind this book.

To summarize: $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ have plus signs because **kij** and **jki** are in the same "***cyclic order***" as **ijk**. (*Anticyclic is minus.*) The z–x–y and y–z–x axes form right-handed triples like x–y–z.

THE FORMULA FOR THE CROSS PRODUCT

We begin the algebra of $\mathbf{A} \times \mathbf{B}$. It is essential for computation, and it comes out beautifully. The square roots in $|\mathbf{A}|\,|\mathbf{B}|\,|\sin\theta|$ will disappear in formula (6) for $\mathbf{A} \times \mathbf{B}$. (The square roots also disappeared in $\mathbf{A} \cdot \mathbf{B}$, which is $|\mathbf{A}|\,|\mathbf{B}|\cos\theta$. But $|\mathbf{A}|\,|\mathbf{B}|\tan\theta$ would be terrible.) Since $\mathbf{A} \times \mathbf{B}$ is a vector we need to find *three components.*

Start with the two-dimensional case. The vectors $a_1\mathbf{i} + a_2\mathbf{j}$ and $b_1\mathbf{i} + b_2\mathbf{j}$ are in the xy plane. Their cross product must go in the z direction. Therefore $\mathbf{A} \times \mathbf{B} = \underline{\ \ ?\ \ }\,\mathbf{k}$ and there is only one nonzero component. It must be $|\mathbf{A}|\,|\mathbf{B}|\sin\theta$ (with the correct sign), but we want a better formula. There are two clean ways to compute $\mathbf{A} \times \mathbf{B}$, either by algebra (***a***) or by a bridge (***b***) to the dot product and geometry:

$$(a)\quad (a_1\mathbf{i} + a_2\mathbf{j}) \times (b_1\mathbf{i} + b_2\mathbf{j}) = a_1b_1\,\mathbf{i} \times \mathbf{i} + a_1b_2\,\mathbf{i} \times \mathbf{j} + a_2b_1\,\mathbf{j} \times \mathbf{i} + a_2b_2\,\mathbf{j} \times \mathbf{j}. \tag{4}$$

On the right are $\mathbf{0}$, $a_1b_2\mathbf{k}$, $-a_2b_1\mathbf{k}$, and $\mathbf{0}$. ***The cross product is*** $(a_1b_2 - a_2b_1)\mathbf{k}$.

(***b***) Rotate $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$ clockwise through $90°$ into $\mathbf{B}^* = b_2\mathbf{i} - b_1\mathbf{j}$. Its length is unchanged (and $\mathbf{B} \cdot \mathbf{B}^* = 0$). Then $|\mathbf{A}|\,|\mathbf{B}|\sin\theta$ equals $|\mathbf{A}|\,|\mathbf{B}^*|\cos\theta$, which is $\mathbf{A} \cdot \mathbf{B}^*$:

$$|\mathbf{A}|\,|\mathbf{B}|\sin\theta = \mathbf{A} \cdot \mathbf{B}^* = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cdot \begin{bmatrix} b_2 \\ -b_1 \end{bmatrix} = a_1b_2 - a_2b_1. \tag{5}$$

> **11F** In the xy plane, $\mathbf{A} \times \mathbf{B}$ equals $(a_1b_2 - a_2b_1)\mathbf{k}$. The parallelogram with sides **A** and **B** has area $|a_1b_2 - a_2b_1|$. The triangle OAB has area $\frac{1}{2}|a_1b_2 - a_2b_1|$.

EXAMPLE 5 For $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{B} = 4\mathbf{i} + 5\mathbf{j}$ the cross product is $(1 \cdot 5 - 2 \cdot 4)\mathbf{k} = -3\mathbf{k}$. Area of parallelogram $= 3$, area of triangle $= 3/2$. The minus sign in $\mathbf{A} \times \mathbf{B} = -3\mathbf{k}$ is absent in the areas.

