MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

8.251: String Theory for Undergraduates

Prof. Barton Zwiebach April 30, 2007

REVIEW NOTES FOR TEST 2

Notes by Alan Guth

I. World-Sheet Currents:

Noether's Theorem:

Suppose than an action of the form

$$S = \int d\xi^0 \, d\xi^1 \, \ldots \, d\xi^k \, \mathcal{L}(\phi^a, \partial_\alpha \phi^a) \tag{1.1}$$

is invariant under an infinitesimal variation of the fields

$$\phi^a(\xi) \to \phi^a(\xi) + \delta\phi^a(\xi) \ , \qquad \text{with } \delta\phi^a(\xi) = \epsilon^i h_i^a(\phi^a, \partial_\alpha \phi^a) \tag{1.2}$$

in the sense that the Lagrangian density is changed at most by a total derivative,

$$\delta\mathcal{L} = \frac{\partial}{\partial \xi^\alpha}\left(\epsilon^i \Lambda_i^\alpha\right) \ . \tag{1.3}$$

Then the currents $j_i^\alpha(\xi)$ defined by

$$\epsilon^i j_i^\alpha \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \, \delta\phi^a - \epsilon^i \Lambda_i^\alpha \tag{1.4}$$

are conserved:

$$\partial_\alpha j_i^\alpha = 0 \qquad \text{(for each } i\text{).} \tag{1.5}$$

This implies that the corresponding charges,

$$Q_i = \int d\xi^1 \, \ldots \, d\xi^k \, j_i^0(\xi) \ , \tag{1.6}$$

are independent of time.

The proof is constructed by replacing $\delta\mathcal{L}$ in Eq. (1.3) by an expansion in terms of the derivatives of $\mathcal{L}$ and the variation of the fields (1.2), and then using the Lagrangian equations of motion.

World-Sheet Application 1: Conservation of Momentum:

Identifying $\xi^0 \equiv \tau$ and $\xi^1 \equiv \sigma$, we can apply this theorem to the string world-sheet. Defining $\dot{X}^\mu \equiv \partial_\tau X^\mu$ and $X^{\mu\prime} \equiv \partial_\sigma X^\mu$, the string action can be written

$$S = -\frac{T_0}{c} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\sigma_1} d\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2} = \int d\xi^0 \, d\xi^1 \, \mathcal{L}(\partial_0 X^\mu, \partial_1 X^\mu) \ , \tag{1.7}$$

which is invariant under the symmetry

$$\delta X^\mu(\tau, \sigma) = \epsilon^\mu \ , \tag{1.8}$$

which describes a uniform spacetime translation of the string coordinates X^μ. (Note that this is really a family of D symmetries, one for each value of the spacetime index μ. But I will continue to describe it as one symmetry, in the sense that it forms one multiplet of symmetries.) The corresponding conserved current is

$$j^\alpha_\mu \equiv (j^0_\mu, j^1_\mu) = \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} , \frac{\partial \mathcal{L}}{\partial X^{\mu\prime}} \right) = \left(\mathcal{P}^\tau_\mu, \mathcal{P}^\sigma_\mu \right) \quad . \tag{1.9}$$

More compactly,

$$j^\alpha_\mu = \mathcal{P}^\alpha_\mu \ . \tag{1.10}$$

The conserved charge is the total spacetime momentum of the string:

$$p_\mu = \int_0^{\sigma_1} \mathcal{P}^\tau_\mu(\tau, \sigma) \, d\sigma \ . \tag{1.11}$$

The above expression gives the conserved momentum as an integral over a line of constant τ, but the reparameterization invariance of the string suggests that there is nothing special about such a line. In fact we found that the conservation equation $\partial_\alpha j^\alpha_\mu = 0$ implies, with the use of the two-dimensional divergence theorem, that we can write

$$p_\mu = \int_\gamma \left(\mathcal{P}^\tau_\mu \, d\sigma - \mathcal{P}^\sigma_\mu \, d\tau \right) \ , \tag{1.12}$$

where γ describes a general curve. For open strings γ must begin at one end of the string and end at the other, and for closed strings it must wind once around the world-sheet.

World-Sheet Application 2: Lorentz Symmetry and its Currents:

Lorentz transformations can be described by

$$\delta X^\mu = \epsilon^{\mu\nu} X_\nu \ , \qquad \text{where } \epsilon^{\mu\nu} = -\epsilon^{\nu\mu} \ . \tag{1.13}$$

The string Lagrangian is invariant under this symmetry, and with Noether's theorem one obtains the conserved world-sheet current

$$\mathcal{M}^\alpha_{\mu\nu} = X_\mu \mathcal{P}^\alpha_\nu - X_\nu \mathcal{P}^\alpha_\mu \ , \qquad \text{where } \ \partial_\alpha \mathcal{M}^\alpha_{\mu\nu} = 0 \ . \tag{1.14}$$

The conserved charge is then

$$M_{\mu\nu} = \int \mathcal{M}^\tau_{\mu\nu}(\tau,\sigma)\, d\sigma = \int \left(X_\mu \mathcal{P}^\tau_\nu - X_\nu \mathcal{P}^\tau_\mu \right) d\sigma \ , \tag{1.15}$$

where $M_{\mu\nu} = -M_{\nu\mu}$, or it can be written as an integral over a general curve,

$$M_{\mu\nu} = \int_\gamma \left(\mathcal{M}^\tau_{\mu\nu}\, d\sigma - \mathcal{M}^\sigma_{\mu\nu}\, d\tau \right) \ , \tag{1.16}$$

analogous to Eq. (1.12). $M_{\mu\nu}$ is conserved for any spacetime dimension. For the familiar case of four dimensions,

$$L_i = \frac{1}{2} \epsilon_{ijk} M_{jk} \tag{1.17}$$

is the total angular momentum of the string. The M^{0i} components can be evaluated easily in static gauge, giving

$$M^{i0} = \int \left(X^i \mathcal{P}^{\tau 0} - X^0 \mathcal{P}^{\tau i} \right) d\sigma = p^0 \left(X^i_{\rm cm} - v^i_{\rm cm} t \right) \ , \tag{1.18}$$

where

$$X^i_{\rm cm} = \frac{1}{p^0} \int d\sigma X^i \mathcal{P}^{\tau 0} \ , \qquad v^i_{\rm cm} = \frac{p^i}{p^0}\, c \ . \tag{1.19}$$

Thus, the conservation of M^{i0} gives an explicitly time-dependent conservation law, implying that the center of mass position $X^i_{\rm cm}$ moves at the fixed velocity $v^i_{\rm cm}$.

