2.00AJ / 16.00AJ Exploring Sea, Space, & Earth: Fundamentals of Engineering Design
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2.00AJ/16.00AJ Reading:
Hydrostatic Pressure Effects and Archimedes' Principle

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Pressure Effects

I.   Hydrostatic Pressure

Fluid forces can arise due to flow stresses (pressure and viscous shear), gravity forces, fluid acceleration, or other body forces. For now, let us consider a fluid in static equilibrium – with no velocity gradients (thus no viscous stresses).

Forces are then due only to:

1. Pressure acting on the fluid volume
2. Gravity acting on the mass of the fluid
3. External body forces

Using standard conventions we will consider pressure to be positive for compression. Recall that we said pressure is isotropic: Consider a triangular volume of fluid with height, *dz*, length, *dx,* and unit width, *b*, into the page:
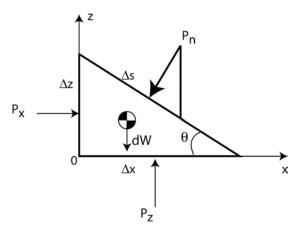

Figure 2.1: Elemental fluid volume

The fluid element can support no shear while at rest (by our definition of a fluid). Thus the sum of the forces on the triangle, in the *x*- and *z*- directions, MUST equal zero:

$$\sum F_x = p_x b(\Delta z) - p_n b(\Delta s)\sin\theta = 0 \tag{2.1}$$

$$\sum F_z = p_z b(\Delta x) - p_n b(\Delta s)\cos\theta - \frac{1}{2}\gamma b(\Delta x \Delta z) = 0 \tag{2.2}$$

We can simplify the above equations using simple geometry:

$$\Delta z = (\Delta s)\sin\theta \text{ and } \Delta x = (\Delta s)\cos\theta, \tag{2.3}$$

such that

$$\sum F_x = \left(p_x - p_n\right) = 0 \tag{2.4}$$

and

$$\sum F_z = p_z - p_n - 1/2\gamma(\Delta z) = 0. \tag{2.5}$$

Taking the limit as Δx, Δz goes to zero (i.e. the triangle goes to a point), we see that

$$p_x = p_z = p_n = p. \tag{2.6}$$

Since θ is arbitrary, pressure at a point in a fluid is independent of orientation and is thus isotropic. Pressure (or any stress for that matter) causes NO net force on a fluid element unless it varies spatially!

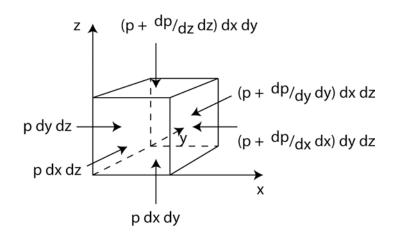Take for example a small fluid element, δx, δy, δz:

Figure 2.2: Fluid Element Volume (z is positive upwards by convention).

We will continue to assume that the only forces are due to gravity and pressure gradients. Let's start by looking at the pressure acting on the top and bottom of the element volume. Using Taylor's series expansion:

$$p_{top} = po + \frac{\delta z}{2} \frac{dp}{dz}\bigg|_0 + \frac{1}{2!}\left(\frac{\delta z}{2}\right)^2 \frac{d^2 p}{dz^2}\bigg|_0 - ... \tag{2.7}$$

Second term is positive b/c we are taking z positive upwards from the center of the element.

$$p_{bot} = P_o - \frac{\delta z}{2} \frac{dp}{dz}\bigg|_0 + \frac{1}{2!}\left(\frac{\delta z}{2}\right)^2 \frac{d^2 p}{dz^2}\bigg|_0 + ... \tag{2.8}$$

Taking the limit as δz goes to zero we are able to ignore the higher order terms and keep only up to second order. The resultant force due to pressure on the face is

$$F_{press.} = p\, dA = p\, dx\, dy. \tag{2.9}$$

$$F_{press} = (p_{bot} - p_{top}) dx \, dy = \left. \frac{dp}{dz} \right|_o dx \, dy \, dz \, . \tag{2.10}$$

F_{press} acts in the positive *z*-direction opposite to gravity. In summing the pressures acting on the top and bottom we see that the second order terms cancel exactly and we are left with the resulting force due to pressure.

