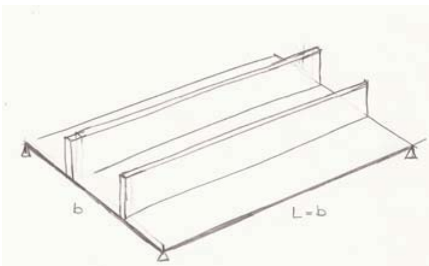
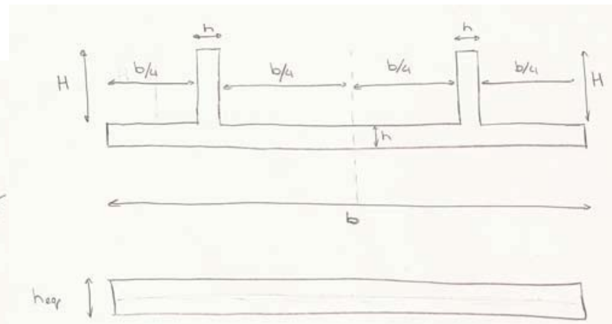
# Lecture 7

# Bending Response of Plates and Optimum Design

**<u>Problem 7-1:</u>**

Consider cylindrical bending of a wide beam (width ‘b’ and length ‘L’ = ‘b’) stiffened by two flat bar stiffeners. The cross section of the beam is sketched below.





a) Find the thickness (h) of an equivalent monolithic plate that will have the same bending rigidity as the actual stiffened plate.

b) Plot the relationship between the ratio H/b and the bending rigidity.

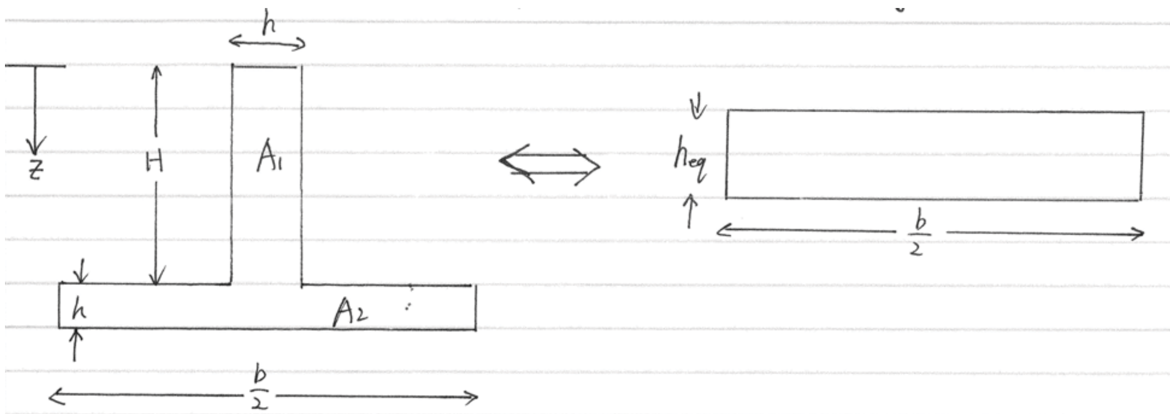
c) Is there a limit on the height of the stiffeners (H) and what could happen if we bend the stiffened plate in a positive (smiling beam) vs. negative beam (sad beam)?

d) How much is the rigidity of the beam reduced when we try to bend the beam in the direction perpendicular to the stiffeners?

**<u>Problem 7-1 Solution:</u>**

a) To find the equivalent thickness plate so that its moment of inertia is equal to that of the plate beam combination, we must first evaluate the moment of inertia $I_{beam}$ of the plate beam combination

The position $z_o$ of the neutral axis has to be first determined from vanishing of the first moment of inertia




$$0 = \int_A (z - z_o)\, dA$$
$$= \int_{A_1} (z - z_o)\, dA_1 + \int_{A_2} (z - z_o)\, dA_2$$

Where $A_1$ and $A_2$ are the areas of the beam and the plate, as indicated in the figure above. Then

$$0 = h \int_0^H (z - z_o)\, dz + \frac{b}{2} \int_H^{H+b} (z - z_o)\, dz$$
$$= \frac{h}{2}\left(H^2 + 2Hz_o\right) + \frac{b}{4}\left[2h\left(H - z_0\right) + h^2\right]$$