II. Tension, Slope Parameter, String Length, $\hbar$, and c:

By analyzing the rigidly rotating open string, we found that the energy E and angular momentum J are related by

$$\frac{J}{\hbar} = \alpha' E^2 \ , \tag{2.1}$$

where

$$\alpha' = \frac{1}{2\pi T_0 \hbar c} \ . \tag{2.2}$$

It is traditional to express string quantities in terms of α', rather than T_0. We will also, from now on, use *natural units*, which means that we define

$$\hbar \equiv c \equiv 1 \ . \tag{2.3}$$

We can then write $\alpha' = 1/(2\pi T_0)$. T_0 has the dimension of energy/length, or (energy)^2 in natural units. α' then has the dimension of $1/(\text{energy})^2$, or equivalently (length)^2, so it makes sense to define a "string length"

$$\ell_s = \sqrt{\alpha'} \ . \tag{2.4}$$

III. The String in Light-Cone Gauge:

(If the next few pages seem familiar, it is because they were adapted from Homework 8 Solutions, Problem 1.)

We begin by choosing the worldsheet parameter τ by setting it equal to a linear combination of coordinate values,

$$\tau \equiv \frac{1}{\lambda} n \cdot X \ , \tag{3.1}$$

where λ is an arbitrary constant. n is an arbitrary timelike or lightlike vector, where here we will take

$$n_\mu = n_{\text{lc},\,\mu} \equiv \left(\frac{1}{\sqrt{2}} \, , \frac{1}{\sqrt{2}} \, , 0 \, , \, \ldots \, , 0 \right) \quad , \tag{3.2a}$$

so $n_{\text{lc}} \cdot X = X^+$, and Eq. (3.1) is called the light-cone gauge condition. In Chapter 6 we used the same formalism, but with

$$n_\mu = n_{\text{static},\,\mu} \equiv (1, 0, \, \ldots \, , 0) \quad , \tag{3.2b}$$

which leads to the static gauge condition. Until further notice, the equations here will apply to both cases, except for statements that refer explicitly to $+$ or $-$ components.

To choose a helpful parameterization of σ, we can first decide that we will define σ for some initial value of τ, and then we will determine σ for other values of τ by insisting that the lines of constant σ are orthogonal to the lines of constant τ. This condition is written mathematically as

$$\frac{\partial X}{\partial \tau} \cdot \frac{\partial X}{\partial \sigma} \equiv \dot{X} \cdot X' = 0. \tag{3.3}$$

The procedure for constructing such lines of constant σ is described in the textbook in the last paragraph starting on p. 154, where it is used to construct the line $\sigma = 0$ for the case of closed strings. But the procedure can be used for both open and closed strings and for all σ, except that for open strings the endpoints must be lines of constant σ, and hence we cannot impose this procedure to construct the lines $\sigma = 0$ or $\sigma = \sigma_1$. We can show, however, that Eq. (3.3) nonetheless holds at the endpoints, as a consequence of the string boundary conditions. To do this we will assume that we always have Neumann boundary conditions in the direction of n,

$$n \cdot \mathcal{P}^{\sigma} = 0 \quad \text{at open string endpoints,} \tag{3.4}$$

a condition necessary for the conservation of $n \cdot p$, where p is the total momentum of the string.

We need to make use of Eq. (3.4) without assuming Eq. (3.3), so that we can demonstrate that Eq. (3.3) holds at the endpoints. We begin by writing the general formulas for $\mathcal{P}^{\tau\mu}$ and $\mathcal{P}^{\sigma\mu}$ from Eqs. (6.49) and (6.50) in the textbook, using natural units ($\hbar \equiv c \equiv 1$) and replacing T_0 by $1/(2\pi\alpha')$ (as in Eq. (2.2):

$$\mathcal{P}^{\tau\mu} = -\frac{1}{2\pi\alpha'} \frac{\left(\dot{X} \cdot X'\right) X^{\mu\prime} - \left(X'\right)^2 \dot{X}^{\mu}}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 \left(X'\right)^2}} \;, \tag{3.5a}$$

$$\mathcal{P}^{\sigma\mu} = -\frac{1}{2\pi\alpha'} \frac{\left(\dot{X} \cdot X'\right) \dot{X}^{\mu} - \left(\dot{X}\right)^2 X^{\mu\prime}}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 \left(X'\right)^2}} \;. \tag{3.5b}$$

Note that Eq. (3.1) implies that $\frac{\partial}{\partial\sigma} n \cdot X = n \cdot X' = 0$, and $n \cdot \dot{X} = \lambda$, so Eq. (3.5b) implies

$$n \cdot \mathcal{P}^{\sigma} = -\frac{1}{2\pi\alpha'} \frac{\left(\dot{X} \cdot X'\right) \lambda}{\sqrt{\left(\dot{X} \cdot X'\right)^2 - \left(\dot{X}\right)^2 \left(X'\right)^2}} \;. \tag{3.6}$$

The denominator of the above expression is never infinite, and $\lambda \neq 0$. Thus, the vanishing of $n \cdot \mathcal{P}^{\sigma}$ at the string endpoints implies that Eq. (3.3) holds at the endpoints, and hence holds everywhere for our parameterization. The validity of Eq. (3.3) at the string endpoints implies that the construction of lines of constant σ using orthogonality to lines of constant τ will join smoothly with the evolution of the string endpoints, which have fixed values of σ.

Using Eq. (3.3), Eqs. (3.5) simplify to

$$\mathcal{P}^{\tau\mu} = \frac{1}{2\pi\alpha'} \frac{\left(X'\right)^2 \dot{X}^\mu}{\sqrt{-\left(\dot{X}\right)^2 \left(X'\right)^2}} \;, \tag{3.7a}$$

$$\mathcal{P}^{\sigma\mu} = \frac{1}{2\pi\alpha'} \frac{\left(\dot{X}\right)^2 X^{\mu\prime}}{\sqrt{-\left(\dot{X}\right)^2 \left(X'\right)^2}} \;, \tag{3.7b}$$

from which it follows immediately (using Eq. (3.1)) that

$$n \cdot \mathcal{P}^{\tau} = \frac{1}{2\pi\alpha'} \frac{\left(X'\right)^2 \lambda}{\sqrt{-\left(\dot{X}\right)^2 \left(X'\right)^2}} \;, \tag{3.8a}$$

$$n \cdot \mathcal{P}^{\sigma} = 0 \;. \tag{3.8b}$$

From Eq. (3.8b) and the equations of motion

$$\frac{\partial \mathcal{P}^{\tau\mu}}{\partial \tau} + \frac{\partial \mathcal{P}^{\sigma\mu}}{\partial \sigma} = 0 \;, \tag{3.9}$$

it follows that

$$\frac{\partial}{\partial \tau} \left[n \cdot \mathcal{P}^{\tau} \right] = 0 \;, \tag{3.10}$$

or, in other words, $n \cdot \mathcal{P}^{\tau}$ is independent of τ.

