PROFESSOR: Let's do E less than V not. So we're back here. And now of the energy e here is v not is x equal 0. X-axis. And that's the situation. Now you could solve this again. And do your calculations once more. But we can do this in an easier way by trusting the principle of analytic continuation.

In this case, it's very clear and very unambiguous. So the big words, analytic continuation, don't carry all the mathematical depth. But it's a nice, simple thing. We first say that the solution is the same for x less than 0. So for x less than 0, we write the same solution.

Because the energy is greater than 0, or all what we said here, the value of k squared, a into the ikhd e to the minus i k x. It's all good. And k squared is still 2 m e over h squared. The problem is the region where x is greater than 0. Because there you have an exponential. But now you must have a decaying exponential.

But we know how that should work. It should really be an e to the minus some kappa x. So how could I achieve that? If I let k bar replace-- everywhere you see k bar, replace it by i kappa. Park then one thing that happens is that-- you learn from here, from this k bar squared, would be minus kappa squared equal to that.

So kappa squared would be 2 m, v not minus e over h squared. The sign is just the opposite from this equation. That's what that equation becomes upon that substitution. Now that substitution would not make any difference, how they put a plus i or minus i, I wouldn't have gotten my sine change, and this.

But if I look at the solution there, the solution psi becomes-- on the region x greater than 0, turns into c, e to the i. kappa bar is a i kappa x. Therefore, it's equal to c, e to the minus kappa x, which is the right thing. And that sine that I chose, of letting k equal i kappa proves necessary to get the right thing.

So it's clear that to get the right thing, you have that. And now you know that, of course, if you would have written the equation from the beginning, you would have said, yes, in this region, there is a decaying thing. And looking at the Schrodinger equation, you have concluded that kappa is given by [? that. ?]

But the place where you now save the time is that, since I just must do this change in the equations, I can do that change in the solutions as well. And I don't have to write the continuity equations again, nor solve them. I can take the solutions and let everywhere that was a kappa bar replaced by i, that was a k bar, replace it by i kappa bar.

So what do we get? It should go here and believe those circuits. OK, so b over a, that used to be. Top blackboard there. Middle, k minus k bar becomes k plus i-- no, minus I k bar, minus i kappa. And k plus i kappa. so it has changed.

Suddenly this ratio has become complex. It's kind of interesting. Well, let's make it clearer by factoring a minus i here. So this becomes kappa. And you need plus k, so this must be plus i k over i. This would be kappa minus i k. So this is just minus kappa plus i k over kappa minus i k.

But when you see that ratio, you're seeing the ratio of two complex numbers of equal length. And therefore, that ratio is just a phase. It's not any magnitude. So this is just a phase, and it deserves a new name. There is a phase shift between the b coefficient and the a coefficient. And we'll write it as e minus-- the minus I'll keep. e to the 2 i delta.

That depends on the energy. I'll put delta of the energy, because after all, kappa, k, everybody depends on the energy. So let's call it 2 i delta of e. And what is delta of e? Well, think of the number kappa plus i k. This is the k, i k here.

The angle-- this complex number has an angle that, in fact, is delta. e to i delta is that phase. And delta is the arc tangent, k over kappa. So I'll write it like that. Delta. Now you get a delta from the numerator, a minus delta, as you can imagine, from the denominator. And that's why you get a total 2 i delta here.

So delta of e is 10 minus 1, k over kappa. And if you look at what k and kappa were, k over kappa is like the ratio of the square root of the energy over v not minus the energy. So delta of e is equal to 10 minus 1 square root of energy over e v not minus and energy.

Now I got the question about current conservation. What happens to current conservation this time? Well, you have all these waves here. But on the region x greater than 0, the solution is real. If the solution is real, there is no probability current on the right. There's really no probability that you get this thing, and you get current flowing there. And you get this pulse, or whatever you send to keep moving and moving and moving to the right.