Solve for $z_o$

$$\boxed{z_o = \frac{2bH^2 + 2bhH + bh^2}{4Hh + 2bh}}$$

The distance from neutral axis to the plat surface

$$\eta = z_0 - H = \frac{bh^2 - 2bH^2}{4Hh + 2bh}$$

$$\frac{\eta}{h} = \frac{1}{2} \frac{1 - \frac{2h}{b}\left(\frac{H}{b}\right)^2}{1 + \frac{2h}{b}\left(\frac{H}{b}\right)^2}$$

The moment of inertia is then evaluated

$$I_{beam} = \int_{A_1} (z - z_o)^2\, dA_1 + \int_{A_2} (z - z_o)^2\, dA_2$$
$$= \frac{h}{3}\left[\left(H - z_o\right)^3 + z_0^{\,3}\right] + \frac{b}{6}\left(H - z_o\right)^3 \left[\left(1 + \frac{h}{H - z_0}\right)^3 - 1\right]$$

Substitute $z_o$ into the above expression, and using $\eta = z_0 - H$

$$I_{beam} = \frac{b}{6}\left(h^3 - 3h^2\eta + 3h\eta^2\right) + \frac{2h}{b}\left(H^3 + 3H^2\eta + 3H\eta^2\right)$$

The moment of inertia of the equivalent beam is

$$I_{eq} = \frac{\frac{b}{2}{h_{eq}}^3}{12}$$

Also $I_{eq} = I_{beam}$

Therefore

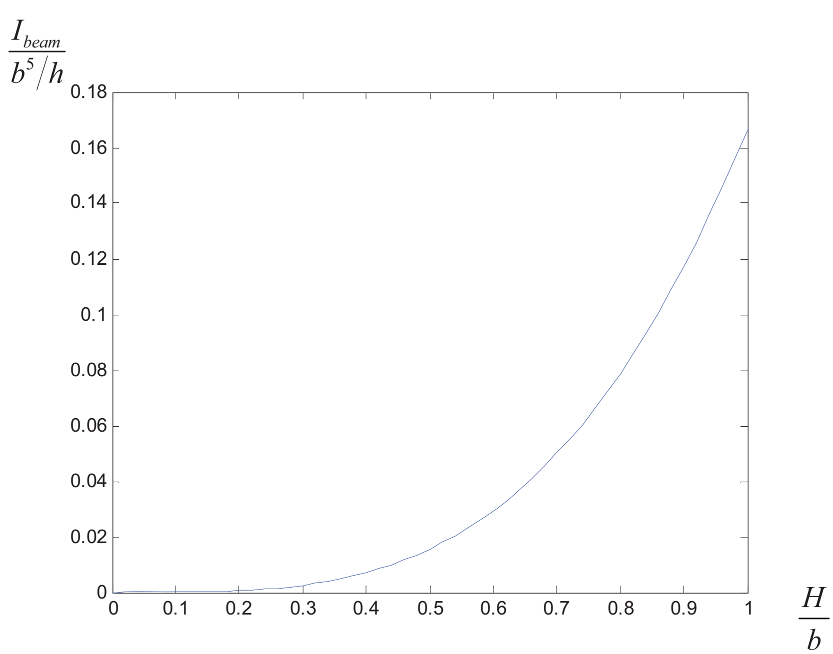
$$h_{eq} = \left(\frac{24 I_{beam}}{b}\right)^{1/3}$$

$$\boxed{I_{eq} = \frac{\frac{b}{2}{h_{eq}}^3}{12}}$$

Normalize by the thickness of the stiffened plate

$$\left(\frac{h_{eq}}{h}\right)^3 = 4\left\{2 - 3\frac{\eta}{h} + 3\left(\frac{\eta}{h}\right)^3 + \frac{2h}{b}\left[\left(\frac{H}{h}\right)^3 + 3\left(\frac{H}{h}\right)^2\frac{\eta}{h} + 3\frac{H}{h}\left(\frac{\eta}{h}\right)^2\right]\right\}$$

b) Consider $h << H$ and $h << b$ , ignore all higher order terms of h $I_{beam} = \frac{b^5}{h}\frac{(H/b)^4}{2(2H/b+1)}$



c) If we bend the stiffened plate in a positive direction(smiling beam), two bar stiffener are in the compressive side, ,they may fail because of buckling

If we bend the stiffened plate in a negative direction, two bar stiffener are in the tensile side, ,they may fail because of yielding or fracture or crippling.

d) When we try to bend the beam in the direction perpendicular to the stiffeners, the stiffen effect will vanish, as a result, the stiffened beam will deforms as a normal beam.