We can simplify these equations further by making a special choice for the definition of σ for the initial value of τ. Specifically, we can insist that $n \cdot \mathcal{P}^{\tau}$ be a constant. Since the total momentum p is given by

$$p^\mu = \int_0^{\sigma_1} d\sigma \, \mathcal{P}^{\tau\mu} \;, \tag{3.11}$$

the constant $n \cdot \mathcal{P}^{\tau}$ prescription is equivalent to assuming that the density of $n \cdot p = p^+$ is uniform in σ. From Eq. (3.8a), we see that this uniformity can be achieved by requiring

$$\frac{\left(X'\right)^2}{\sqrt{-\left(\dot{X}\right)^2 \left(X'\right)^2}} = \text{constant} \;, \tag{3.12}$$

where for convenience we will choose the constant to be one. Solving Eq. (3.12) for $\left(X'\right)^2$ gives

$$X'^2 = -\dot{X}^2 \ , \tag{3.13}$$

a condition that can always be achieved by a redefinition of σ. That is, if the condition does not already hold, we can introduce a new variable $\sigma'(\sigma)$ so that

$$\left(\frac{\partial X}{\partial \sigma'}\right)^2 = \left(\frac{\partial X}{\partial \sigma}\right)^2 \left(\frac{d\sigma}{d\sigma'}\right)^2 = -\dot{X}^2 \ ,$$

or

$$\frac{d\sigma'}{d\sigma} = \sqrt{-\frac{X'^2}{\dot{X}^2}} \ , \tag{3.14}$$

which can be integrated to determine $\sigma'(\sigma)$. Once the new σ' is defined, the old σ can be forgotten and the new parameter can be renamed σ.

With $n \cdot \mathcal{P}^\tau$ now constant at the initial time τ, Eq. (3.10) can be invoked to show that $n \cdot \mathcal{P}^\tau$ has a constant value on the entire string worldsheet, which from Eqs. (3.8a) and (3.12) is given by

$$n \cdot \mathcal{P}^\tau = \frac{\lambda}{2\pi\alpha'} \ . \tag{3.15}$$

From Eq. (3.11),

$$n \cdot p = \frac{\sigma_1 \lambda}{2\pi\alpha'} \ . \tag{3.16}$$

It is customary to choose $\sigma_1 = \pi$ for open strings, and 2π for closed strings, so we can write

$$\lambda = \frac{2\pi}{\sigma_1}(n \cdot p)\alpha' = \beta(n \cdot p)\alpha' \ , \tag{3.17}$$

where

$$\beta = \begin{cases} 2 & \text{for open strings} \\ 1 & \text{for closed strings} \ . \end{cases} \tag{3.18}$$

The light-cone gauge condition (3.1) is then written in final form as

$$\boxed{n \cdot X(\tau, \sigma) = \beta\alpha'(n \cdot p)\tau \ .} \tag{3.19}$$

Furthermore, Eqs. (3.7) now simplify to

$$\boxed{\begin{aligned} \mathcal{P}^{\tau\mu} &= \frac{1}{2\pi\alpha'}\dot{X}^\mu \ , \\ \mathcal{P}^{\sigma\mu} &= -\frac{1}{2\pi\alpha'}X^{\mu\prime} \ , \end{aligned}} \tag{3.20}$$

and the equation of motion (3.9) simplifies to the wave equation,

$$\boxed{\ddot{X}^\mu - X^{\mu\prime\prime} = 0\ .} \tag{3.21}$$

The full set of equations can be further simplified by combining the σ–τ orthogonality condition of Eq. (3.3) with the constant $n \cdot \mathcal{P}^\tau$ condition of Eq. (3.13), giving

$$\boxed{\left(\dot{X} \pm X'\right)^2 = 0\ .} \tag{3.22}$$

Eqs. (3.19), (3.21), and (3.22) then give the full set of equations of motion of the string in light-cone gauge. Eqs. (3.20) give simple expressions for the world-sheet momentum current $\mathcal{P}^\alpha_\mu$ in this gauge.

Until this point the equations were valid for any timelike or null vector n in the gauge condition of Eq. (3.17), but now we will specialize to the light-cone gauge, with $n = n_{\rm lc}$ as defined by Eq. (3.2a), and

$$x^\pm \equiv \frac{1}{\sqrt{2}}(x^0 \pm x^1)\ . \tag{3.23}$$

The important advantage of light-cone gauge is that it allows us to use Eq. (3.22) to solve explicitly for one of the components of X^μ. Specifically, Eq. (3.22) becomes

$$-2\left(\dot{X}^+ \pm X^{+\prime}\right)\left(\dot{X}^- \pm X^{-\prime}\right) + \left(\dot{X}^I \pm X^{I\prime}\right)^2 = 0\ , \tag{3.24}$$

where I is summed over the D-2 indices other than $+$ and $-$. Eq. (3.19) implies that

$$\dot{X}^+ = n \cdot \dot{X} = \beta\alpha'\, p^+\ , \qquad X^{+\prime} = 0\ . \tag{3.25}$$

Eq. (3.24) can then be solved for the derivatives of X^-, giving

$$\boxed{\left(\dot{X}^- \pm X^{-\prime}\right) = \frac{1}{2\beta\alpha' p^+}\left(\dot{X}^I \pm X^{I\prime}\right)^2\ .} \tag{3.26}$$

By taking linear combinations of the above expression we can find $\dot{X}^-$ and $X^{-\prime}$, which can be integrated to determine $X^-(\tau,\sigma)$ up to an integration constant. Note that if we had used static gauge we still could have formally expressed one component of X^μ in terms of the others, but the expression would involve a square root of a sum of operators, which is much more difficult to deal with.

Classical Solution for the Open String:

We begin by writing equations for the open string with a space-filling D-brane, so all the string coordinates X^μ satisfy free boundary conditions.