Note Splitting $\mathbf{A} \times \mathbf{B}$ into four separate cross products is correct, but it does not follow easily from $|\mathbf{A}|\,|\mathbf{B}|\sin\theta$. Method (***a***) is not justified until Remark 1 below. An algebraist would change the definition of $\mathbf{A} \times \mathbf{B}$ to start with the distributive law (splitting rule) and the anticommutative law:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) + (\mathbf{A} \times \mathbf{C}) \quad \text{and} \quad \mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A}).$$

THE CROSS PRODUCT FORMULA (3 COMPONENTS)

We move to three dimensions. The goal is to compute all three components of $\mathbf{A} \times \mathbf{B}$ (not just the length). Method (***a***) splits each vector into its **i**, **j**, **k** components, making nine separate cross products:

$$(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) = a_1b_2(\mathbf{i} \times \mathbf{i}) + a_1b_2(\mathbf{i} \times \mathbf{j}) + \text{seven more terms.}$$

Remember $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$. Those three terms disappear. The other six terms come in pairs, and ***please notice the cyclic pattern***:

$$\textbf{FORMULA} \quad \mathbf{A} \times \mathbf{B} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}. \tag{6}$$

The **k** component is the 2×2 answer, when $a_3 = b_3 = 0$. The **i** component involves indices 2 and 3, **j** involves 3 and 1, **k** involves 1 and 2. The cross product formula is written as a "determinant" in equation (14) below—many people use that form to compute $\mathbf{A} \times \mathbf{B}$.

EXAMPLE 6 $(\mathbf{i}+2\mathbf{j}+3\mathbf{k}) \times (4\mathbf{i}+5\mathbf{j}+6\mathbf{k}) = (2 \cdot 6 - 3 \cdot 5)\mathbf{i} + (3 \cdot 4 - 1 \cdot 6)\mathbf{j} + (1 \cdot 5 - 2 \cdot 4)\mathbf{k}$. The $\mathbf{i}, \mathbf{j}, \mathbf{k}$ components give $\mathbf{A} \times \mathbf{B} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$. Never add the -3, 6, and -3.

Remark* 1** The three-dimensional formula (6) is still to be matched with $\mathbf{A} \times \mathbf{B}$ from geometry. One way is to rotate $\mathbf{B}$ into $\mathbf{B}^*$ as before, staying in the plane of $\mathbf{A}$ and $\mathbf{B}$. Fortunately there is an easier test. The vector in equation (6) satisfies all four geometric requirements on $\mathbf{A} \times \mathbf{B}$: ***perpendicular to* A**, ***perpendicular to* B**, ***correct length, ***right-hand rule***. The length is checked in Problem 16—here is the zero dot product with $\mathbf{A}$:

$$\mathbf{A} \cdot (\mathbf{A} \times \mathbf{B}) = a_1(a_2b_3 - a_3b_2) + a_2(a_3b_1 - a_1b_3) + a_3(a_1b_2 - a_2b_1) = 0. \tag{7}$$

***Remark* 2** (Optional) There is a wonderful extension of the Pythagoras formula $a^2 + b^2 = c^2$. Instead of sides of a triangle, we go to *areas of projections* on the yz, xz, and xy planes. $3^2 + 6^2 + 3^2$ is the square of the parallelogram area in Example 6.

For triangles these areas are cut in half. Figure 11.15a shows three projected triangles of area $\frac{1}{2}$. Its Pythagoras formula is $(\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 = (\text{area of } PQR)^2$.

EXAMPLE 7 $P = (1, 0, 0)$, $Q = (0, 1, 0)$, $R = (0, 0, 1)$ lie in a plane. Find its equation.

***Idea for any* P, Q, R: *Find vectors* A *and* B *in the plane. Compute the normal* $\mathbf{N} = \mathbf{A} \times \mathbf{B}$.**

Solution The vector from P to Q has components $-1, 1, 0$. It is $\mathbf{A} = \mathbf{j} - \mathbf{i}$ (subtract to go from P to Q). Similarly the vector from P to R is $\mathbf{B} = \mathbf{k} - \mathbf{i}$. Since $\mathbf{A}$ and $\mathbf{B}$ are in the plane of Figure 11.15, $\mathbf{N} = \mathbf{A} \times \mathbf{B}$ is perpendicular:

$$(\mathbf{j} - \mathbf{i}) \times (\mathbf{k} - \mathbf{i}) = (\mathbf{j} \times \mathbf{k}) - (\mathbf{i} \times \mathbf{k}) - (\mathbf{j} \times \mathbf{i}) + (\mathbf{i} \times \mathbf{i}) = \mathbf{i} + \mathbf{j} + \mathbf{k}. \tag{8}$$

***The normal vector is* $\mathbf{N} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. *The equation of the plane is* $1x + 1y + 1z = d$.**

With the right choice $d = 1$, this plane contains P, Q, R. The equation is $x + y + z = 1$.

EXAMPLE 8 What is the area of this same triangle PQR?

Solution The area is half of the cross-product length $|\mathbf{A} \times \mathbf{B}| = |\mathbf{i} + \mathbf{j} + \mathbf{k}| = \sqrt{3}$.

