

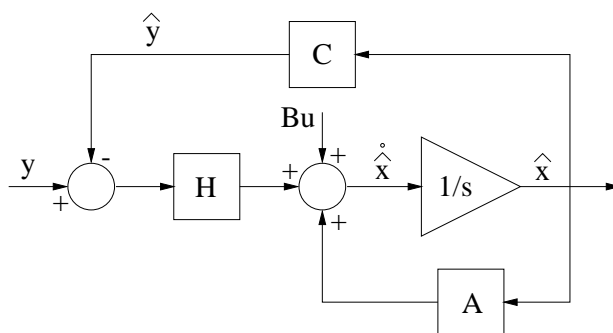
20 KALMAN FILTER

20.1 Introduction

In the previous section, we derived the linear quadratic regulator as an optimal solution for the full-state feedback control problem. The inherent assumption was that each state was known perfectly. In real applications, the measurements are subject to disturbances, and may not allow reconstruction of all the states. This *state estimation* is the task of a model-based estimator having the form:

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x}) \quad (241)$$

The vector \hat{x} represents the state estimate, whose evolution is governed by the nominal A and B matrices of the plant, and a correction term with the estimator gain matrix H . H operates on the estimation error mapped to the plant output y , since $C\hat{x} = \hat{y}$. Given statistical properties of real plant disturbances and sensor noise, the *Kalman Filter* designs an optimal H .



20.2 Problem Statement

We consider the state-space plant model given by:

$$\begin{aligned} \dot{x} &= Ax + Bu + W_1 \\ y &= Cx + W_2. \end{aligned} \quad (242)$$

There are n states, m inputs, and l outputs, so that A has dimension $n \times n$, B is $n \times m$, and C is $l \times n$. The plant is subject to two random input signals, W_1 and W_2 . W_1 represents

disturbances to the plant, since it drives \dot{x} directly; W_2 denotes sensor noise, which corrupts the measurement y .

An important assumption of the Kalman Filter is that W_1 and W_2 are each vectors of unbiased, independent white noise, and that all the $n + l$ channels are uncorrelated. Hence, if $E(\cdot)$ denotes the expected value,

$$E(W_1(t_1)W_1(t_2)^T) = V_1\delta(t_1 - t_2) \quad (243)$$

$$E(W_2(t_1)W_2(t_2)^T) = V_2\delta(t_1 - t_2) \quad (244)$$

$$E(W_1(t)W_2(t)^T) = 0_{n \times l}. \quad (245)$$

Here $\delta(t)$ represents the impulse (or delta) function. V_1 is an $n \times n$ diagonal matrix of intensities, and V_2 is an $l \times l$ diagonal matrix of intensities.

The estimation error may be defined as $e = x - \hat{x}$. It can then be verified that

$$\begin{aligned} \dot{e} &= [Ax + Bu + W_1] - [A\hat{x} + Bu + H(y - C\hat{x})] \\ &= (A - HC)e + (W_1 - HW_2). \end{aligned} \quad (246)$$

The eigenvalues of the matrix $A - HC$ thus determine the stability properties of the estimation error dynamics. The second term above, $W_1 + HW_2$ is considered an external input.

The Kalman filter design provides H that minimizes the scalar cost function

$$J = E(e^T W e), \quad (247)$$

where W is an unspecified symmetric, positive definite weighting matrix. A related matrix, the symmetric *error covariance*, is defined as

$$\Sigma = E(ee^T). \quad (248)$$

There are two main steps for deriving the optimal gain H .

20.3 Step 1: An Equation for $\dot{\Sigma}$

The evolution of Σ follows from some algebra and the convolution form of $e(t)$. We begin with

$$\begin{aligned} \dot{\Sigma} &= E(\dot{e}e^T + e\dot{e}^T) \\ &= E[(A - HC)ee^T + (W_1 - HW_2)e^T + ee^T(A^T - C^T H^T) + e(W_1^T - W_2^T H^T)]. \end{aligned} \quad (249)$$

The last term above can be expanded, using the property that

$$e(t) = e^{(A-HC)t}e(0) + \int_0^t e^{(A-HC)(t-\tau)}(W_1(\tau) - HW_2(\tau)) d\tau.$$