The wave equation (3.21) implies that the most general solution can be written as the sum of an arbitrary wave moving to the left and an arbitrary wave moving to the right:

$$X^\mu(\tau,\sigma) = \frac{1}{2}\Big(f^\mu(\tau+\sigma) + g^\mu(\tau-\sigma)\Big) \ , \tag{3.27}$$

where f^μ and g^μ are arbitrary functions of a single argument. The boundary condition at $\sigma = 0$ implies that

$$\frac{\partial X^\mu}{\partial\sigma}(\tau,0) = \frac{1}{2}\Big(f^{\mu\prime}(\tau) - g^{\mu\prime}(\tau)\Big) = 0 \ , \tag{3.28}$$

so f^μ and g^μ are equal up to a constant, which can be absorbed into a redefinition of f. Eq. (3.27) can therefore be written as

$$X^\mu(\tau,\sigma) = \frac{1}{2}\Big(f^\mu(\tau+\sigma) + f^\mu(\tau-\sigma)\Big) \ . \tag{3.29}$$

The boundary condition at $\sigma = \pi$ then implies that

$$\frac{\partial X^\mu}{\partial\sigma}(\tau,\pi) = \frac{1}{2}\Big(f^{\mu\prime}(\tau+\pi) - f^{\mu\prime}(\tau-\pi)\Big) = 0 \ , \tag{3.30}$$

which must hold for all τ, so $f^{\mu\prime}$ must be periodic with period 2π.

Since $f^{\mu\prime}$ is periodic with period 2π, we can write it as a Fourier sum:

$$f^{\mu\prime}(u) = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha_n^\mu \, e^{-inu} \quad , \tag{3.31}$$

where reality implies that

$$\alpha_{-n}^\mu = \alpha_n^{\mu *} \ . \tag{3.32}$$

The constant factor $\sqrt{2\alpha'}$ is inserted by convention, and serves the purpose of making the expansion coefficients α_n^μ dimensionless. To see this, note that X^μ has units of length, σ and τ are dimensionless, and α' has dimensions of (length)^2. This Fourier expansion can immediately be related to the expansion for the derivatives of $X^\mu(\tau,\sigma)$, since from Eq. (3.29) one can see that

$$\dot{X}^\mu \pm X^{\mu\prime} = f^{\mu\prime}(\tau \pm \sigma) \ , \tag{3.33}$$

from which it follows that

$$\boxed{\dot{X}^\mu \pm X^{\mu\prime} = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha_n^\mu \, e^{-in(\tau\pm\sigma)} \ .} \tag{3.34}$$

Integrating Eq. (3.31) and inserting the result into Eq. (3.29), we have

$$\boxed{X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{1}{n}\,\alpha_n^\mu \, e^{-in\tau}\ \cos n\sigma \ ,} \tag{3.35}$$

where x_0^μ is the constant arising from the integration of (3.31).

Eq. (3.35) holds for every component X^μ of the string coordinates, but the coefficients that appear in the expansion are not all independent. One relationship comes from using Eq. (3.35) to calculate the total momentum of the string, which is given by Eqs. (1.11) and (3.20) as

$$p^\mu = \int_0^\pi d\sigma \, \frac{\dot{X}^\mu}{2\pi\alpha'} = \frac{\alpha_0^\mu}{\sqrt{2\alpha'}} \quad , \tag{3.36}$$

so

$$\boxed{\alpha_0^\mu = \sqrt{2\alpha'}\, p^\mu \ .} \tag{3.37}$$

For $\mu = +$, the expansion is already fixed by the light-cone gauge condition (3.19), $X^+ = 2\alpha' p^+ \tau$, so

$$\boxed{x_0^+ = 0 \ , \qquad \alpha_0^+ = \sqrt{2\alpha'}\, p^+ \ , \qquad \alpha_n^+ = 0 \ \text{ for } n \neq 0,} \tag{3.38}$$

where the α_0^+ relation is a special case of Eq. (3.37). Furthermore, the $\mu = -$ components, except for the zero mode x_0^-, can be found from Eq. (3.26). Using Eq. (3.34) to re-express both sides of Eq. (3.26), one finds that

$$\boxed{\alpha_n^- = \frac{1}{\sqrt{2\alpha'}\, p^+} L_n^\perp \ , \qquad \text{where } \ L_n^\perp \equiv \frac{1}{2} \sum_{p=-\infty}^{\infty} \alpha_{n-p}^I \, \alpha_p^I \ .} \tag{3.39}$$

Recall that I is summed over the "transverse" indices, meaning all indices other than $+$ or $-$. Thus, the complete solution is specified by choosing values for each of the α_n^I, for $n \in \mathbb{Z}$ and $I = 2, 3, \ldots, D-1$, subject to the reality constraint of Eq. (3.32), and also choosing the values of x_0^μ for $\mu = -, 2, 3, \ldots, D-1$, and finally choosing the value of p^+.

Classical Solution for the Closed String:

The closed string has the same equations of motion as the open string, but the boundary conditions are different. In analogy to Eq. (3.27), we can start with the general solution to the wave equation,

$$X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma) \ . \tag{3.40}$$

(The names we are using for the two functions are more elaborate than the simple f and g used in Eq. (2.27) for the open string, and there is a reason: these names will remain in use for longer, so we are choosing them to be more descriptive.) In this case we have no endpoint conditions, but we do insist on periodicity. We choose the period to be 2π, so

$$X^\mu(\tau,\sigma+2\pi) = X^\mu(\tau,\sigma) \quad \text{for all } \tau \text{ and } \sigma. \tag{3.41}$$

For now our strings move in Minkowski space, and Eq. (3.41) is certainly the correct periodicity condition in that case. It is worth mentioning, however, that in Chapter 17 we will consider strings that move in spaces that are not simply connected, spaces that contain loops which cannot be continuously shrunk to a point. It is useful in such cases to use coordinates for the unshrinkable loops which are not single-valued, and then Eq. (3.41) would need to be modified.

Defining

$$u \equiv \tau+\sigma \ , \qquad v \equiv \tau - \sigma \ \ , \tag{3.42}$$

the periodicity condition becomes

$$X_L^\mu(u+2\pi) - X_L^\mu(u) = X_R^\mu(v) - X_R^\mu(v-2\pi) \ \ . \tag{3.43}$$

Since the left-hand side does not depend on v, and the right-hand side does not depend on u, both sides must be constant. Thus neither $X_L^\mu(u)$ nor $x_R^\mu(v)$ are required to be periodic, but the amount by which they change when their argument increases by 2π must be the same for both functions, and independent of the argument. The derivatives of the functions must be strictly periodic, so we can expand

$$\boxed{\begin{aligned} X_L^{\mu\prime}(u) &= \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \bar\alpha_n^\mu e^{-inu} \\ X_R^{\mu\prime}(v) &= \sqrt{\frac{\alpha'}{2}} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-inv} \ . \end{aligned}} \tag{3.44}$$

