8.251 Test

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Only personal 2 page notes allowed. Test duration: 60 minutes.

Problem 1. (10 points) Boundary conditions for open strings.
Consider two static D2-branes in four dimensional spacetime (ct, x^1, x^2, x^3). The first one is at $x^3 = 0$. The second one is parallel to the first and is located at $x^3 = a > 0$. Sketch the branes. Consider open strings with $\sigma \in [0, \sigma_1]$ that stretch from the second brane ($\sigma = 0$) to the first brane ($\sigma = \sigma_1$).
State the boundary conditions (Free or Dirichlet, with value) for the string coordinates $X^\mu(t, \sigma_*)$ (list the eight conditions – $\mu = 0, 1, 2, 3$ and $\sigma_* = 0, \sigma_1$).

Problem 2. (10 points) Spaces constructed by identifications.

Give a (simple!) fundamental domain $\mathcal{F}$ and describe the resulting space $\mathcal{M}$ for each of the following (single) identifications acting on the complex plane $z = x + iy$:

(a) $z \sim z + i$.

(b) $z \sim 2z$.

Problem 3. (15 points) Variation of an action
Consider the Chern-Simons action for three-dimensional electromagnetism:

$$S = \int dt \int d^2x \big(A_0 F_{12} + A_1 F_{20} + A_2 F_{01} \big) \,.$$

Recall that the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Find the equation of motion resulting from the variation of the gauge field component A_0 (as usual, ignore boundary terms). The equation of motion can be written fully in terms of field strengths.

Problem 4. (10 points) How heavy is a cosmic string?
A nearby relativistic cosmic string of tension T_0 produces a cylindrical gravitational lens in which two images of a single faraway source would be separate by an angle

$$\delta = 8\pi G T_0 \,. \tag{1}$$

This formula is given in units where c and $\hbar$ are set equal to one, the angle δ is measured in radians, and $G \simeq 6.7 \times 10^{-11}\, \mathrm{m}^3/\mathrm{kg}\cdot\mathrm{s}^2$ is Newton's constant ($c = 3 \times 10^8 \mathrm{m/s}$, $\hbar = 1.06 \times 10^{-34}$ kg.m^2/s.)

(a) Complete (1) by adding whatever factors of c and/or $\hbar$ are needed.

(b) A string produces the plausible value of $\delta = 0.5$ arc-seconds (degree = 60 arc-minutes, arc-minute = 60 arc-seconds). What is the linear mass density of such string in kg/m?

Problem 5. (20 points) Angular momentum of a rotating open string.

An open string of length ℓ and energy E rotates rigidly with angular velocity ω. Recall that $\omega\frac{\ell}{2} = c$ and $\ell = \frac{2}{\pi}\frac{E}{T_0}$.

(a) Introduce a radial length r along the string and let dr denote a small piece of string a distance r from the center. What is the magnitude dp of the (relativistic) momentum carried by this small piece of string ? What is the magnitude dJ of the angular momentum carried by this small piece of string? Both answers should be in terms of ω, T_0, r, dr and constants.

(b) Use integration to calculate the total angular momentum carried by the rotating string. Give your answer in terms of the energy E of the string and the string tension T_0.

Useful integral: $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^2}} = \frac{\pi}{4}$.

Problem 6. (25 points) Momentum of closed strings.

For a free closed string we have

$$\vec{X}(t,\sigma) = \frac{1}{2}\Big(\vec{F}(u) + \vec{G}(v)\Big), \quad \text{with} \quad u = ct + \sigma\,,\; v = ct - \sigma\,. \tag{1}$$

(a) Demonstrate that the periodicity condition $\sigma \sim \sigma + \sigma_1$ ($\sigma_1 = E/T_0$) relates the lack of periodicity of $\vec{F}(u)$ to the lack of periodicity of $\vec{G}(v)$.

(b) We now write

$$\vec{F}(u) = \vec{f}(u) + \vec{\alpha}\,u\,, \quad \text{and} \quad \vec{G}(v) = \vec{g}(v) + \vec{\beta}\,v\,,$$

where $\vec{f}$ and $\vec{g}$ are strictly periodic functions with period σ_1 and $\vec{\alpha}$ and $\vec{\beta}$ are constant vectors.

How does the result in (a) relate $\vec{\alpha}$ and $\vec{\beta}$?

Plug back in (1) to find $\vec{X}(t,\sigma)$ in terms of $\vec{f}(u), \vec{g}(v), \vec{\alpha}, ct$, and possibly σ.

(c) The momentum density (per unit σ) carried of the string is $\vec{\mathcal{P}}^\tau = \frac{T_0}{c^2}\frac{\partial \vec{X}}{\partial t}$. Calculate the total momentum $\vec{p}$ carried by the string in terms of the vector $\vec{\alpha}$ and other constants.

Problem 7. (10 points) You learned that a closed string stretched along a circle and released with zero initial velocity will contract to zero size at some later time. Consider a closed string that is stretched along an *ellipse* and is released with zero initial velocity. Will it contract to zero size? If yes, why? if not, why not? A complete answer requires a precise justification but, in fact, no calculation.