We have

$$\begin{aligned} E(e(W_1^T - W_2^T H^T)) &= e^{(A-HC)t}E(e(0)(W_1^T - W_2^T H^T)) + \\ &\quad \int_0^t e^{(A-HC)(t-\tau)}E((W_1(\tau) - HW_2(\tau)) \\ &\quad (W_1^T(t) - W_2^T(t)H^T)) d\tau \\ &= \int_0^t e^{(A-HC)(t-\tau)}E((W_1(\tau) - HW_2(\tau)) \\ &\quad (W_1^T(t) - W_2^T(t)H^T)) d\tau \\ &= \int_0^t e^{(A-HC)(t-\tau)}(V_1\delta(t-\tau) + HV_2H^T\delta(t-\tau)) d\tau \\ &= \frac{1}{2}V_1 + \frac{1}{2}HV_2H^T. \end{aligned}$$

To get from the first right-hand side to the second, we note that the initial condition $e(0)$ is uncorrelated with $W_1^T - W_2^T H^T$. The fact that W_1 and HW_2 are uncorrelated leads to the third line, and the final result follows from

$$\int_0^t \delta(t-\tau)d\tau = \frac{1}{2},$$

i.e., the written integral includes only *half* of the impulse.

The final expression for $E(e(W_1^T - W_2^T H^T))$ is symmetric, and therefore appears in Equation 249 twice, leading to

$$\dot{\Sigma} = (A - HC)\Sigma + \Sigma(A^T - C^T H^T) + V_1 + HV_2H^T. \quad (250)$$

This equation governs propagation of the error covariance. It is independent of the initial condition $e(0)$, and depends on the (as yet) unknown estimator gain matrix H .

20.4 Step 2: H as a Function of Σ

We now make the connection between $\Sigma = E(ee^T)$ (a matrix) and $J = E(e^T W e)$ (a scalar). The *trace* of a matrix is the sum of its diagonal elements, and it can be verified that

$$J = E(e^T W e) = \text{trace}(\Sigma W). \quad (251)$$

We now introduce an auxiliary cost function defined as

$$J' = \text{trace}(\Sigma W + \Lambda F), \quad (252)$$

where F is an $n \times n$ matrix of zeros, and Λ is an $n \times n$ matrix of unknown Lagrange multipliers. Note that since F is zero, $J' = J$, so minimizing J' solves the same problem. Lagrange multipliers provide an ingenious mechanism for drawing constraints into the optimization; the constraint we invoke is the evolution of Σ , Equation 250:

$$J' = \text{trace} \left(\Sigma W + \Lambda(-\dot{\Sigma} + A\Sigma - HC\Sigma + \Sigma A^T - \Sigma C^T H^T + V_1 + HV_2 H^T) \right) \quad (253)$$

If J' is an optimal cost, it follows that $\partial J'/\partial H = 0$, i.e., the correct choice of H achieves an extremal value. We need the following lemmas, which give the derivative of a trace with respect to a constituent matrix:

$$\begin{aligned} \frac{\partial}{\partial H} \text{trace}(-\Lambda HC\Sigma) &= -\Lambda^T \Sigma C^T \\ \frac{\partial}{\partial H} \text{trace}(-\Lambda \Sigma C^T H^T) &= -\Lambda \Sigma C^T \\ \frac{\partial}{\partial H} \text{trace}(\Lambda H V_2 H^T) &= \Lambda^T H V_2 + \Lambda H V_2. \end{aligned}$$

Proofs of the first two are given at the end of this section; the last lemma uses the chain rule, and the previous two lemmas. Next, we enforce $\Lambda = \Lambda^T$, since the values are arbitrary. Then the condition $\partial J'/\partial H = 0$ leads to

$$\begin{aligned} 0 &= 2\Lambda(-\Sigma C^T + H V_2), \text{ satisfied if} \\ H &= \Sigma C^T V_2^{-1}. \end{aligned} \quad (254)$$

Hence the estimator gain matrix H can be written as a function of Σ . Inserting this back into Equation 250, we obtain

$$\dot{\Sigma} = A\Sigma + \Sigma A^T + V_1 - \Sigma C^T V_2^{-1} C \Sigma. \quad (255)$$

Equations 254 and 255 represent the practical solution to the Kalman filtering problem, which minimizes the squared-norm of the estimation error. The evolution of Σ is always stable, and depends only on the constant matrices $[A, C, V_1, V_2]$. Notice also that the result is independent of the weighting matrix W , which might as well be the identity.