Note that the overbar in the above equation does not denote complex conjugation — instead the coefficients $\bar{\alpha}_n^\mu$ are a completely independent set of variables from the coefficients α_n^μ. Both obey reality conditions

$$\boxed{\alpha_{-n}^\mu = \alpha_n^{\mu *} \ , \qquad \bar{\alpha}_{-n}^\mu = \bar{\alpha}_n^{\mu *} \ ,} \tag{3.45}$$

and the condition (3.43) implies that

$$\bar{\alpha}_0^\mu = \alpha_0^\mu \ . \tag{3.46}$$

Integrating and using Eq. (3.46), the string coordinates $X^\mu(\tau,\sigma)$ can be expanded as

$$\boxed{X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}\left(\alpha_n^\mu\,e^{in\sigma} + \bar{\alpha}_n^\mu\,e^{-in\sigma}\right) \ .} \tag{3.47}$$

The calculation of the total string momentum p^μ is similar to the previous one, but this time we integrate σ from 0 to 2π:

$$p^\mu = \int_0^{2\pi} d\sigma\,\frac{\dot{X}^\mu}{2\pi\alpha'} = \sqrt{\frac{2}{\alpha'}}\,\alpha_0^\mu \ , \tag{3.48}$$

so

$$\boxed{\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\,p^\mu \ .} \tag{3.49}$$

For $\mu = +$, the expansion is fixed by the light-cone gauge condition (3.19), so

$$\boxed{x_0^+ = 0 \ , \qquad \alpha_0^+ = \sqrt{\frac{\alpha'}{2}}\,p^+ \ , \qquad \alpha_n^+ = \bar{\alpha}_n^+ = 0 \ \text{ for } n \neq 0,} \tag{3.50}$$

where the α_0^+ relation is a special case of Eq. (3.49). The coefficients of the X^- expansion, except for x_0^-, are again determined by Eq. (3.26), but the explicit calculation is a bit different. From Eq. (3.40) one has

$$\begin{aligned} \dot{X}^\mu &= X_L^{\mu\prime}(\tau+\sigma) + X_R^{\mu\prime}(\tau-\sigma) \\ X^{\mu\prime} &= X_L^{\mu\prime}(\tau+\sigma) - X_R^{\mu\prime}(\tau-\sigma) \ , \end{aligned} \tag{3.51}$$

so with Eqs. (3.44) one has

$$\boxed{\begin{aligned} \dot{X}^\mu + X^{\mu\prime} &= 2X_L^{\mu\prime}(\tau+\sigma) = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \bar{\alpha}_n^\mu e^{-in(\tau+\sigma)} \\ \dot{X}^\mu - X^{\mu\prime} &= 2X_R^{\mu\prime}(\tau-\sigma) = \sqrt{2\alpha'} \sum_{n=-\infty}^{\infty} \alpha_n^\mu e^{-in(\tau-\sigma)} \ . \end{aligned}} \tag{3.52}$$

Using the above relations to evaluate each side of Eq. (3.26), and recalling that $\beta = 1$ for closed strings, one finds

$$\boxed{\alpha_n^- = \frac{1}{p^+}\sqrt{\frac{2}{\alpha'}}\, L_n^\perp \ , \qquad \bar{\alpha}_n^- = \frac{1}{p^+}\sqrt{\frac{2}{\alpha'}}\, \bar{L}_n^\perp \ ,} \tag{3.53}$$

where

$$\boxed{L_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty} \alpha_{n-p}^I\, \alpha_p^I \ , \qquad \bar{L}_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty} \bar{\alpha}_{n-p}^I\, \bar{\alpha}_p^I \ .} \tag{3.54}$$

For the case of $\mu = -$, Eq. (3.46) produces the nontrivial constraint

$$\boxed{\bar{L}_0^\perp = L_0^\perp \ ,} \tag{3.55}$$

a relation that has no analogue for the open string.

For the closed string, the complete solution is specified by choosing values for each of the α_n^I and $\bar{\alpha}_n^I$, for $n \in \mathbb{Z}$ and $I = 2, 3, \ldots, D-1$, subject to the reality constraints of Eq. (3.45), and subject to the constraints $\bar{\alpha}_0^I = \alpha_0^I$ and $\bar{L}_0^\perp = L_0^\perp$. One must also choose the values of x_0^μ for $\mu = -, 2, 3, \ldots, D-1$, and also the value of p^+.

IV. Quantization of the String in Light-Cone Gauge:

Quantization of the Open String:

Basic Operators and Commutation Relations:

We now discuss the quantization of the open string solution, corresponding to the classical solutions described in Sec. III.

The beauty of the light-cone gauge is that we can treat the transverse components $X^I(\tau,\sigma)$ of the string coordinates as unconstrained dynamical variables, while the $\mu = +$ and $\mu = -$ components are for the most part determined by the transverse components. We found only two exceptions to this rule: x_0^- and p^+ were not determined by the transverse operators. Since

$$\mathcal{P}^{\tau I} = \frac{\partial \mathcal{L}}{\partial \dot{X}^I} \tag{4.1}$$

is the canonical momentum variable conjugate to X^I, we will choose the following set of classical variables to promote to Schrödinger operators:

$$\boxed{\text{Schrödinger operators:} \quad \Big(X^I(\sigma)\,,\; x_0^-\,,\; \mathcal{P}^{\tau I}(\sigma)\,,\; p^+ \Big)\ .} \tag{4.2}$$

We adopt the usual canonical commutation relations,

$$\left[X^I(\sigma)\,,\,\mathcal{P}^{\tau J}(\sigma')\right] = i\eta^{IJ}\delta(\sigma-\sigma') = i\delta^{IJ}\delta(\sigma-\sigma')\ , \tag{4.3}$$

$$\left[X^I(\sigma)\,,\,X^J(\sigma')\right] = \left[\mathcal{P}^{\tau I}(\sigma)\,,\,\mathcal{P}^{\tau J}(\sigma')\right] = 0\ , \tag{4.4}$$

$$\left[x_0^-\,,\,p^+\right] = i\eta^{-+} = -i\ , \tag{4.5}$$

$$\left[x_0^-\,,\,X^I(\sigma)\right] = \left[x_0^-\,,\,\mathcal{P}^{\tau I}(\sigma)\right] = \left[p^+\,,\,X^I(\sigma)\right] = \left[p^+\,,\,\mathcal{P}^{\tau I}(\sigma)\right] = 0\ . \tag{4.6}$$

Treating τ as the time variable for the system, we will also introduce Heisenberg operators which also depend on τ:

$$\boxed{\text{Schrödinger operators:} \quad \Big(X^I(\tau,\sigma)\,,\; x_0^-(\tau)\,,\; \mathcal{P}^{\tau I}(\tau,\sigma)\,,\; p^+(\tau) \Big)\ .} \tag{4.7}$$

From the classical motion, however, we expect that $x_0^-(\tau)$ and $p^+(\tau)$ will actually be independent of their arguments. For the Heisenberg operators, the nonzero commutators will be

$$\left[X^I(\tau,\sigma)\,,\,\mathcal{P}^{\tau J}(\tau,\sigma')\right] = i\eta^{IJ}\delta(\sigma-\sigma') = i\delta^{IJ}\delta(\sigma-\sigma')\ , \tag{4.8}$$

$$\left[x_0^-(\tau)\,,\,p^+(\tau)\right] = i\eta^{-+} = -i\ \ , \tag{4.9}$$

where the other commutators are zero, as in Eqs. (4.4) and (4.6).