Acting in tandem to the pressure force is the force due to gravity. The fluid density and volume dictate the mass of the element (***F=mg***).

$$F_{weight} = \int_{top}^{bot} \rho(z) g \, dxdydz = g \, dxdy \int_{top}^{bot} \rho(z) dz \tag{2.11}$$

$$= g \, dxdy \frac{1}{2}(\rho_{top} + \rho_{bot}) dz \tag{2.12}$$

Using Taylor's series expansion similar to the pressure terms above we get:

$$\rho_{top} + \rho_{bot} = 2\rho_o \tag{2.13}$$

For static equilibrium, pressure forces MUST balance gravitational/body forces:

$$F_{weight} + F_{press} = 0 \tag{2.14}$$

$$\left. \frac{dp}{dz} \right|_o (dxdydz) = -\rho g (dxdydz) \tag{2.15}$$

We can drop the subscript "o" from the pressure gradient term since we have taken the limit to a small element of fluid. This equation is valid at any point in the fluid. Canceling the elemental volume terms we are left with the hydrostatic equation.

Hydrostatic Equation: $$\frac{dp}{dz} = -\rho g \tag{2.16}$$

The negative sign is valid here since the z-direction is taken positive pointing up, in the opposite direction to the gravitational force. If z-direction coincided with gravity then the negative sign would be dropped.

In this case, there are no acting forces or pressure changes in the x- or y-directions therefore dp/dz is representative of the pressure gradient:

$$\frac{dp}{dz} = \nabla p \, . \tag{2.17}$$

We can rewrite the hydrostatic equation as:

$$\bar{\nabla}p = -\rho\bar{g} \qquad\qquad (2.18)$$

If there were additional gradients in the x- and y-directions similar steps could be followed and the resulting equation would be similar. Pressure gradient is always balanced by gravity, acceleration, viscous forces, and any other external body forces.

II.   Gauge and Vacuum Pressure

Pressure is usually referred to in one of two ways:

      1) Absolute, or total, pressure

      2) Relative to ambient (atmospheric) pressure

Since most pressure instruments are differential measurement devices, meaning they measure the pressure in the fluid relative to atmospheric pressure, absolute pressure is a commonly used quantity.

Pressure is either greater or less than the ambient (atmospheric) pressure:

    1) $p > p_a$ Gage Pressure   *p(gage) = p − p_a*

    2) $p < p_a$ Vacuum pressure   *p(vacuum) = p_a − p*

In order to get the total (absolute) pressure, the atmospheric pressure must be known.

P (Pascals)

120,000 ──────── High Pressure

30,000

90,000 ──────── Local Atmospheric

40,000

50,000 ──────── Vacuum Pressure
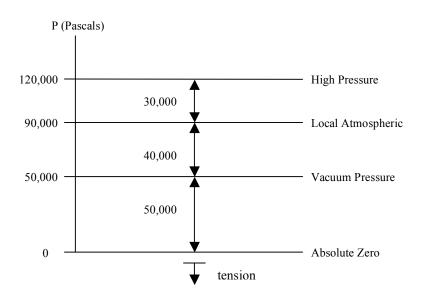
50,000

0 ──────── Absolute Zero

tension

Figure 2.3: Relative pressure chart

III.   Hydrostatic Forces on a Wall

Recall the equation for the pressure gradient in a liquid is

$$\bar{\nabla}p = -\rho\bar{g} \,.\tag{2.19}$$

Which, in the vertical direction this translates to

$$\frac{dp}{dz} = -\rho g \,,\tag{2.20}$$

or

$$dp = -\rho g\, dz \,.\tag{2.21}$$

Integrating in the z-direction we get pressure as a function of depth:

$$\int_{p}^{p_a} dp = -\int_{z}^{H} \rho g\, dz\tag{2.22}$$

$$p - p_a = -\rho g\left(z - H\right) = \rho g\left(H - z\right)\tag{2.23}$$

Note that pressure increases with depth with a constant slope, ρg . Pressure is either considered relative to a *reference* pressure or in *absolute* terms. Most pressure gauges are differential measurement devices that measure pressure relative to ambient (or

atmospheric) pressure. Thus it is important to keep in mind the effects due to atmospheric pressure in your laboratories and calculations.

