8.03 Lecture 3

Summary: solution of $\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = 0$

(0): $\Gamma = 0$ No damping:

$$\theta(t) = A\cos(\omega_0 t + \alpha)$$

(1): $\omega_0^2 > \frac{\Gamma^2}{4}$ Underdamped Oscillator:

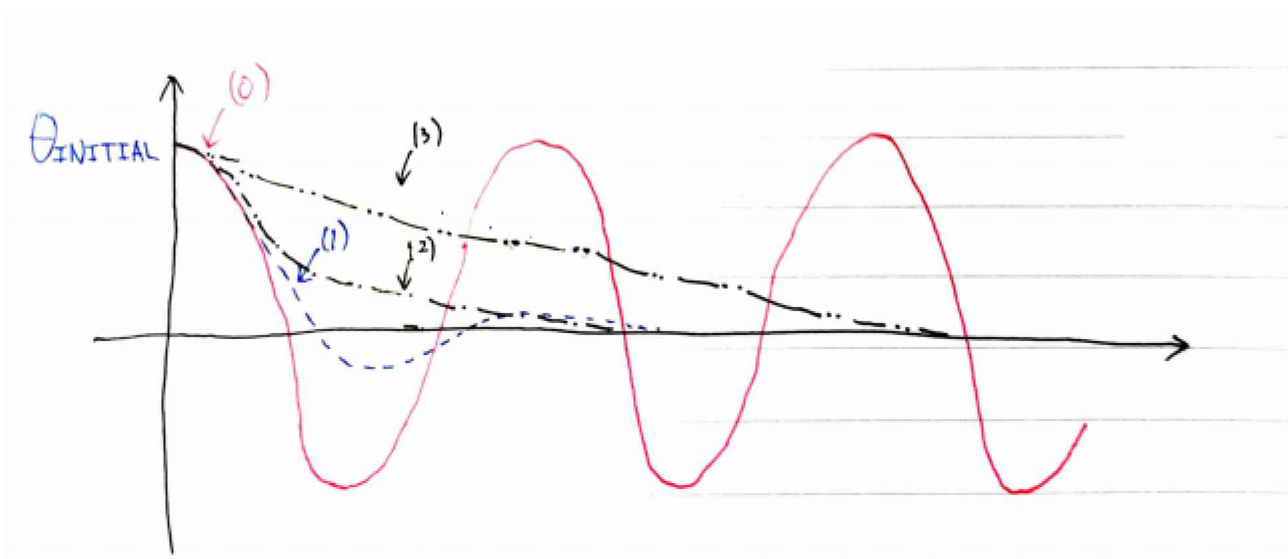
$$\theta(t) = Ae^{-\Gamma t/2}\cos(\omega t + \alpha) \text{ where } \omega = \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}$$

(2): $\omega_0^2 = \frac{\Gamma^2}{4}$ Critically damped oscillator:

$$\theta(t) = (A + Bt)e^{-\Gamma t/2}$$

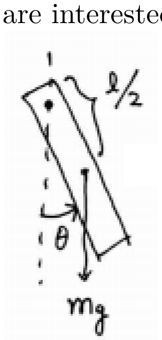
(3): $\omega_0^2 < \frac{\Gamma^2}{4}$ Overdamped Oscillator:

$$\theta(t) = Ae^{-(\Gamma/2+\beta)t} + Be^{-(\Gamma/2-\beta)t} \text{ where } \beta = \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}$$



Continue from lecture 2:

Now we are interested in giving a driving force to this rod:



Assume that the force produces a torque:

$$\tau_{DRIVE} = d_0\cos\omega_d t$$

Total torque:

$$\tau(t) = \tau_g(t) + \tau_{DRAG}(t) + \tau_{DRIVE}(t)$$

Equation of motion: $\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = \frac{d_0}{I}\cos\omega_d t$
Where, from last lecture, we have defined:

$$\Gamma \equiv \frac{3b}{ml^2} \quad \omega_0 \equiv \sqrt{\frac{3g}{2l}}$$

Where Γ is the size of the drag force and ω_0 is the natural angular frequency (i.e., without drive). Also, define $f_0 \equiv \frac{d_0}{I}$. Now our equation of motion reads:

$$\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = f_0\cos\omega_d t$$

We would like to construct something to "cancel" $\cos\omega_d t$. Idea: use complex notation:

$$\ddot{z} + \Gamma\dot{z} + \omega_0^2 z = f_0 e^{i\omega_d t}$$

Guess:

$$z(t) = Ae^{i(\omega_d t - \delta)}$$

where the δ is designed to cancel $e^{i\omega_d t}$. It takes some time for the system to "feel" the driving torque. Taking our derivatives gives us:

$$\dot{z}(t) = i\omega_d z$$
$$\ddot{z}(t) = -\omega_d^2 z$$

Insert these results into the equation of motion:

$$\begin{aligned}(-\omega_d^2 + i\omega_d\Gamma + \omega_0^2)z(t) &= f_0 e^{i\omega_d t}\\ (-\omega_d^2 + i\omega_d\Gamma + \omega_0^2)Ae^{i(\omega_d t-\delta)} &= f_0 e^{i\omega_d t}\\ (-\omega_d^2 + i\omega_d\Gamma + \omega_0^2)A &= f_0 e^{i\delta}\\ &= f_0(\cos\delta + i\sin\delta)\end{aligned}$$

