Massachusetts Institute of Technology

Department of Physics

Course: 8.701 – Introduction to Nuclear and Particle Physics
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Discussion Problems

from recitation on **October 7th, 2020**

Problem 1:

Show that the adjoint spinor $\bar{u}$ and $\bar{v}$ satisfy $\bar{u}(\gamma^\mu p_\mu - m) = 0$ and $\bar{v}(\gamma^\mu p_\mu + m) = 0$.

$$(\gamma^\mu p_\mu - mc)u = 0 \Longrightarrow u^\dagger(\gamma^{\mu\dagger} p_\mu - mc) = 0 \Longrightarrow u^\dagger(\gamma^{\mu\dagger}\gamma^0 p_\mu - \gamma^0 mc) = 0.$$

But $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$ (see below), so $u^\dagger\gamma^0(\gamma^\mu p_\mu - mc) = 0,$ or $\bar{u}(\gamma^\mu p_\mu - mc) = 0.$

Similarly, $(\gamma^\mu p_\mu + mc)v = 0 \Rightarrow \bar{v}(\gamma^\mu p_\mu + mc) = 0$ (same as above, with sign of m reversed).

Proof that $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$:

$$\gamma^{0\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0, \text{ so it holds for } \mu = 0.$$

$$(\gamma^i)^\dagger = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -(\sigma^i)^\dagger \\ (\sigma^i)^\dagger & 0 \end{pmatrix}.$$

But $(\sigma^i)^\dagger = \sigma^i$:

$$(\sigma^1)^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^1;$$

$$(\sigma^2)^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma^2;$$

$$(\sigma^3)^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^3.$$

So $(\gamma^i)^\dagger = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} = -\gamma^i$. But γ^i anticommutes with γ^0, so $\gamma^i\gamma^0 = -\gamma^0\gamma^i$.
Therefore $(\gamma^i)^\dagger\gamma^0 = -\gamma^i\gamma^0 = \gamma^0\gamma^i$. So it holds for $\mu = i = 1,2,3$ also. ✓

Problem 2:

Show that the normalization condition simplifies to $\bar{u}u = -\bar{v}v = 2m$.

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$$\bar{u}u = u^\dagger\gamma^0 u = N^2\left(u_A^\dagger\ u_B^\dagger\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}u_A\\ u_B\end{pmatrix} = N^2\left(u_A^\dagger u_A - u_B^\dagger u_B\right).$$

In particular, for $u^{(1)}$:

$$\begin{aligned}
\bar{u}u &= N^2\left[(1\ 0)\begin{pmatrix}1\\0\end{pmatrix} - \frac{c^2}{(E+mc^2)^2}\left(p_z\ (p_x - ip_y)\right)\begin{pmatrix}p_z\\ p_x + ip_y\end{pmatrix}\right]\\
&= N^2\left[1 - \frac{c^2}{(E+mc^2)^2}(p_z^2 + p_x^2 + p_y^2)\right] = \frac{N^2}{(E+mc^2)^2}\left[(E+mc^2)^2 - c^2\mathbf{p}^2\right]\\
&= \frac{(E+mc^2)}{c}\frac{1}{(E+mc^2)^2}\left[E^2 + 2Emc^2 + m^2c^4 - c^2\mathbf{p}^2\right]\\
&= \frac{1}{c(E+mc^2)}\left(2Emc^2 + 2m^2c^4\right) = \frac{1}{c(E+mc^2)}2mc^2(E+mc^2) = 2mc.\ \checkmark
\end{aligned}$$

$$\begin{aligned}
\bar{v}v &= N^2\left[\frac{c^2}{(E+mc^2)^2}\left((p_x + ip_y)\ -p_z\right)\begin{pmatrix}p_x - ip_y\\ -p_z\end{pmatrix} - (0\ 1)\begin{pmatrix}0\\1\end{pmatrix}\right]\\
&= N^2\left[\frac{c^2}{(E+mc^2)^2}(p_x^2 + p_y^2 + p_z^2) - 1\right] = \frac{N^2}{(E+mc^2)^2}\left[c^2\mathbf{p}^2 - (E+mc^2)^2\right]\\
&= \frac{(E+mc^2)}{c}\frac{1}{(E+mc^2)^2}\left[c^2\mathbf{p}^2 - E^2 - 2Emc^2 - m^2c^4\right]\\
&= \frac{-1}{c(E+mc^2)}\left(2Emc^2 + 2m^2c^4\right) = \frac{-1}{c(E+mc^2)}2mc^2(E+mc^2) = -2mc.\ \checkmark
\end{aligned}$$

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