Pressure is isotropic, and therefore pressure is the same on a vertical or horizontal surface at depth *h*. Take for instance the ocean bottom and a vertical seawall that extends to the bottom. At depth H, the horizontal pressure acting on the wall is equivalent to that pressure acting along the entire seafloor at that same depth.
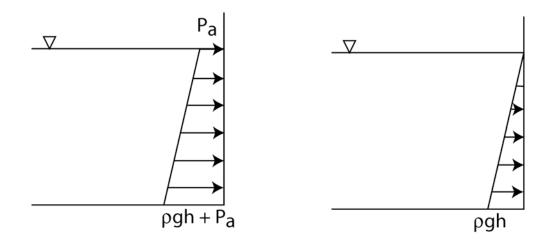
Horizontal Force acting on a vertical surface:

Figure 2.4: Absolute Pressure vs. Gauge pressure

Using absolute pressure formulation we can find the force on the wall from the pressure as a function of depth:

$$p - p_a = -\rho g (z - h) \tag{2.24}$$

Thus the elemental force acting in the x-direction due to the pressure is

$$dF_x = p\, dA = p\, w\, dz, \tag{2.25}$$

where *w* is the width of the wall into the page.

To determine the force on the wall we must consider the pressure acting on both sides of the wall. Let's assume that on the left of the wall water of depth, h, is exerting force F_1 on the wall. The right side of the wall is open to the air with atmospheric pressure acting

over the height of the wall. Under this setup we must account for the atmospheric pressure on both sides of the wall. The elemental force in the x-direction at some depth z below the free surface is formulated on the left and right sides of the wall to determine the total force acting on the wall:

Left:
$$dF_{x1} = (P_a + \rho g(h - z))w\,dz$$

Right:
$$dF_{x2} = -P_a\,w\,dz$$

Total force:
$$dF_x = dF_{x1} + dF_{x2} = \underline{P_a w\,dz} + \rho g(h - z)w\,dz - \underline{P_a\,w\,dz}$$

The two underlined terms cancel and we are left only with the effect due to the presence of the water. Integrating the pressure over the depth

$$F = \int_0^h \rho g(h - z)dz = \tfrac{1}{2}\rho gwh^2 \qquad (2.26)$$

If use gauge pressure to calculate the force on the wall, we already take into account the effect of atmospheric pressure and can directly calculate the force from:

$$p_g = \rho g(h - z)\,(\text{gauge pressure}) \qquad (2.27)$$

$$dF = p_g dA = \rho g(h - z)w\,dz \qquad (2.28)$$

Resulting in the same force found when we used absolute pressure.

The force acting on the wall is a "distributed load" which acts along the entire depth of the wall.

However it is possible to determine how this force "acts" on the wall using a simple moment balance:

$$dMo = z \times dF \qquad (2.29)$$

where *Mo* is the moment on the wall about the origin (for now lets consider the origin the bottom of the wall) and z is the moment arm perpendicular to the force direction.
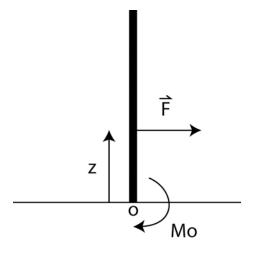
Figure 2.5: Moment balance on the wall about the origin o with z taken positive from the ground up.

The elemental moment about the origin due to the hydrostatic pressure is:

$$dMo = \rho g(h-z)w\,z\,dz \qquad (2.30)$$
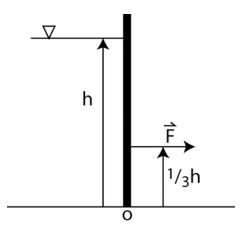
Integrating leads to the total moment about the origin:
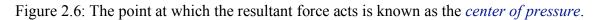
$$Mo = \tfrac{1}{6}\rho gwh^{3} \qquad (2.31)$$

By definition the product of the force F times the moment arm $\overline{z}$ must equal the moment M:

$$M = F\overline{z} \qquad (2.32)$$

Such that the point at which the force acts on the wall is $\overline{z}$:

$$\overline{z} = \frac{\tfrac{1}{6}\rho gwh^{3}}{\tfrac{1}{2}\rho gwh^{2}} = \tfrac{1}{3}h \qquad (2.33)$$

Figure 2.6: The point at which the resultant force acts is known as the *center of pressure*.

