Topics: Coordinate Systems; Gradients; Line and Surface Integrals
Related Reading:
        Spring 2006 Math Review Presentation
        Hale Bradt's Spring 2001 8.02 Mathematics Supplement

Topic Introduction

Today we go over some of the more advanced mathematical concepts we will need in the course, so that you see the mathematics before being introduced to the physics. Maxwell's equations as we will state them involve line and surface integrals over open and closed surfaces. A closed surface has an inside and an outside, e.g. a basketball, and there is no two dimensional contour that "bounds" the surface. In contrast, an open surface has no inside and outside, e.g. a flat infinitely thin plate, and there is a two dimensional contour that bounds the surface, e.g. the rim of the plate. There are four Maxwell's equations:

$$(1)\ \oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{in}}{\varepsilon_0} \qquad\qquad (2)\ \oiint_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}} = 0$$

$$(3)\ \oint_C \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -\frac{d\Phi_B}{dt} \qquad\qquad (4)\ \oint_C \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{enc} + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

Equations (1) and (2) apply to closed surfaces. Equations (3) and (4) apply to open surfaces, and the contour C represents the line contour that bounds those open surfaces.

There is not need to understand the details of the electromagnetic application right now; we simply want to cover the mathematics in this problem solving session.

Line Integrals
The line integral of a scalar function $f(x, y, z)$ along a path C is defined as

$$\int_C f(x, y, z)\, ds = \lim_{\substack{N \to \infty \\ \Delta s_i \to 0}} \sum_{i=1}^{N} f(x_i, y_i, z_i) \Delta s_i$$

where C has been subdivided into N segments, each with a length Δs_i.

Line Integrals Involving Vector Functions
For a vector function

$$\vec{\mathbf{F}} = F_x\, \hat{\mathbf{i}} + F_y\, \hat{\mathbf{j}} + F_z\, \hat{\mathbf{k}}$$

the line integral along a path C is given by

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_C \left(F_x\, \hat{\mathbf{i}} + F_y\, \hat{\mathbf{j}} + F_z\, \hat{\mathbf{k}} \right) \cdot \left(dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}} + dz\, \hat{\mathbf{k}} \right) = \int_C F_x dx + F_y dy + F_z dz$$

where

$$d\vec{\mathbf{s}} = dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}} + dz\, \hat{\mathbf{k}}$$

is the differential line element along C.

Surface Integrals

A function $F(x, y)$ of two variables can be integrated over a surface S, and the result is a double integral:

$$\iint_S F(x, y)\, dA = \iint_S F(x, y)\, dx\, dy$$

where $dA = dx\, dy$ is a (Cartesian) differential area element on S. In particular, when $F(x, y) = 1$, we obtain the area of the surface S:

$$A = \iint_S dA = \iint_S dx\, dy$$

Surface Integrals Involving Vector Functions

For a vector function $\vec{\mathbf{F}}(x, y, z)$, the integral over a surface S is is given by

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{A}} = \iint_S \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}\, dA = \iint_S F_n\, dA$$

where $d\vec{\mathbf{A}} = dA\, \hat{\mathbf{n}}$ and $\hat{\mathbf{n}}$ is a unit vector pointing in the normal direction of the surface. The dot product $F_n = \vec{\mathbf{F}} \cdot \hat{\mathbf{n}}$ is the component of $\vec{\mathbf{F}}$ parallel to $\hat{\mathbf{n}}$. The above quantity is called "flux." For an electric field $\vec{\mathbf{E}}$, the electric flux through a surface is

$$\Phi_E = \iint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \iint_S E_n\, dA$$

Important Equations

The line integral of a vector function:

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = \int_C \left(F_x\, \hat{\mathbf{i}} + F_y\, \hat{\mathbf{j}} + F_z\, \hat{\mathbf{k}} \right) \cdot \left(dx\, \hat{\mathbf{i}} + dy\, \hat{\mathbf{j}} + dz\, \hat{\mathbf{k}} \right) = \int_C F_x dx + F_y dy + F_z dz$$

The flux of a vector function: $\Phi_E = \iint_S \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, dA = \iint_S E_n\, dA$