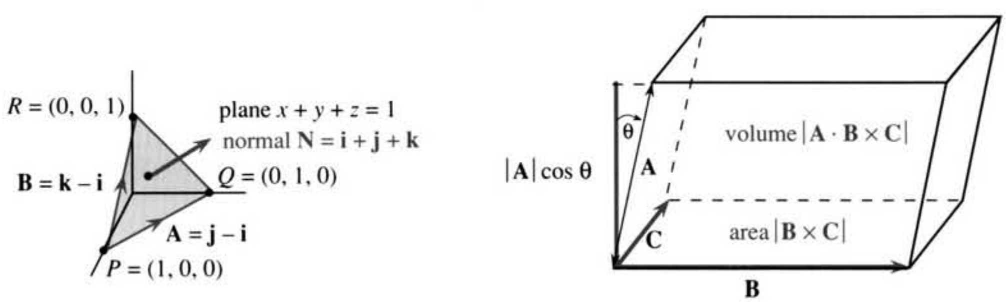


Fig. 11.15 Area of PQR is $\sqrt{3}/2$. $\mathbf{N}$ is $PQ \times PR$. ***Volume of box is*** $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.

DETERMINANTS AND VOLUMES

We are close to good algebra. The two plane vectors $a_1\mathbf{i} + a_2\mathbf{j}$ and $b_1\mathbf{i} + b_2\mathbf{j}$ are the sides of a parallelogram. Its area is $a_1b_2 - a_2b_1$, possibly with a sign change. There is a special way to write these four numbers—in a "***square matrix***." There is also a name for the combination that leads to area. It is the "***determinant of the matrix***":

$$\textit{The matrix is } \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, \quad \textit{its determinant is } \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

This is a 2 by 2 matrix (notice brackets) and a 2 by 2 determinant (notice vertical bars). The matrix is an array of four numbers and the determinant is one number:

$$\text{Examples of determinants:} \quad \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} = 6 - 4 = 2, \quad \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

The second has no area because $\mathbf{A} = \mathbf{B}$. The third is a unit square ($\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{j}$).

Now move to three dimensions, where determinants are most useful. The parallelogram becomes a parallelepiped. The word "box" is much shorter, and we will use it, but remember that *the box is squashed*. (Like a rectangle squashed to a parallelogram, the angles are generally not 90°.) The three edges from the origin are $\mathbf{A} = (a_1, a_2, a_3)$, $\mathbf{B} = (b_1, b_2, b_3)$, $\mathbf{C} = (c_1, c_2, c_3)$. Those edges are at right angles only when $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C} = \mathbf{B} \cdot \mathbf{C} = 0$.

Question: ***What is the volume of the box***? The right-angle case is easy—it is length times width times height. The volume is $|\mathbf{A}|$ times $|\mathbf{B}|$ times $|\mathbf{C}|$, when the angles are 90°. For a squashed box (Figure 11.15) we need the perpendicular height, not the sloping height.

There is a beautiful formula for volume. $\mathbf{B}$ and $\mathbf{C}$ give a parallelogram in the base, and $|\mathbf{B} \times \mathbf{C}|$ is the base area. This cross product points straight up. The third vector $\mathbf{A}$ points up at an angle—its perpendicular height is $|\mathbf{A}| \cos \theta$. Thus the volume is area $|\mathbf{B} \times \mathbf{C}|$ times $|\mathbf{A}|$ times $\cos \theta$. ***The volume is the dot product of* A *with* B × C.**

11G The ***triple scalar product*** is $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. ***Volume of box*** $= |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.

Important: $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is a number, not a vector. This volume is zero when $\mathbf{A}$ is in the same plane as $\mathbf{B}$ and $\mathbf{C}$ (the box is totally flattened). Then $\mathbf{B} \times \mathbf{C}$ is perpendicular to $\mathbf{A}$ and their dot product is zero.

$$\textit{Useful facts:} \quad \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}).$$

All those come from the same box, with different sides chosen as base—but no change in volume. Figure 11.15 has $\mathbf{B}$ and $\mathbf{C}$ in the base but it can be $\mathbf{A}$ and $\mathbf{B}$ or $\mathbf{A}$ and $\mathbf{C}$. ***The triple product*** $\mathbf{A} \cdot (\mathbf{C} \times \mathbf{B})$ ***has opposite sign***, since $\mathbf{C} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{C})$. This order **ACB** is not cyclic like **ABC** and **CAB** and **BCA**.