The Heisenberg fields $X^\mu(\tau,\sigma)$ can be expanded exactly as in Eq. (3.35)

$$\boxed{X^\mu(\tau,\sigma) = x_0^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu\,\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\,\alpha_n^\mu\,e^{-in\tau}\,\cos n\sigma\ ,} \tag{3.35}$$

except now the coefficients x_0^μ and α_n^μ are operators, rather than numbers. For $\mu = I$, we can treat them as independent operators, and we can use the canonical commutation relations (4.7) and (4.8) to determine the commutation relations for the new operators α_n^μ. The simplest link between the α's and the X's is found in Eq. (3.34),

$$\boxed{\dot{X}^\mu \pm X^{\mu\prime} = \sqrt{2\alpha'}\sum_{n=-\infty}^{\infty}\alpha_n^\mu\,e^{-in(\tau\pm\sigma)}\ .} \tag{3.34}$$

Our goal is to express the sum on the right in terms of Heisenberg operators $X^\mu(\tau,\sigma)$, but to get it right we must be careful about the range of σ. The sum on the right (for either the upper or lower sign) defines a function which is manifestly periodic in σ with period 2π, but the equation makes sense only for σ in its allowed physical range, which is $\sigma \in [0,\pi]$. Let us define

$$A^I(\tau,\sigma) \equiv \sqrt{2\alpha'}\sum_{n=-\infty}^{\infty}\alpha_n^\mu\,e^{-in(\tau+\sigma)}\ . \tag{4.10}$$

Eq. (3.34) then implies that

$$\dot{X}^I(\tau,\sigma) + X^{\mu\prime}(\tau,\sigma) = A^I(\tau,\sigma)\ \text{ for } \sigma\in[0,\pi], \tag{4.11}$$

but also

$$\dot{X}^I(\tau,\sigma) - X^{\mu\prime}(\tau,\sigma) = A^I(\tau,-\sigma)\ \text{ for } \sigma\in[0,\pi]. \tag{4.12}$$

The above equation can be rewritten as

$$\dot{X}^I(\tau,-\sigma) - X^{\mu\prime}(\tau,-\sigma) = A^I(\tau,\sigma)\ \text{ for } \sigma\in[-\pi,0]. \tag{4.13}$$

Eqs. (4.11) and (4.13) can then be used to write an equation for $A^I(\tau,\sigma)$ that is valid over its full period, which can be taken as $\sigma\in[-\pi,\pi]$:

$$A^I(\tau,\sigma) = \begin{cases}\dot{X}^I(\tau,\sigma) + X^{\mu\prime}(\tau,\sigma) & \text{if } \sigma\in[0,\pi]\\ \dot{X}^I(\tau,-\sigma) - X^{\mu\prime}(\tau,-\sigma) & \text{if } \sigma\in[-\pi,0].\end{cases} \tag{4.14}$$

The elementary commutator is found by differentiating Eq. (4.8) with respect to σ, and replacing $\mathcal{P}^{\tau J}$ by $\dot{X}^J/(2\pi\alpha')$, as specified in Eq. (3.20). Thus

$$\left[X^{I\prime}(\tau,\sigma)\,,\,\dot{X}^J(\tau,\sigma')\right] = 2\pi i\alpha'\,\delta^{IJ}\,\frac{d}{d\sigma}\delta(\sigma-\sigma')\ . \tag{4.15}$$

If you are not familiar with the derivative of a δ-function, don't be frightened. Expressions containing δ-functions are defined by what happens when one integrates over them, perhaps after multiplying by a smooth function. Derivatives of a δ-function are then defined by an integration by parts. For example,

$$\int dx F(x)\frac{d}{dx}\delta(x-a) = -\int dx\frac{dF}{dx}\delta(x-a) = -\frac{dF}{dx}(a)\quad . \tag{4.16}$$

By combining Eqs. (4.14) and (4.15), one finds

$$\left[A^I(\tau,\sigma)\,,\,A^J(\tau,\sigma')\right] = 4\pi i\alpha'\,\delta^{IJ}\,\frac{d}{d\sigma}\delta(\sigma-\sigma')\ . \tag{4.17}$$

To check Eq. (4.17) one needs to consider four separate cases: $(\sigma\in[0,\pi],\ \sigma'\in[0,\pi])$, $(\sigma\in[0,\pi],\ \sigma'\in[-\pi,0])$, $(\sigma\in[-\pi,0],\ \sigma'\in[0,\pi])$, and $(\sigma\in[-\pi,0],\ \sigma'\in[-\pi,0])$. You should find, however, that it works in each case. In evaluating this expression it helps to recall that $\delta(x-y)=\delta(y-x)$, and $\frac{d}{dx}\delta(x-y) = -\frac{d}{dy}\delta(x-y)$.

One can then invert Eq. (4.10) to give

$$\alpha_n^I = \frac{1}{2\pi\sqrt{2\alpha'}}\int_0^{2\pi} d\sigma\, e^{in(\tau+\sigma)}A^I(\tau,\sigma)\quad . \tag{4.18}$$

Using Eq. (4.17), the commutator is then given by

$$\begin{aligned}\left[\alpha_n^I\,,\,\alpha_m^J\right] &= \frac{1}{8\pi^2\alpha'}\int_0^{2\pi} d\sigma' e^{im(\tau+\sigma')}\int_0^{2\pi} d\sigma e^{in(\tau+\sigma)}\left[A^I(\tau,\sigma)\,,\,A^J(\tau,\sigma')\right]\\ &= \frac{4\pi i\alpha'}{8\pi^2\alpha'}\,\delta^{IJ}\int_0^{2\pi} d\sigma' e^{im(\tau+\sigma')}\int_0^{2\pi} d\sigma e^{in(\tau+\sigma)}\frac{d}{d\sigma}\delta(\sigma-\sigma')\\ &= \frac{i}{2\pi}\,\delta^{IJ}\int_0^{2\pi} d\sigma' e^{im(\tau+\sigma')}(-in)e^{in(\tau+\sigma')}\\ &= n\delta^{IJ}\delta_{m+n,0}\ ,\end{aligned} \tag{4.19}$$

so finally

$$\boxed{\left[\alpha_n^I\,,\,\alpha_m^J\right] = n\delta_{m+n,0}\,\delta^{IJ}\ .} \tag{4.20}$$

Given Eqs. (4.18) and (4.14), one sees that the α_n^I are constructed entirely from operators $X^{I\prime}$ and $\dot{X}^I$, and therefore they commute with x_0^- and p^+.