We can use a similar approach to the problem of pressure on a sloped wall. This is left for a homework exercise. Since atmospheric pressure acts everywhere then gage pressure is the ideal pressure to use in these exercises. The resultant forces and moments on the sloped wall can be found using simple geometry and then extended to the case of a "V"-shaped ship hull.
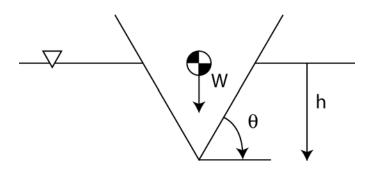

Figure 2.7: V-shaped Hull

Archimedes's Principle and Static Stability for Floating Vessels

IV.   Archimedes's Principle:

The force on a body due to pressure alone (in the absence of viscous forces) acts in the normal direction (recall pressure is isotropic!). The force can be calculated by integrating the pressure around the body:

$$\vec{F} = \iint_S p\,\hat{n}\,ds$$

where pressure is a function of depth below the free surface. Hydrostatic pressure is found using the equation: $p(z) = -\rho g z$. Thus the force due to pressure can be simplified as follows:

$$\vec{F} = \iint_S p\,\hat{n}\,ds = -\rho g \iint_S z\,\hat{n}\,ds$$

By calculus, the surface integral can be converted into a volume integral:

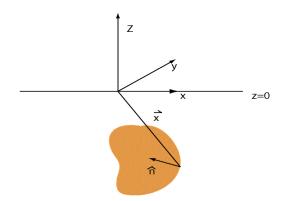$$\iint_S z\,\hat{n}\,ds = \iiint_V \nabla z\,d\forall$$

So that the force becomes:

$$\vec{F} = \underline{\iint_S p\,\hat{n}\,ds} = -\rho g \iint_S z\,\hat{n}\,ds = \rho g \iiint_v d\forall = \underline{\rho g \forall \hat{k}}$$

We can see now that the buoyancy force acts to counterbalance the displaced volume ($\square$) of fluid. For a semi-submerged body (partially submerged) the area of the water plane must be accounted for in the integration.

V.   Moment on a body (Ideal Fluid)

The moment on a submerged body follows directly from structural mechanics or dynamics methodologies.

$$\vec{M} = \iint_S p\,(\bar{x} \times \hat{n})\,ds \equiv (M_1, M_2, M_3) = (M_x, M_y, M_z)$$

Recall from vector calculus:

$$\vec{x} \times \hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ n_1 & n_2 & n_3 \end{vmatrix} = \begin{matrix} \hat{i}(y\,n_3 - z\,n_2) \\ -\hat{j}(x\,n_3 - z\,n_1) \\ +\hat{k}(x\,n_2 - y\,n_1) \end{matrix}$$
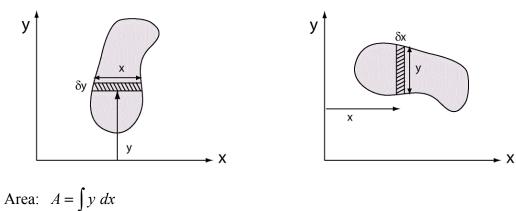
So we get the moments as follows:

$$M_1 = \iint_s p(y\,n_3 - z\,n_2)\,ds \text{ about the x-axis}$$

We can also calculate in a similar fashion: M_2 about the y-axis and M_3 about the z-axis.

VI.   Center of Buoyancy

Calculating the Center of buoyancy, it is first necessary to find the center of area.

Area: $A = \int y\,dx$

When calculating a ships area, an approximate area is used since there is often no mathematical formula for the ships geometry.

First moment of the area is found as follows:

$$M_{yy} = \int y_1 \, x \, dx$$

$$M_{xx} = \int x_1 \, y \, dy$$

The coordinates for the center of area are then found using the above equations:

$$\bar{x} = \frac{M_{yy}}{A} = \frac{1}{A} \int x \, y_1 \, dx$$

$$\bar{y} = \frac{M_{xx}}{A} = \frac{1}{A} \int y \, x_1 \, dy$$

Center of buoyancy is the point at which the buoyancy force acts on the body and is equivalently the geometric center of the submerged portion of the hull.
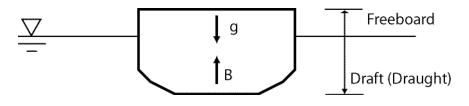
VII.   Static Equilibrium: Gravity and buoyancy balance

A body at rest is considered to be in equilibrium.