To compute this triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$, we take $\mathbf{B} \times \mathbf{C}$ from equation (6):

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1). \tag{9}$$

The numbers a_1, a_2, a_3 multiply 2 by 2 determinants to give a 3 by 3 determinant! There are three terms with plus signs (like $a_1b_2c_3$). The other three have minus signs (like $-a_1b_3c_2$). The plus terms have indices 123, 231, 312 in cyclic order. The minus terms have anticyclic indices 132, 213, 321. Again there is a special way to write the nine components of **A**, **B**, **C**—as a "3 by 3 matrix." The combination in (9), which gives volume, is a "3 by 3 determinant:"

$$\textbf{matrix} = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}, \quad \textbf{determinant} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

A single number is produced out of nine numbers, by formula (9). The nine numbers are multiplied three at a time, as in $a_1b_1c_2$—except this product is not allowed. ***Each row and column must be represented once.*** This gives the six terms in the determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{matrix} a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \\ - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 \end{matrix} \tag{10}$$

The trick is in the $\pm$ signs. Products down to the right are "*plus*":

$$\begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{matrix} 2\cdot2\cdot2 + 1\cdot1\cdot1 + 1\cdot1\cdot1 \\ -2\cdot1\cdot1 - 1\cdot1\cdot2 - 1\cdot2\cdot1 \end{matrix} = \begin{matrix} 8+1+1 \\ -2-2-2 \end{matrix} = 4.$$

With practice the six products like $2\cdot2\cdot2$ are done in your head. Write down only $8+1+1-2-2-2=4$. This is the determinant and the volume.

Note the special case when the vectors are **i**, **j**, **k**. The box is a unit cube:

$$\textbf{volume of cube} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{matrix} 1+0+0 \\ -0-0-0 \end{matrix} = 1.$$

If* A, B, C *lie in the same plane, the volume is zero. A zero determinant is the test to see whether three vectors lie in a plane. Here row **A** = row **B** − row **C**:

$$\begin{vmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{matrix} 0\cdot1\cdot1 + 1\cdot0\cdot(-1) + (-1)\cdot(-1)\cdot0 \\ -0\cdot0\cdot0 - 1\cdot(-1)\cdot1 - (-1)\cdot1\cdot(-1) \end{matrix} = 0. \tag{11}$$

Zeros in the matrix simplify the calculation. All three products with plus signs—down to the right—are zero. The only two nonzero products cancel each other.

If the three -1's are changed to $+1$'s, the determinant is -2. The determinant can be negative when all nine entries are positive! A negative determinant only means that the rows **A**, **B**, **C** form a "left-handed triple." This extra information from the sign—right-handed vs. left-handed—is free and useful, but the volume is the absolute value.

The determinant yields the volume also in higher dimensions. In physics, four dimensions give space-time. Ten dimensions give superstrings. Mathematics uses all dimensions. The 64 numbers in an 8 by 8 matrix give the volume of an eight-dimensional box—with $8! = 40{,}320$ terms instead of $3! = 6$. Under pressure from my class I omit the formula.

Question When is the point (x, y, z) on the plane through the origin containing **B** and **C**? For the vector $\mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ to lie in that plane, the volume $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ must be zero. The equation of the plane is ***determinant = zero***.

Follow this example for $\mathbf{B} = \mathbf{j} - \mathbf{i}$ and $\mathbf{C} = \mathbf{k} - \mathbf{i}$ to find the plane parallel to **B** and **C**:

$$\begin{vmatrix} x & y & z \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \begin{matrix} x \cdot 1 \cdot 1 + y \cdot 0 \cdot (-1) + z \cdot 0 \cdot (-1) \\ -x \cdot 0 \cdot 0 - y \cdot 1 \cdot (-1) - z \cdot 1 \cdot (-1) \end{matrix} = 0. \tag{12}$$

This equation is $x + y + z = 0$. The normal vector $\mathbf{N} = \mathbf{B} \times \mathbf{C}$ has components 1, 1, 1.

