PROFESSOR: Delta function potential. So it's still a one-dimensional potential-- potential is a function of x. We'll write it this way-- minus alpha delta of x, where alpha is positive. So this is a delta function in a negative direction.

So if you want to draw the potential-- there's no way to draw really nicely a delta function. So you just do a thick arrow with it pointing down. It's a representation of a potential that, somehow, is rather infinite at x equals zero-- but infinite and negative.

It can be thought of as the limit of a square well that is becoming deeper and deeper. And, in fact, that could be a way to analytically calculate the energy levels-- by taking carefully the limit of a potential. It is becoming thinner and thinner, but deeper and deeper, which is the way you define or regulate the delta function.

You can imagine the delta function as a sequence of functions, in which it's becoming more and more narrow-- but deeper at the same time. So that the area under the curve is still the same.

So, at any rate, the delta function potential is a potential that should be understood as 0 everywhere , else except at the delta function where it becomes infinite. And there are all kinds of questions we can ask.

OK. Are there bound states? What are bound states in this case? They are energy eigenstates with energy less than zero. So bound states, which means e less than zero. Do they exist? Does this potential have bound states?

And, if it does, how many bound states? 1, 2, 3? Does It depend on the intensity of the delta function? When you get more bound states, the deeper the potential is. Well, we'll try to figure out.

In fact, there's a lot that can be figured out without calculating, too much. And it's a good habit to try to do those things before you-- not to be so impatient that you begin, and within a second start writing the differential equation trying to solve it. Get a little intuition about how any state could look like, and how could the answer for the energy eigenstates-- the energies-- what could they be?

Could you just reason your way and conclude there's no bound states? Or one bound state?

Or two? All these things are pretty useful. So one way, as you can imagine, is to think of units.

And what are the constants in this problem? In this problem we'll have three constants. Alpha, the mass of the particle, and h-bar. So with alpha, the mass and the particle, and h-bar you can ask, how do I construct the quantity with units of energy?

If there there's only one way to construct the quantity with units of energy, then the energy of a bound state will be proportional to that quantity-- because that's the only quantity that can carry the units.

And here, indeed, there's only one way to construct that quantity with units of energy-- from these three. That's to be expected. With three constants that are not linearly dependent-- whatever that is supposed to mean-- you can build anything that has units of length, mass, or time. And from that you can build something that has units of energy.

So you can now decide, well, what are the units of alpha? The units of alpha have give you energy, but the delta function has units of one over length. This has one over length. , Remember if you integrate over x the delta function gives you 1. So this has units of 1 over length. And, therefore, alpha has to have units of energy times length.

So this is not quite enough to solve the problem, because I want to write e-- think of finding how do you get units of energy from these quantities? But l-- we still don't have a length scale either. So we have to do a little more work.

So from here we say that units of energy is alpha over l. There should be a way to say that this is an equality between units. I could put units or leave it just like that. So in terms of units, it's this. But in terms of units, energy-- you should always remember-- is p squared over m. And p is h-bar over a length. So that's p squared and that's m. So that's also units of energy

From these two you can get what has units of length. Length. You pass the l to this side-- the l squared to this left-hand side. Divide. So you get l is h squared over m alpha. And if I substitute back into this l here, e would be alpha over l, which is h squared, alpha squared, m.

So that's the quantity that has units of energy. M alpha squared over h squared has units of energy. If this has units of energy-- the bound state energy. Now, if you have a bounce state here, it has to decay in order to be normalizable. In order to be normalizable it has to decay, so it has to be in the forbidden region throughout x.

So the energy as we said is negative, energy of a bound state-- if it exists. And this bound state energy would have to be negative some number m alpha squared over h squared. And that's very useful information.

The whole problem has been reduced to calculating a number. It better be and the answer cannot be any other way. There's no other way to get the units of energy. So if a bound state exists it has to be that. And that number could be pi, it could be 1/3, 1/4, it could be anything.

There's a naturalness to that problem in that you don't expect that number to be a trillion. Nor do you expect that number to be 10 to the minus 6. Because there's no way-- where would those numbers appear? So this number should be a number of order one, and we're going to wait and see what it is.

So that's one thing we know already about this problem. The other thing we can do is to think of the regulated delta function. So we think of this as a potential that has this form. So here is v of x, and here is x. And for this potential-- if you have a bound state-- how would the wave function look?

Well, it would have to-- suppose you have a ground state-- it's an even potential. The delta function is even, too. It's in the middle. It's symmetric. There's nothing asymmetric about the delta function. So if it's an even potential the ground states should be even, because the ground state is supposed to have no nodes. And it's supposed to be even if the potential is even.

So how will it look? Well, it shouldn't be decaying in this region. So, presumably, it decays here. It decays there-- symmetrically. And in the middle it curves in the other direction. It is in an allowed region-- and you remember that's kind of allowed this way. So that's probably the way it looks.

Now, if that bound state exists, somehow, as I narrow this and go down-- as it becomes even more narrow, very narrow now, but very deep. This region becomes smaller. And I would pretty much expect the wave function to have a discontinuity.

You basically don't have enough power to see the curving that is happening here. Especially because the curving is going down. The distance is going down. So if this bound state exists, as you approach the limit in which this becomes a delta function the energy moves a little, but stays finite at some number. And the curvature that is created by the delta function is not visible, and the thing looks just discontinuous in its derivative.

So this is an intuitive way to understand that the wave function we're looking for is going to be discontinuous on its derivative. Let's write the differential equation, even though we're still not going to solve it. So what is the differential equation? Minus h squared over m, psi double prime, is equal to E psi.

And, therefore-- and I write this, and you say, oh, what are you writing? I'm writing the differential equation when x is different from 0. No potential when x is different from 0. So this applies for positive x and negative x. It doesn't apply at x equals 0. We'll have to deal with that later.

So then, no potential for x different from 0. And this differential equation becomes psi double prime equals minus 2m e over h squared psi. And this is equal to kappa squared psi, where kappa squared is minus 2me over h squared. And it's positive. Let's make that positive.

It's positive because the energy is negative and we're looking for bound states. So we're looking for bound states only. Kappa squared is positive. And this differential equation is just this. I'll copy it again here. Kappa squared psi.

And the solutions of this equation are-- solutions-- are e to the minus kappa x and e to the kappa x. Or, if you wish, cosh kappa x and sinh kappa x-- whichever you prefer.

This is something we now have to use in order to produce a solution. But now, let's see if I can figure out how many bound states there are. If there is one bound state, it's going to be even. It's the ground state. It has no nodes. It has to be even, because the potential is even. If I have the first excited state after the ground state, it will have to be odd. It would have to vanish at x equals 0, because it's odd. There is it's node-- it has to have one node.

For an odd bound state-- or first excited state-- you'd have to have psi equals 0 at x equals 0. And the way to do that would be to have a sinh, because this doesn't vanish at zero. This doesn't vanish at zero. And cosh doesn't vanish at zero.

So you would need psi of x equals sinh of kappa x. But that's not good. sinh of kappa x is like this and blows up. Blows down. It has to go like this. It is in a forbidden region, so it has to be convex towards the axis. And convex here. But it blows up. So there's no such solution. No such solution.

You cannot have an odd bound state. So since the bound states alternate-- even, odd, even, odd, even, odd-- you're stuck. You only will have a ground state-- if we're lucky-- but no excited state that is bound, while a finite square will.

You remember this quantity z0 that tells you how many bound states you can have. Probably you're anticipating that in the case of the delta function potential, you can only have one bound state, if any. The first excited state would not exist. So, enough preliminaries. Let's just solve that now.