To find the commutators of the x_0^I, we can start by rewriting the commutator $\left[X^I(\tau,\sigma)\,,\,\mathcal{P}^{\tau J}\right]$ of Eq. (4.8) in terms of $\dot{X}^J$ by using Eq. (3.20), and integrating σ from 0 to π to simplify the mode expansion of $X^I(\tau,\sigma)$, as given by Eq. (3.35). Note that although the $\cos n\sigma$ factor in Eq. (3.35) has period 2π and not π, it still vanishes when integrated from 0 to π. The result is

$$\left[x_0^I + \sqrt{2\alpha'}\,\alpha_0^I\,\tau\,,\,\dot{X}^J(\tau,\sigma')\right] = 2\alpha' i\delta^{IJ} \quad . \tag{4.21}$$

From Eq. (4.20) we know that α_0^I commutes with the sum of α_n^J operators that appear in the expansion for $\dot{X}^J$, so the only contribution comes from x_0^I. Inserting the expansion for $\dot{X}^J$,

$$\sum_{n=-\infty}^{\infty}\left[x_0^I\,,\,\alpha_n^J\right]\cos n\sigma' e^{-in\tau} = \sqrt{2\alpha'}\,i\,\delta^{IJ} \ . \tag{4.22}$$

This expression must hold for all τ, but the left hand side has the form of a Fourier expansion in τ, with period 2π. Since Fourier expansions are unique, the two sides must match term by term. The right-hand side has only an $n=0$ term, so we have

$$\boxed{\left[x_0^I\,,\,\alpha_0^J\right] = \sqrt{2\alpha'}\,i\,\delta^{IJ}\ , \qquad \left[x_0^I\,,\,\alpha_n^J\right] = 0 \text{ if } n \neq 0.\ .} \tag{4.23}$$

Recalling that $\alpha_0^J = \sqrt{2\alpha'}\,p^J$ (Eq. (3.37)), the first of the relations above is equivalent to

$$\boxed{\left[x_0^I\,,\,p^J\right] = i\delta^{IJ}\ ,} \tag{4.24}$$

which is what we would expect.

Note that the commutation relations (4.20) imply that the α_n^I behave essentially as creation and annihilation operators, except that they are not normalized in the standard way. When quantized, the reality condition $\alpha_{-n}^\mu = \alpha_n^{\mu *}$ of Eq. (3.32) becomes

$$\alpha_{-n}^I = \alpha_n^{I\dagger}\ , \tag{4.25}$$

and the α_n^I can be related to a new set of operators a_n, with

$$\alpha_n^I = \sqrt{n}\,a_n^I\ , \qquad \alpha_{-n}^I = \sqrt{n}\,a_n^{I\dagger} \qquad \text{for } n \geq 1, \tag{4.26}$$

so that the commutation relations for the new operators are

$$\left[a_m^I\,,\,a_n^{J\dagger}\right] = \delta^{IJ}\,\delta_{mn} \quad , \tag{4.27}$$

exactly the form of the standard commutation relations for creation and annihilation operators.

Solving for X^- and the Virasoro Algebra:

Now that we understand the operators associated with the transverse coordinates $X^I(\tau,\sigma)$, we can think about solving for the dependent $X^-(\tau,\sigma)$ operators. The equations

$$\boxed{\left(\dot{X}^- \pm X^{-\prime}\right) = \frac{1}{2\beta\alpha' p^+}\left(\dot{X}^I \pm X^{I\prime}\right)^2} \tag{3.26}$$

and

$$\boxed{\dot{X}^\mu \pm X^{\mu\prime} = \sqrt{2\alpha'}\sum_{n=-\infty}^{\infty}\alpha_n^\mu\, e^{-in(\tau\pm\sigma)}} \tag{3.34}$$

are still expected to hold as operator equations, but Eq. (3.26) leads to ordering ambiguities, since $\dot{X}^I$ and $X^{I\prime}$ do not commute. One can proceed to solve for α_n^- obtaining as before the result

$$\boxed{\alpha_n^- = \frac{1}{\sqrt{2\alpha'}\,p^+}L_n^\perp\ , \qquad \text{where}\ \ L_n^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty}\alpha_{n-p}^I\,\alpha_p^I\ ,} \tag{3.39}$$

but now the ordering of the operator products in the expression on the right is ambiguous. For $n \neq 0$ the operators commute, but for $n = 0$ they do not. Since the commutator is a c-number, the operator Lp_0 is ambiguous in that it might contain an arbitrary c-number. To be able to at least discuss the ambiguity quantitatively, we **define**

$$\boxed{L_0^\perp \equiv \frac{1}{2}\sum_{p=-\infty}^{\infty} :\alpha_{n-p}^I\alpha_p^I:\ ,} \tag{4.28}$$

where

$$:expression: \ \equiv \text{Normal ordered form of } expression. \tag{4.29}$$

In this notation, given Eq. (4.26), normal ordering means that positive indices are placed to the right. Given the normal ordering prescription, $:\alpha_{n-p}^I\alpha_p^I:$ has the same value for p and $-p$, so $L_0^\perp$ can be expanded as

$$L_0^\perp = \frac{1}{2}\alpha_0^I\alpha_0^I + \sum_{p=1}^{\infty}\alpha_{-p}^I\alpha_p^I = \alpha' p^I p^I + \sum_{p=1}^{\infty} p a_p^{I\dagger} p_p^I\ . \tag{4.30}$$

With this definition, we allow for the ambiguous c-number in α_0^- by writing

$$\boxed{\alpha_n^- = \sqrt{2\alpha'}\,p^- = \frac{1}{\sqrt{2\alpha'}\,p^+}\left(L_0^\perp + a\right) \quad ,} \tag{4.31}$$

where a is an as yet unknown constant. The constant a is intimately tied to the spectrum of states in string theory, since the invariant M^2 for the string state can be written as

$$\begin{aligned} M^2 = -p^2 = 2\,p^+\,p^- - p^I\,p^I &= \frac{1}{\alpha'}\left(L_0^\perp + a\right) - p^I\,p^I \\ &= \boxed{\frac{1}{\alpha'}\left(N^\perp + a\right) \quad ,} \end{aligned} \tag{4.32}$$

where

$$\boxed{N^\perp \equiv \sum_{n=1}^{\infty} n\,a_n^{I\dagger}\,a_n^I \ .} \tag{4.33}$$