THE CROSS PRODUCT AS A DETERMINANT

There is a connection between 3 by 3 and 2 by 2 determinants that you have to see. The numbers in the top row multiply determinants from the other rows:

$$\begin{vmatrix} \underline{a_1} & a_2 & a_3 \\ b_1 & \underline{b_2} & \underline{b_3} \\ c_1 & \underline{c_2} & \underline{c_3} \end{vmatrix} = \underline{a_1} \begin{vmatrix} \underline{b_2} & \underline{b_3} \\ \underline{c_2} & \underline{c_3} \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}. \tag{13}$$

The highlighted product $a_1(b_2c_3 - b_3c_2)$ gives two of the six terms. ***All six products contain an a and b and c from different columns.*** There are $3! = 6$ different orderings of columns 1, 2, 3. Note how a_3 multiplies a determinant from columns 1 and 2.

Equation (13) is identical with equations (9) and (10). We are meeting the same six terms in different ways. The new feature is the minus sign in front of a_2—and the common mistake is to forget that sign. In a 4 by 4 determinant, $a_1, -a_2, a_3, -a_4$ would multiply 3 by 3 determinants.

Now comes a key step. We write $\mathbf{A} \times \mathbf{B}$ as a determinant. The vectors **i, j, k** go in the top row, the components of **A** and **B** go in the other rows. ***The "determinant" is exactly*** $\mathbf{A} \times \mathbf{B}$:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \underline{a_1} & a_2 & \underline{a_3} \\ \underline{b_1} & b_2 & \underline{b_3} \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} \underline{a_1} & \underline{a_3} \\ \underline{b_1} & \underline{b_3} \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}. \tag{14}$$

This time we highlighted the **j** component with its minus sign. There is no great mathematics in formula (14)—it is probably illegal to mix **i, j, k** with six numbers but it works. This is the good way to remember and compute $\mathbf{A} \times \mathbf{B}$. In the example $(\mathbf{j} - \mathbf{i}) \times (\mathbf{k} - \mathbf{i})$ from equation (8), those two vectors go into the last two rows:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \underline{\mathbf{k}} \\ \underline{-1} & \underline{1} & 0 \\ \underline{-1} & \underline{0} & 1 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 0 \\ -1 & 1 \end{vmatrix} + \underline{\mathbf{k}} \begin{vmatrix} \underline{-1} & \underline{1} \\ \underline{-1} & \underline{0} \end{vmatrix} = \mathbf{i} + \mathbf{j} + \underline{\mathbf{k}}.$$

The **k** component is highlighted, to see $a_1b_2 - a_2b_1$ again. Note the change from equation (11), which had 0, 1, -1 in the top row. That triple product was a number (zero). This cross product is a vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$.

Review question 1 With the **i**, **j**, **k** row changed to 3, 4, 5, what is the determinant?
Answer $3 \cdot 1 + 4 \cdot 1 + 5 \cdot 1 = 12$. That triple product is the volume of a box.

Review question 2 When is $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ and when is $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 0$? Zero vector, zero number.
Answer When **A** and **B** are on the same line. When **A**, **B**, **C** are in the same plane.

Review question 3 Does the parallelogram area $|\mathbf{A} \times \mathbf{B}|$ equal a 2 by 2 determinant?
Answer If **A** and **B** lie in the xy plane, *yes*. Generally *no*.

Review question 4 What are the *vector* triple products $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$?
Answer Not computed yet. These are two new vectors in Problem 47.

Review question 5 Find the plane through the origin containing $\mathbf{A} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + \mathbf{k}$. Find the cross product of those same vectors **A** and **B**.
Answer The position vector $\mathbf{P} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is perpendicular to $\mathbf{N} = \mathbf{A} \times \mathbf{B}$:

$$\mathbf{P} \cdot (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = x + y - z = 0. \qquad \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} - \mathbf{k}.$$

11.3 EXERCISES

Read-through questions

The cross product $\mathbf{A} \times \mathbf{B}$ is a __a__ whose length is __b__. Its direction is __c__ to **A** and **B**. That length is the area of a __d__, whose base is $|\mathbf{A}|$ and whose height is __e__. When $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$, the area is __f__. This equals a 2 by 2 __g__. In general $|\mathbf{A} \cdot \mathbf{B}|^2 + |\mathbf{A} \times \mathbf{B}|^2 =$ __h__.

The rules for cross products are $\mathbf{A} \times \mathbf{A} =$ __i__ and $\mathbf{A} \times \mathbf{B} = -($__j__$)$ and $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} +$ __k__. In particular $\mathbf{A} \times \mathbf{B}$ needs the __l__-hand rule to decide its direction. If the fingers curl from **A** towards **B** (not more than 180°), then __m__ points __n__. By this rule $\mathbf{i} \times \mathbf{j} =$ __o__ and $\mathbf{i} \times \mathbf{k} =$ __p__ and $\mathbf{j} \times \mathbf{k} =$ __q__.