The rigidity of the beam will reduce from the rigidity of beam+stiffener to only the rigidity of beam.

The ratio of the moment of inertia of the stiffened plate bending in the weaker direction to the stronger direction is

$$\frac{h}{H}^{4} \frac{1}{24} \; 2 \; \frac{2H}{b} \; +1$$

$H$ is of the order of $b$. Therefore the above ratio is proportional to $\frac{h}{H}^{4}$

**Note:** There is no correct limiting case with $H \to 0$, because we neglect higher order terms in the derivation of stiffened beam rigidity

**Problem 7-2:**

Another way of designing a weight-efficient structure is to replace the stiffened plate by a sandwich plate.

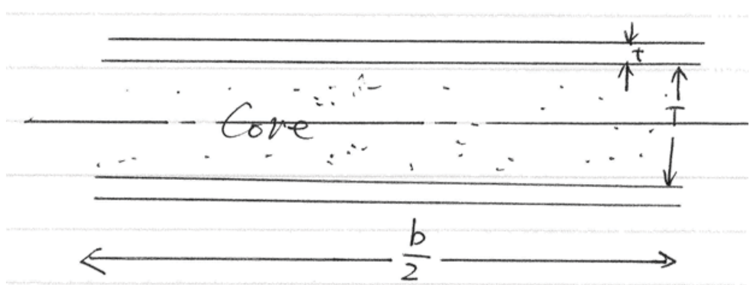
a) Design the sandwich plate of an identical weight of a stiffened plate (determine the phase plate thickness (t) and the core thickness (T)).

b) Compare the bending stiffness of the sandwich with that of the stiffened plate.

**Problem 7-2 Solution:**

The Young's modulus and density of cone material are usually two orders of magnitude smaller than the face plates. So we neglect the weight and stiffness of core plate in the following deduction

a) We want a sandwich plate to have an identical weight of a stiffened plate, the volumes of face material must be equal




$$V_{beam} = b \quad bH + h\frac{b}{2}$$

$$V_{sandwich} = 2\times\frac{b}{2}\times t\times b$$

$$V_{beam} = V_{sandwich} \Rightarrow \boxed{t = \frac{h}{2}+\frac{hH}{b}}$$

b) Use Parallel Axis Theorem, moments of inertia of sandwich plate is

$$I_s = 2\times \quad \frac{1}{12}\times\frac{b}{2}\times t^3 + \frac{b}{2}\times t\times \quad \frac{t}{2} \quad ^2$$

The first term is much smaller than the second term.

$$I_s = \frac{1}{4}btT^2 = \frac{h}{4} \quad \frac{b}{2}+H \quad T^2$$

Revisit the moments of inertia of a stiffened beam

$$I_{beam} = \frac{b^5}{h}\frac{(H/b)^4}{2(H/b+1)}$$

The ratio of the moment of inertia of a sandwich plate to the stiffened plate is

$$R = \frac{I_s}{I_{beam}} = \frac{1}{4}\left(1+\frac{2H}{b}\right)^2\left(\frac{hT}{H^2}\right)^2$$

The plot of the ratio $R$ versus $\frac{hT}{H^2}$ for three different values of $\frac{2H}{b}$ is showed in the figure below. In particular, both types of structural have an identically strength when $R=1$, that is when

$$\frac{hT}{H^2} = \frac{2}{1+2\frac{H}{b}}$$

Solve for $T$

$$T = \frac{2H^2}{h\left(1+2\frac{H}{b}\right)}$$

For example, if $b=4H$, $T=\frac{4}{3}\frac{H^2}{h}$