The mass of the object (vessel) is equal to the mass of an equivalent volume of water (displacement). When a vessel is in static equilibrium, the center of gravity and buoyancy act on the same vertical line. Buoyancy acts perpendicular to the water line (upwards) at the center of buoyancy.

Center of gravity usually lies amidships and near to the water line.

Center of buoyancy changes dependant on the volume of the submerged portion of the vessel. As a ship rolls, pitches, etc. the center of buoyancy moves according to the "new" shape of the submerged block. Take for example a container ship: as containers are loaded the weight of the ship increases and it submerges further into the water. With the increased draft the center of buoyancy shifts.
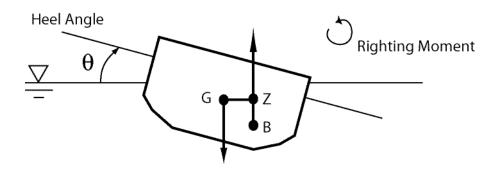

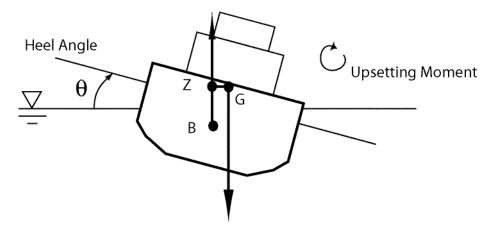*A vessel in static equilibrium with the centers of gravity*

and buoyancy aligned in the same vertical line.

VIII.   Righting moments: The moment required/available to right a vessel.

An object disturbed from a static position that returns to its original position is considered statically stable or in *static equilibrium.* Such vessels are "self-righting". However if the vessel is unable to return to its equilibrium position and continues to turn over (capsize) it is considered unstable. It a vessel is in equilibrium at any position then it is neutrally stable (take for example a simple cylinder on its side – turn it to any angle and it will stay).

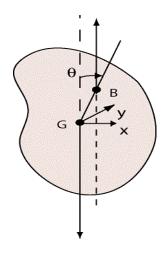*A statically stable vessel with a positive righting arm.*

Statically unstable vessel with a negative righting arm.

The righting moment is equivalent to the Displacement of the vessel times the righting arm:

$$RM = W \times \overline{GZ}$$

IX.   Small rotation about the y-axis equation for stability:

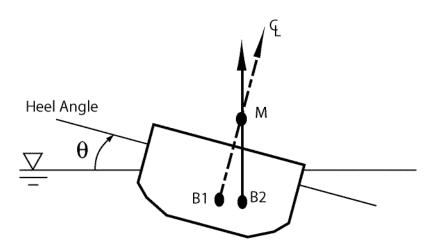Sum of the torques about the center of rotation must be zero (conservation of angular momentum!)

We can write the equation of motion in terms of the rotational displacement, θ, and rotational acceleration, $\ddot{\theta}$:

$$I\ddot{\theta} + l\theta = 0$$

where $l = |GB|$. This is the equation of simple harmonic motion. The motion is in stable equilibrium when $l > 0$ and unstable when $l < 0$. (You'll see this later in dynamics class!)

X.   Metacenter: A ships metacenter is the intersection of two lines of action of the buoyancy force.

What does this mean? Well, basically it means that as a ship rolls (heels) through an angle, θ, the center of buoyancy, *B,* shifts along a semi-elliptical path (depending on geometry). Since the buoyancy always acts vertically, the two lines of action for two different heel angles must cross at a point. This point is the metacenter of the ship, denoted by *M* in the figure below.

*The metacenter is the intersection of two distinct lines of
action of the buoyancy force.*

When on at an angle of roll (heel) the center of buoyancy and center of gravity no longer act on the same vertical line of action.

Transverse Metacenter: Roll/Heel
Longitudinal Metacenter: Pitch/Trim

The concept of the metacenter is only really valid for small angles of motion.

Metacentric height is the distance measured from the metacenter to the center of gravity, $\overline{GM}$. If the metacentric height is large, then the vessel is considered to be "stiff" in roll – indicating that there will be a large righting moment as a result of small roll angles. In contrast, if the metacentric height is small then the ship rolls slowly due to a smaller righting arm.