The vectors $a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ have cross product __r__**i** + __s__**j** + __t__**k**. The vectors $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{B} = \mathbf{i} + \mathbf{j}$ have $\mathbf{A} \times \mathbf{B} =$ __u__. (*This is also the 3 by 3 determinant* __v__.) Perpendicular to the plane containing (0, 0, 0), (1, 1, 1), (1, 1, 0) is the normal vector $\mathbf{N} =$ __w__. The area of the triangle with those three vertices is __x__, which is half the area of the parallelogram with fourth vertex at __y__.

Vectors **A**, **B**, **C** from the origin determine a __z__. Its volume $|\mathbf{A} \cdot ($__A__$)|$ comes from a 3 by 3 __B__. There are six terms, __C__ with a plus sign and __D__ with minus. In every term each row and __E__ is represented once. The rows (1, 0, 0), (0, 0, 1), and (0, 1, 0) have determinant = __F__. That box is a __G__, but its sides form a __H__-handed triple in the order given.

If **A**, **B**, **C** lie in the same plane then $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is __I__. For $\mathbf{A} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ the first row contains the letters __J__. So the plane containing **B** and **C** has the equation __K__ = 0. When $\mathbf{B} = \mathbf{i} + \mathbf{j}$ and $\mathbf{C} = \mathbf{k}$ that equation is __L__. $\mathbf{B} \times \mathbf{C}$ is __M__.

A 3 by 3 determinant splits into __N__ 2 by 2 determinants. They come from rows 2 and 3, and are multiplied by the entries in row 1. With **i**, **j**, **k** in row 1, this determinant equals the __O__ product. Its **j** component is __P__, including the __Q__ sign which is easy to forget.

Compute the cross products 1–8 from formula (6) or the determinant (14). Do one example both ways.

1 $(\mathbf{i} \times \mathbf{j}) \times \mathbf{k}$

2 $(\mathbf{i} \times \mathbf{j}) \times \mathbf{i}$

3 $(2\mathbf{i} + 3\mathbf{j}) \times (\mathbf{i} + \mathbf{k})$

4 $(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \times (2\mathbf{i} + 3\mathbf{j} - \mathbf{k})$

5 $(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) \times (\mathbf{i} - \mathbf{j} - \mathbf{k})$

6 $(\mathbf{i} + \mathbf{j} - \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k})$

7 $(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \times (4\mathbf{i} - 9\mathbf{j})$

8 $(\mathbf{i} \cos \theta + \mathbf{j} \sin \theta) \times (\mathbf{i} \sin \theta - \mathbf{j} \cos \theta)$

9 When are $|\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}|$ and $|\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})| = |\mathbf{A}||\mathbf{B}||\mathbf{C}|$?

10 True or false:

(a) $\mathbf{A} \times \mathbf{B}$ never equals $\mathbf{A} \cdot \mathbf{B}$.

(b) If $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ and $\mathbf{A} \cdot \mathbf{B} = 0$, then either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

(c) If $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$ and $\mathbf{A} \neq \mathbf{0}$, then $\mathbf{B} = \mathbf{C}$.

In 11–16 find $|\mathbf{A} \times \mathbf{B}|$ by equation (1) and then by computing $\mathbf{A} \times \mathbf{B}$ and its length.

11 $\mathbf{A} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i}$

12 $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{i} - \mathbf{j}$

13 $\mathbf{A} = -\mathbf{B}$

14 $\mathbf{A} = \mathbf{i} + \mathbf{j}$, $\mathbf{B} = \mathbf{j} + \mathbf{k}$

15 $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j}$

16 $\mathbf{A} = (a_1, a_2, a_3)$, $\mathbf{B} = (b_1, b_2, b_3)$

In Problem 16 (the general case), equation (1) proves that the length from equation (6) is correct.

17 True or false, by testing on $\mathbf{A} = \mathbf{i}$, $\mathbf{B} = \mathbf{j}$, $\mathbf{C} = \mathbf{k}$:
(a) $\mathbf{A} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{0}$ (b) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}$
(c) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{B} \times \mathbf{A})$
(d) $(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B}) = 2(\mathbf{A} \times \mathbf{B})$.