Indeed, the solution decays. And it looks like the one of the bound state in this region. So eventually there's no current here, because there's no current here. No current there. No current there. Because the solution drops down. But it's a real solution anyway. So there's no current there.

So Ja-- Jc is equal to 0. 0. Solution is real for x greater than 0, and any way goes to 0 at infinity. So the fact that it's real is a mathematical nicety that help us realize that it must be 0. But the fact that there's no current far away essentially telling you better be 0.

So if the current Jc is 0, Ja must be equal to Jb. And therefore that means a squared is equal to b squared. And happily that's what happened because b over a is a complex number of magnitude one. So the fact that b and a differ by just the phase was required by current conservation.

A over b is a number that has norm equal to 1. So that's a consistent picture. This phase is very important. So what happens for resolution for x less than 0? Well, psi of x would be a into the i k x, plus b all the way to the left there.

Your solution is a plus b equal minus i k x. Of course, we now know what the b is, so this is minus a from the ratio over there. e to the 2 i delta of e. E to the minus i k x. For x less than 0. And for x greater than 0, psi of x is going to c e to the minus kappa x. And I'm not bothering to write the coefficient c in Terms of a.

Now this expression for x less than 0 can be simplified a little. You can factor an a. But it's very nice, and you should have an eye for those kind of simplifications. It's very nice to factor more than an a and to factor one phase like an i delta. I delta of e, because in that way, you get e to the i k x minus delta of e from the first term, where the two delta appearances cancel each other.

Because the first term didn't have a delta. But then the second term will have the same argument here, of the exponential but with a minus sign. Minus i k x. And I claim also a minus delta of e. And this time minus and minus gives you plus. And the other e to the i delta gives you back the 2.

But now you've created the trigonometric function, which is simpler to work with. So psi effects is equal to 2 i a e to i delta of e, sine of k x minus delta of e. And if you wish psi squared, the probability density is proportional to 4 a squared, sine squared of k x minus delta of e.

So this can be plotted. Sine squared is like that. And where is x equal 0. OK, I'll say x equal 0, say is here. So this is not really true anymore. But this point here is x 0. It vanishes. Would be the point at which k x 0 is equal to delta of e. And the sine squared vanishes. So this is not a solution either.

Solution is like that. And then this is for psi squared. And then it must couple to the k exponential on this side. So that the true solution must somehow be like this and well, whatever. I don't know how it looks. That is the e to the minus 2 kappa x decay and exponential.

It must decay. There's continuity of the derivative and continuity of the wave function. So that's how this should look. A couple more things we can say about this solution that will play a role later. I want to get just a little intuition about this phase, delta of e, this phase shift.

So we have it there. Delta of e, I'll write it here, so you won't have to-- 10 minus 1, square root of e over v not minus e. So this is interesting. This phase shift just applies for energies up to v not. And that corresponds to the fact that we've been solving for energies under the barrier. And if we solve for energies under the barrier, well, the solutions as we're writing with these complex numbers, apply up to energies equal to the barrier, but no more.

So we shouldn't plot beyond this place. And here is delta of e. The phase shift. And when the energy is 0, when your particle you're sending in, or the packet eventually is very low energy here. Then the phase shift is the arctangent of 0, which is 0.

As the energy goes to the value v not, then the denominator goes to 0. The ratio goes to infinity. And the arctangent is pi over 2. So it's a curve that goes from here to here. And it's not quite like a straight line. But because of the square roots, it sort of begins kind of vertical, then goes like this. It's not flat either, in the middle. So maybe my curve doesn't look too good. Those more vertical here. Wow, I'm having a hard time with this. Something like this.

In fact, it's kind of interesting to plug the derivative, d delta, d energy. A little calculation will give you this expression. You can do this with mathematica or v not minus e. And shows, in fact, that here is v not, and here is v delta, v e.

We could call it delta prime of e, because we wrote the phase shift as a function of the energy. So that the delta, d e is really delta prime of e. And it sort of infinite-- goes to a minimum and infinite again, in that direction. That's how it behaves.