Now that we have an unambigous definition of the operators $L_n^\perp$, called the Virasoro operators, we can calculate their commutators. They are calculated in the textbook, and in a set of notes that are posted on the course home page, so I will not repeat the calculation here. The result, however, is

$$\boxed{\left[L_m^\perp\,,\,L_n^\perp\right] = (m-n)L_{m+n}^\perp + \frac{1}{12}m(m^2-1)(D-2)\delta_{m+n,0} \ .} \tag{4.34}$$

Other commutators that may prove useful are the following:

$$\boxed{\left[L_m^\perp\,,\,\alpha_n^J\right] = -n\alpha_{m+n}^J} \tag{4.35}$$

and

$$\boxed{\left[L_m^\perp\,,\,x_0^I\right] = -i\sqrt{2\alpha'}\,\alpha_m^I \ .} \tag{4.36}$$

When the Virasoro generators act on string coordinates, they generate reparameterizations of the string:

$$\boxed{\left[L_m^\perp\,,\,X^I(\tau,\sigma)\right] = \xi_m^\tau\,\dot{X}^I + \xi_m^\sigma\,X^{I\prime}\ ,} \tag{4.37}$$

where

$$\boxed{\begin{aligned} \xi_m^\tau(\tau,\sigma) &= -ie^{im\tau}\,\cos m\sigma\ , \\ \xi_m^\sigma(\tau,\sigma) &= e^{im\tau}\,\sin m\sigma\ \ . \end{aligned}} \tag{4.38}$$

This is a reparameterization in the sense that the change in X^I is proportional to the derivative of X^I with respect to σ or τ, so the effect is the same as shifting these parameters by an infinitesimal amount. That is,

$$\left[\epsilon L_m^\perp\,,\,X^I(\tau,\sigma)\right] = X^I(\tau+\epsilon\xi_m^\tau,\sigma+\epsilon\xi_m^\sigma) - X^I(\tau,\sigma) + \mathcal{O}(\epsilon^2)\ . \tag{4.39}$$

Testing Lorentz Invariance:

We now confront an important issue, which is central to the question of whether this theory is well-defined at all. The action that we started with was manifestly Lorentz-invariant, and we intended to construct a Lorentz-invariant theory, consistent with the fact that no violations of Lorentz invariance have ever been seen. In quantizing the theory, however, we found it convenient to ignore the Lorentz symmetry, choosing a light-cone gauge in which the 0 and 1-directions are treated in a special way. There is no guarantee, therefore, that the theory that we have constructed is in fact Lorentz-invariant. To see if the theory is invariant, we will attempt to construct the operators that generate Lorentz transformations.

We begin to construct the Lorentz generators by using Eq. (1.15),

$$M_{\mu\nu} = \int \mathcal{M}^\tau_{\mu\nu}(\tau,\sigma)\,d\sigma = \int \left(X_\mu \mathcal{P}^\tau_\nu - X_\nu \mathcal{P}^\tau_\mu\right) d\sigma\ . \tag{1.15}$$

I guess we have not shown it in this course, but in general a quantity that is conserved by virtue of a symmetry will also the generator of that symmetry. For example, angular momentum is conserved as a consequence of rotational invariance, and angular momentum in turn is the generator of rotations. Inserting the mode expansion (3.35) into Eq. (1.15), using Eq. (3.20) for $\mathcal{P}^\tau_\mu$, one soon finds

$$M^{\mu\nu} = x_0^\mu p^\nu - x_0^\nu p^\mu - i\sum_{n=1}^\infty \frac{1}{n}\left(\alpha^\mu_{-n}\,\alpha^\nu_n - \alpha^\nu_{-n}\,\alpha^\mu_n\right)\ \ . \tag{4.40}$$

We consider in particular M^{-I}, which to be consistent with Lorentz symmetry should satisfy

$$\left[M^{-I}\,,\,M^{-J}\right]=0\ . \tag{4.41}$$

From Eq. (4.40), the expression for M^{-I} should be

$$M^{-I}=x_0^-p^I-x_0^Ip^--i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^-\,\alpha_n^I-\alpha_{-n}^I\,\alpha_n^-\right)\ . \tag{4.42}$$

This expression is not Hermitian, however, since x_0^I and p^- do not commute. It can be made Hermitian, however, by symmetrizing the product.

$$M^{-I}=x_0^-p^I-\frac{1}{2}(x_0^Ip^-+p^-x_0^I)-i\sum_{n=1}^{\infty}\frac{1}{n}\left(\alpha_{-n}^-\,\alpha_n^I-\alpha_{-n}^I\,\alpha_n^-\right)\ . \tag{4.43}$$

Finally, to clarify the meaning, we replace the α_n^- operators by their expression in terms of Virasoro operators, including the constant a:

$$M^{-I}=x_0^-p^I-\frac{1}{4\alpha' p^+}\Big(x_0^I\left(L_0^\perp+a\right)+\left(L_0^\perp+a\right)x_0^I\Big)-\frac{i}{\sqrt{2\alpha'}\,p^+}\sum_{n=1}^{\infty}\frac{1}{n}\left(L_{-n}^\perp\,\alpha_n^I-\alpha_{-n}^I\,L_n^\perp\right)\ . \tag{4.44}$$

The calculation of the commutator is too complicated to be included in the textbook, and I have not tried it yet myself, but the answer is claimed to be

$$\boxed{\begin{aligned}\left[M^{-I}\,,\,M^{-J}\right]=&-\frac{1}{\alpha'\,p^{+2}}\sum_{m=1}^{\infty}\left(\alpha_{-m}^I\,\alpha_m^J-\alpha_{-m}^J\,\alpha_m^I\right)\\&\times\left\{m\left[1-\frac{1}{24}(D-2)\right]+\frac{1}{m}\left[\frac{1}{24}(D-2)+a\right]\right\}\ .\end{aligned}} \tag{4.45}$$

This quantity will vanish identically only if $D=26$ and $a=-1$. Thus, we have found that the bosonic string is a consistent Lorentz-invariant theory only in 26 dimensions!

V. Unfinished Business:

There are still some important topics that I did not get to, including construction of the state space, tachyons and D-brane decay, and the quantization of closed strings. Although I didn't get that far, these topics are important. Be sure to study them on your own. I also did not touch on the particle mechanics or field theory discussions of Chapters 10 and 11.