18 (a) From $\mathbf{A} \times \mathbf{B} = -(\mathbf{B} \times \mathbf{A})$ deduce that $\mathbf{A} \times \mathbf{A} = \mathbf{0}$.
(b) Split $(\mathbf{A} + \mathbf{B}) \times (\mathbf{A} + \mathbf{B})$ into four terms, to deduce that $(\mathbf{A} \times \mathbf{B}) = -(\mathbf{B} \times \mathbf{A})$.

What are the normal vectors to the planes 19–22?

19 $(2, 1, 0) \cdot (x, y, z) = 4$

20 $3x + 4z = 5$

21 $\begin{vmatrix} x & y & z \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2$

22 $\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 0$

Find N and the equation of the plane described in 23–29.

23 Contains the points (2, 1, 1), (1, 2, 1), (1, 1, 2)

24 Contains the points (0, 1, 2), (1, 2, 3), (2, 3, 4)

25 Through (0, 0, 0), (1, 1, 1), (a, b, c) [What if $a = b = c$?]

26 Parallel to $\mathbf{i} + \mathbf{j}$ and $\mathbf{k}$

27 $\mathbf{N}$ makes a 45° angle with $\mathbf{i}$ and $\mathbf{j}$

28 $\mathbf{N}$ makes a 60° angle with $\mathbf{i}$ and $\mathbf{j}$

29 $\mathbf{N}$ makes a 90° angle with $\mathbf{i}$ and $\mathbf{j}$

30 The triangle with sides $\mathbf{i}$ and $\mathbf{j}$ is ________ as large as the parallelogram with those sides. The tetrahedron with edges $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is ________ as large as the box with those edges. Extra credit: In four dimensions the "simplex" with edges $\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}$ has volume = ________.

31 If the points (x, y, z), (1, 1, 0), and (1, 2, 1) lie on a plane through the origin, what determinant is zero? What equation does this give for the plane?

32 Give an example of a right-hand triple and left-hand triple. Use vectors other than just $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

33 When $\mathbf{B} = 3\mathbf{i} + \mathbf{j}$ is rotated 90° clockwise in the xy plane it becomes $\mathbf{B}^* =$ ________. When rotated 90° counterclockwise it is ________. When rotated 180° it is ________.

34 From formula (6) verify that $\mathbf{B} \cdot (\mathbf{A} \times \mathbf{B}) = 0$.

35 Compute

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{vmatrix}, \quad \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix}.$$

36 Which of the following are equal to $\mathbf{A} \times \mathbf{B}$?
$(\mathbf{A} + \mathbf{B}) \times \mathbf{B}$, $(-\mathbf{B}) \times (-\mathbf{A})$, $|\mathbf{A}||\mathbf{B}||\sin\theta|$, $(\mathbf{A} + \mathbf{C}) \times (\mathbf{B} - \mathbf{C})$, $\frac{1}{2}(\mathbf{A} - \mathbf{B}) \times (\mathbf{A} + \mathbf{B})$.

37 Compare the six terms on both sides to prove that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

The matrix is "*transposed*"—same determinant.

38 Compare the six terms to prove that

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -b_1\begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + b_2\begin{vmatrix} a_1 & a_3 \\ c_1 & c_3 \end{vmatrix} - b_3\begin{vmatrix} a_1 & a_2 \\ c_1 & c_2 \end{vmatrix}.$$

This is an "expansion on row 2." Note minus signs.

39 Choose the signs and 2 by 2 determinants in

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \pm c_1\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \pm c_2 \_\_\_\_\_\_\_\_ \pm c_3 \_\_\_\_\_\_\_\_.$$

40 Show that $(\mathbf{A} \times \mathbf{B}) + (\mathbf{B} \times \mathbf{C}) + (\mathbf{C} \times \mathbf{A})$ is perpendicular to $\mathbf{B} - \mathbf{A}$ and $\mathbf{C} - \mathbf{B}$ and $\mathbf{A} - \mathbf{C}$.

Problems 41–44 compute the areas of triangles.

41 The triangle PQR in Example 7 has squared area $(\sqrt{3}/2)^2 = (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2$, from the 3D version of Pythagoras in Remark 2. Find the area of PQR when $P = (a, 0, 0)$, $Q = (0, b, 0)$, and $R = (0, 0, c)$. Check with $\frac{1}{2}|\mathbf{A} \times \mathbf{B}|$.

42 A triangle in the xy plane has corners at (a_1, b_1), (a_2, b_2) and (a_3, b_3). Its area A is half the area of a parallelogram. Find two sides of the parallelogram and explain why

$$A = \tfrac{1}{2}|(a_2 - a_1)(b_3 - b_1) - (a_3 - a_1)(b_2 - b_1)|.$$

43 By Problem 42 find the area A of the triangle with corners (2, 1) and (4, 2) and (1, 2). Where is a fourth corner to make a parallelogram?