Since this is a complex equation, we can solve for A and δ
Real part: $(\omega_0^2 - \omega_d^2)A = f_0\cos\delta$
Imaginary part: $\omega_d\Gamma A = f_0\sin\delta$
Squaring both of these equations and adding them together yields:

$$A^2\left[(\omega_0^2 - \omega_d^2) + \omega_d^2\Gamma^2\right] = f_0^2$$
$$A(\omega_d) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_d^2) + \omega_d^2\Gamma^2}}$$

Dividing the imaginary part by the real part yields:

$$\tan\delta = \frac{\Gamma\omega_d}{\omega_0^2 - \omega_d^2}$$

$$\Rightarrow \theta(t) = \text{Re}[z(t)] = A(\omega_d)\cos\left(\omega_d t - \delta(\omega_d)\right) \tag{1}$$

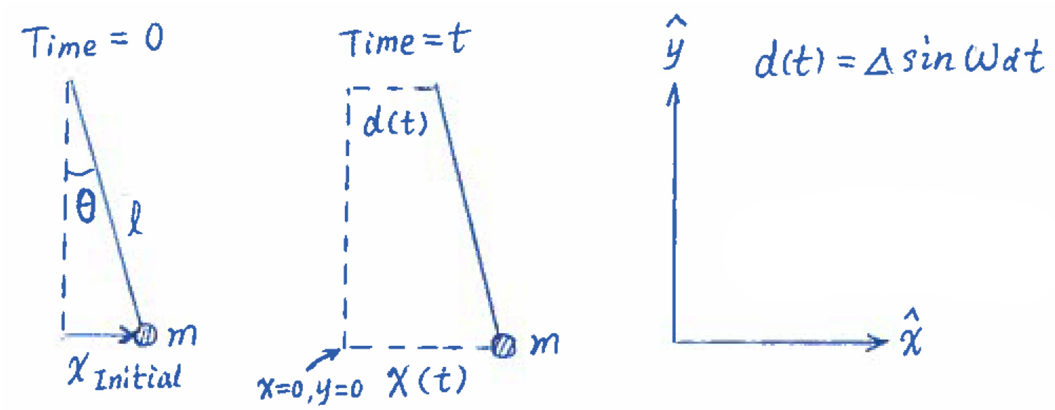
Where both $A(\omega_d)$ and $\delta(\omega_d)$ are functions of ω_d.
No free parameter?! Actually, this is the a particular solution. The full solution (if we prepare the system in the "underdamped" mode) is:

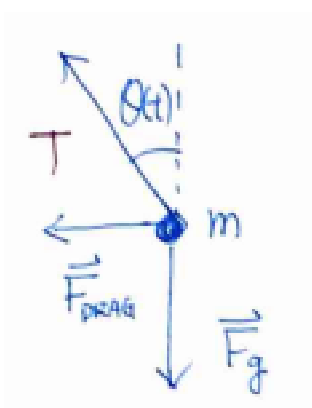
$$\theta(t) = A(\omega_d)\cos(\omega_d t - \delta) + Be^{-\Gamma t/2}\cos(\omega t + \alpha)$$

Where the left side with amplitude A is the steady state solution and the right side with amplitude B will die out as $t \to \infty$.

You may be confused with so many different ω's!! To clarify:
ω_0 is the "natural angular frequency." In our example with the rod, $\omega_0 = \sqrt{3g/2l}$
ω: this frequency is lower if there is a drag force. It is defined by the equation $\omega = \sqrt{\omega_0^2 - \Gamma^2/4}$
ω_d is the frequency of the driving torque or force

Example: Driving a pendulum





Force diagram

$$\vec{F}_{DRAG} = -b\dot{x}\hat{x}$$
$$\vec{F}_g = -mg\hat{y}$$
$$\vec{T} = -T\sin\theta\hat{x} + T\cos\theta\hat{y}$$

Take a small angle approximation: $\sin\theta \approx \theta = \frac{x-d}{l}$ and $\cos\theta \approx 1$
This implies:

$$\vec{T} \approx -T\frac{x-d}{l}\hat{x} + T\hat{y}$$

In the $\hat{x}$ direction we have

$$m\ddot{x} = -b\dot{x} - T\frac{x-d}{l}$$

and in the $\hat{y}$ direction we have

$$0 = m\ddot{y} = -mg + T$$

where the force has to be zero because there is no vertical motion (assuming a small angle). We now know $mg = T$.
Setting up our equation of motion we have

$$m\ddot{x} + b\dot{x} + \frac{mg}{l}x = \frac{mg}{l}\Delta\sin\omega_d t$$
$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{g}{l}x = \frac{g\Delta}{l}\sin\omega_d t$$