20.5 Properties of the Solution

The solution involves a matrix Riccati equation, like the LQR, suggesting a duality with the LQR problem. This is in fact the case, and the same analysis and numerical tools can be applied to both methodologies.

The steady-state solution for Σ is valid for time-invariant systems, leading to a more common MARE form of Equation 255:

$$0 = A\Sigma + \Sigma A^T + V_1 - \Sigma C^T V_2^{-1} C \Sigma. \quad (256)$$

Duality of Linear Quadratic Regulator and Kalman Filter

Linear Quadratic Regulator	Kalman Filter
$\dot{x} = Ax + Bu$	$\dot{x} = Ax + Bu + W_1$
$u = -Kx$	$y = Cx + W_2$
$2J = \int_0^\infty (x^T Qx + u^T Ru) dt$	$\hat{\dot{x}} = A\hat{x} + Bu + H(y - C\hat{x})$
$Q \geq 0, R > 0$	$J = E(e^T We)$
$K = R^{-1}B^T P$	$V_1 \geq 0, V_2 > 0$
$PA + A^T P + Q - PBR^{-1}B^T P = 0$	$H = \Sigma C^T V_2^{-1}$
	$\Sigma A^T + A\Sigma + V_1 - \Sigma C^T V_2^{-1} C \Sigma = 0$

The Kalman Filter is guaranteed to create a stable nominal dynamics $A - HC$, as long as the plant is fully state-observable. This is dual to the stability guarantee of the LQR loop, when the plant is state-controllable. Furthermore, like the LQR, the KF loop achieves 60° phase margin, and infinite gain margin, for all the channels together or independently.

The qualitative dependence of the estimator gain $H = \Sigma C^T V_2^{-1}$ on the other parameters can be easily seen. Recall that V_1 is the intensity matrix of the plant disturbance, V_2 is the intensity of the sensor noise, and Σ is the error covariance matrix.

- A large uncertainty Σ creates large H , placing emphasis on the corrective action of the filter.
- A small disturbance V_1 , and large sensor noise V_2 creates a small H , weighting the model dynamics $A\hat{x} + Bu$ more.
- A large disturbance V_1 , and small sensor noise V_2 creates a large H , so that the filter's correction is dominant.

The limiting closed-loop poles of the Kalman filter are similar, and dual to those of the LQR:

- $V_2 \ll V_1$: good sensors, large disturbance, $H \gg 1$, *dual to cheap-control problem*. Some closed-loop poles go to the stable plant zeros, or the mirror image of unstable plant zeros. The remaining poles follow a Butterworth pattern whose radius increases with increasing V_1/V_2 .
- $V_2 \gg V_1$: poor sensors, small disturbance, H small, *dual to expensive-control problem*. Closed-loop poles go to the stable plant poles, and the mirror images of the unstable plant poles.

20.6 Combination of LQR and KF

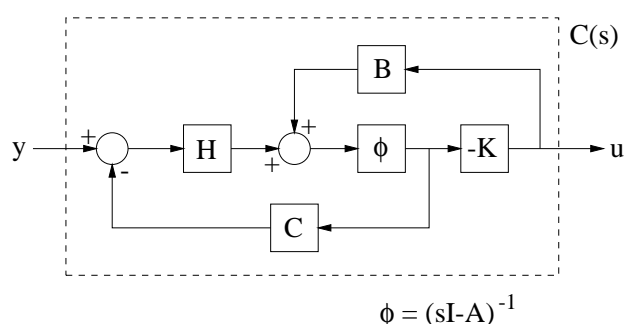
An optimal output feedback controller is created through the use of a Kalman filter coupled with an LQR full-state feedback gain. This combination is usually known as the Linear Quadratic Gaussian design, or LQG. For the plant given as

$$\begin{aligned}\dot{x} &= Ax + Bu + W_1 \\ y &= Cx + W_2,\end{aligned}$$

we put the Kalman Filter and controller gain G together as follows:

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - C\hat{x}) \quad (257)$$

$$u = -K\hat{x}. \quad (258)$$



There are two central points to this construction:

1. **Separation Principle:** The eigenvalues of the nominal closed-loop system are made of up the eigenvalues of $(A - HC)$ and the eigenvalues of $(A - BK)$, separately. See proof below.
2. **Output Tracking:** This compensator is a stand-alone system that, as written, tries to drive its input y to zero. It can be hooked up to receive tracking error $e(s) = r(s) - y(s)$ as an input instead, so that it is not limited to the regulation problem alone. In this case, \hat{x} no longer represents an estimated *state*, but rather an estimated state tracking error. We use the output error as a control input in the next section, on loopshaping via loop transfer recovery.