44 Lifting the triangle of Problem 42 up to the plane $z = 1$ gives corners $(a_1, b_1, 1)$, $(a_2, b_2, 1)$, $(a_3, b_3, 1)$. The area of the triangle times $\frac{1}{3}$ is the volume of the upside-down pyramid from (0, 0, 0) to these corners. This pyramid volume is $\frac{1}{6}$ the box volume, so $\frac{1}{3}$ (area of triangle) $= \frac{1}{6}$ (volume of box):

$$\textit{area of triangle} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}.$$

Find the area A in Problem 43 from this determinant.

45 (1) The projections of $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ onto the xy plane are ________.
(2) The parallelogram with sides $\mathbf{A}$ and $\mathbf{B}$ projects to a parallelogram with area ________.
(3) General fact: The projection onto the plane normal to the unit vector $\mathbf{n}$ has area $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n}$. Verify for $\mathbf{n} = \mathbf{k}$.

46 (a) For $\mathbf{A} = \mathbf{i} + \mathbf{j} - 4\mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + \mathbf{j}$, compute $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{i}$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{j}$ and $(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{k}$. By Problem 45 those are the areas of projections onto the yz and xz and xy planes.
(b) Square and add those areas to find $|\mathbf{A} \times \mathbf{B}|^2$. This is the Pythagoras formula in space (Remark 2).

47 (a) The triple cross product $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ is in the plane of $\mathbf{A}$ and $\mathbf{B}$, because it is perpendicular to the cross product ________.
(b) Compute $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ when $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$, $\mathbf{C} = \mathbf{i}$.
(c) Compute $(\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}$ when $\mathbf{C} = \mathbf{i}$. The answers in (b) and (c) should agree. This is also true if $\mathbf{C} = \mathbf{j}$ or $\mathbf{C} = \mathbf{k}$ or $\mathbf{C} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. That proves the tricky formula

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{B} \cdot \mathbf{C})\mathbf{A}. \qquad (*)$$

48 Take the dot product of equation (*) with $\mathbf{D}$ to prove

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{B} \cdot \mathbf{C})(\mathbf{A} \cdot \mathbf{D}).$$

49 The plane containing $P = (0, 1, 1)$ and $Q = (1, 0, 1)$ and $R = (1, 1, 0)$ is perpendicular to the cross product $\mathbf{N} =$ ________. Find the equation of the plane and the area of triangle PQR.

50 Let $P = (1, 0, -1)$, $Q = (1, 1, 1)$, $R = (2, 2, 1)$. Choose S so that $PQRS$ is a parallelogram and compute its area. Choose T, U, V so that $OPQRSTUV$ is a box (parallelepiped) and compute its volume.

11.4 Matrices and Linear Equations

We are moving from geometry to algebra. Eventually we get back to calculus, where functions are nonlinear—but linear equations come first. In Chapter 1, $y = mx + b$ produced a line. Two equations produce two lines. If they cross, the intersection point solves both equations—and we want to find it.

Three equations in three variables x, y, z produce three planes. Again they go through one point (*usually*). Again the problem is to find that intersection point —which solves the three equations.

The ultimate problem is to solve n equations in n unknowns. There are n hyperplanes in n-dimensional space, which meet at the solution. We need a test to be sure they meet. We also want the solution. These are the objectives of ***linear algebra***, which joins with calculus at the center of pure and applied mathematics.†

Like every subject, linear algebra requires a good notation. To state the equations and solve them, we introduce a "*matrix*." ***The problem will be*** $A\mathbf{u} = \mathbf{d}$. ***The solution will be*** $\mathbf{u} = A^{-1}\mathbf{d}$. It remains to understand where the equations come from, where the answer comes from, and what the matrices A and A^{-1} stand for.

TWO EQUATIONS IN TWO UNKNOWNS

Linear algebra has no reason to choose one variable as special. The equation $y - y_0 = m(x - x_0)$ separates y from x. A better equation for a line is $ax + by = d$. (A vertical

†Linear algebra dominates some applications while calculus governs others. Both are essential. A fuller treatment is presented in the author's book *Linear Algebra and Its Applications* (Harcourt Brace Jovanovich, 3rd edition 1988), and in many other texts.

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Resource: Calculus Online Textbook
Gilbert Strang

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