To compare with our previous solution, define $\Gamma \equiv b/m$, $\omega_0^2 \equiv g/l$, and $f_0 \equiv g\Delta/l$ to give

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = f_0\sin\omega_d t$$

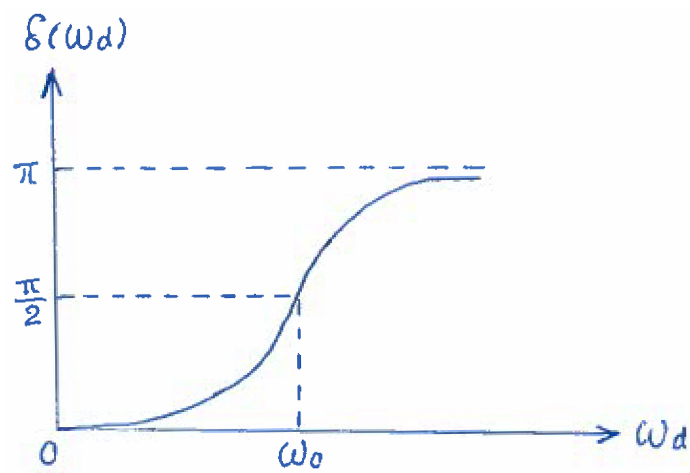
Let us examine the amplitude:

$$A(\omega_d) = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_d^2) + \omega_d^2\Gamma^2}}$$

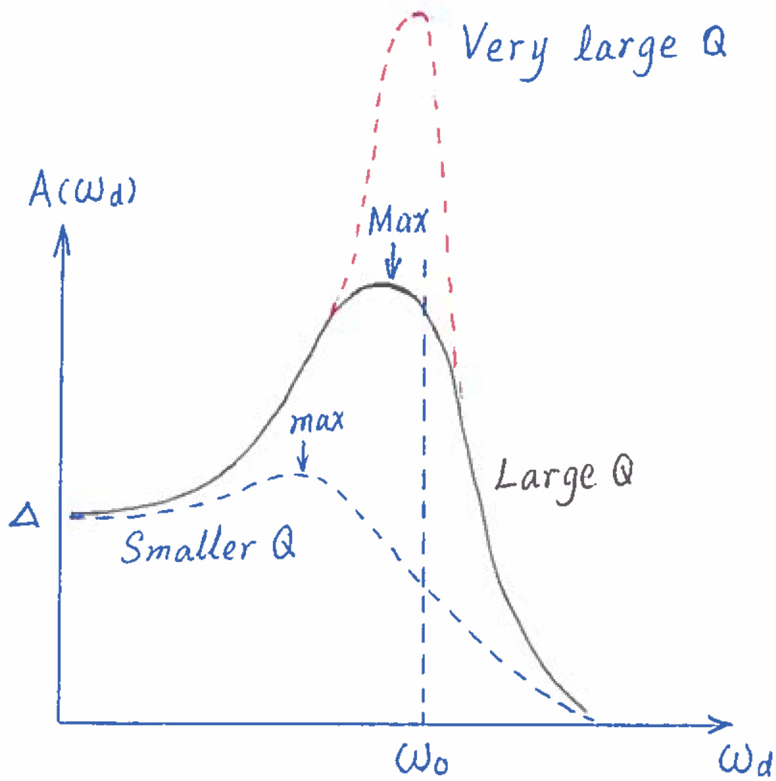
There are a few cases we need to consider:
(1) $\omega_d \to 0$

$$A(\omega_d) = \frac{f_0}{\omega_0^2} = \frac{g\Delta/l}{g/l} = \Delta$$

The amplitude will simply be the amplitude of the initial displacement. If the drive frequency is zero then $\tan\delta = 0 \to \delta = 0$.
(2) $\omega_d \to \infty$
$A(\omega_d) \Rightarrow 0$ and $\tan\delta \to \infty$ therefore $\delta = \pi$



A plot of the phase as a function of the drive frequency.



A plot of the amlitude as a function of the drive frequency.

There is a third possibility:
(3) $\omega_d \approx \omega_0$
This is called driving "on resonance." Even a small Δ can produce a large A, amplitude:

$$A(\omega_0) = \frac{f_0}{\omega_0 \Gamma} = \frac{\omega_0^2 \Delta}{\omega_0 \Gamma} = \frac{\omega_0}{\Gamma}\Delta = Q\Delta$$

Where $Q \equiv \omega_0/\Gamma$ and is a large parameter which gives a large amplitude.