20.7 Proofs of the Intermediate Results

20.7.1 Proof that $E(e^T W e) = \text{trace}(\Sigma W)$

$$\begin{aligned}E(e^T W e) &= E \left(\sum_{i=1}^n e_i \left(\sum_{j=1}^n W_{ij} e_j \right) \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \Sigma_{ij} W_{ji},\end{aligned}$$

the transpose of W being valid since it is symmetric. Now consider the diagonal elements of the product ΣW :

$$\Sigma W = \begin{bmatrix} \Sigma_{11}W_{11} + \Sigma_{12}W_{21} + \cdots & \cdot & \cdot \\ \cdot & \Sigma_{21}W_{12} + \Sigma_{22}W_{22} + \cdots & \cdot \\ \cdot & \cdot & \cdots \end{bmatrix} \rightarrow$$

$$\text{trace}(\Sigma W) = \sum_{i=1}^n \sum_{j=1}^n \Sigma_{ij}W_{ji}. \quad \square$$

20.7.2 Proof that $\frac{\partial}{\partial H} \text{trace}(-\Lambda HC\Sigma) = -\Lambda^T \Sigma C^T$

$$\begin{aligned} \text{trace}(AHB) &= \text{trace} \left[\sum_{j=1}^n A_{ij} \sum_{k=1}^l H_{jk} B_{kl} \right], \text{ the } il'\text{'th element} \\ &= \sum_{i=1}^n \sum_{j=1}^n A_{ij} \sum_{k=1}^l H_{jk} B_{ki}, \end{aligned}$$

where the second form is a sum over i of the ii' 'th elements. Now

$$\begin{aligned} \frac{\partial}{\partial H_{j_0 k_0}} \text{trace}(AHB) &= \sum_{i=1}^n A_{ij_0} B_{k_0 i} \\ &= (BA)_{k_0 j_0} \\ &= (BA)_{j_0 k_0}^T \longrightarrow \\ \frac{\partial}{\partial H} \text{trace}(AHB) &= (BA)^T \\ &= A^T B^T. \quad \square \end{aligned}$$

20.7.3 Proof that $\frac{\partial}{\partial H} \text{trace}(-\Lambda \Sigma C^T H^T) = -\Lambda \Sigma C^T$

$$\begin{aligned} \text{trace}(AH^T) &= \text{trace} \left[\sum_{j=1}^n A_{ij} H_{jl}^T \right], \text{ the } il'\text{'th element} \\ &= \text{trace} \left[\sum_{j=1}^n A_{ij} H_{lj} \right] \\ &= \sum_{i=1}^n \sum_{j=1}^n A_{ij} H_{ij}, \end{aligned}$$

where the last form is a sum over the ii' 'th elements. It follows that

$$\begin{aligned}\frac{\partial}{\partial H_{i_o j_o}} \text{trace}(AH^T) &= A_{i_o j_o} \longrightarrow \\ \frac{\partial}{\partial H} \text{trace}(AH^T) &= A. \quad \square\end{aligned}$$

20.7.4 Proof of the Separation Principle

Without external inputs W_1 and W_2 , the closed-loop system evolves according to

$$\frac{d}{dt} \begin{Bmatrix} x \\ \hat{x} \end{Bmatrix} = \begin{bmatrix} A & -BK \\ HC & A - BK - HC \end{bmatrix} \begin{Bmatrix} x \\ \hat{x} \end{Bmatrix}.$$

Using the definition of estimation error $e = x - \hat{x}$, we can write the above in another form:

$$\frac{d}{dt} \begin{Bmatrix} x \\ e \end{Bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - HC \end{bmatrix} \begin{Bmatrix} x \\ e \end{Bmatrix}.$$

If A' represents this compound A -matrix, then its eigenvalues are the roots of $\det(sI - A') = 0$. However, the determinant of an upper triangular block matrix is the product of the determinants of each block on the diagonal: $\det(sI - A') = \det(sI - (A - BK))\det(sI - (A - HC))$, and hence the separation of eigenvalues follows.