18.600: Lecture 34

Martingales and the optional stopping theorem

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Outline

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Martingales and stopping times

Optional stopping theorem

Martingale definition

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- If Z is any random variable, we let $E[Z|\mathcal{F}_n]$ denote the conditional expectation of X given all the information that is available to us on the nth stage. If we don't specify otherwise, we assume that this information consists precisely of the values $X_0, X_1, \dots, X_n$, so that $E[Z|\mathcal{F}_n] = E[Z|X_0, X_1, \dots, X_n]$. (In some applications, one could imagine there are other things known as well at stage n.)

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- "Taking into account all the information I have at stage n, the expected value at stage $n+1$ is the value at stage n."

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- Question: If you are given a mathematical description of a process $X_0, X_1, X_2, \ldots$ then how can you check whether it is a martingale?
- Consider all of the information that you know after having seen $X_0, X_1, \ldots, X_n$. Then try to figure out what additional (not yet known) randomness is involved in determining X_{n+1}. Use this to figure out the conditional expectation of X_{n+1}, and check to see whether this is necessarily equal to the known X_n value.

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- Answer: yes. To see this, note that $E[X_{n+1}|\mathcal{F}_n] = E[X_n + A_{n+1}|\mathcal{F}_n] = E[X_n|\mathcal{F}_n] + E[A_{n+1}|\mathcal{F}_n]$, by additivity of conditional expectation (given $\mathcal{F}_n$).

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- Since X_n is known at stage n, we have $E[X_n|\mathcal{F}_n] = X_n$. Since we know nothing more about A_{n+1} at stage n than we originally knew, we have $E[A_{n+1}|\mathcal{F}_n] = 0$. Thus $E[X_{n+1}|\mathcal{F}_n] = X_n$.

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- Informally, I'm just tossing a new fair coin at each stage to see if X_n goes up or down one step. If I know the information available up to stage n, and I know $X_n = 10$, then I see $X_{n+1} = 11$ and $X_{n+1} = 9$ as equally likely, so $E[X_{n+1}|\mathcal{F}_n] = 10 = X_n$.

Another martingale example

- What if each A_i is 1.01 with probability .5 and .99 with probability .5 and we write $X_0 = 1$ and $X_n = \prod_{i=1}^n A_i$ for $n > 0$? Then is X_n a martingale?

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- Informally, I'm just tossing a new fair coin at each stage to see if X_n goes up or down by a percentage point of its current value. If I know all the information available up to stage n, and I know $X_n = 5$, then I see $X_{n+1} = 5.05$ and $X_{n+1} = 4.95$ as equally likely, so $E[X_{n+1}|\mathcal{F}_n] = 5$.

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- **Two classic martingale examples:** sums of independent random variables (each with mean zero) and products of independent random variables (each with mean one).

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- Does this mean that X_n is a martingale?
- No. If $n \geq 1$, then given the information available up to stage n, I can figure out what A must be, and can hence deduce exactly what X_{n+1} will be — and it is not the same as X_n. In particular, $E[X_{n+1}|\mathcal{F}_n] = -X_n \neq X_n$.

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- Informally, X_n alternates between 1 and -1. Each time it goes up and hits 1, I know it will go back down to -1 on the next step.

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- Think of T as giving the time the asset will be sold if the price sequence is $X_0, X_1, X_2, \ldots$.
- Say that T is a **stopping time** if the event that $T = n$ depends only on the values X_i for $i \leq n$. In other words, the decision to sell at time n depends only on prices up to time n, not on (as yet unknown) future prices.

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- Which of the following is a stopping time?
  1. The smallest T for which $|X_T| = 50$
  2. The smallest T for which $X_T \in \{-10, 100\}$
  3. The smallest T for which $X_T = 0$.
  4. The T at which the X_n sequence achieves the value 17 for the 9th time.
  5. The value of $T \in \{0, 1, 2, \ldots, 100\}$ for which X_T is largest.
  6. The largest $T \in \{0, 1, 2, \ldots, 100\}$ for which $X_T = 0$.

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- Answer: first four, not last two.

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Optional stopping overview

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- Precisely, if you buy the asset at some time and adopt any strategy at all for deciding when to sell it, then the expected price at the time you sell is the price you originally paid.
- If market price is a martingale, you cannot make money in expectation by "timing the market."

Doob's Optional Stopping Theorem: statement

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- Theorem can be proved by induction if *stopping time* T is bounded. Unbounded T requires a limit argument. (This is where boundedness of martingale is used.)

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- But what about interest, risk premium, etc.?
- According to the **fundamental theorem of asset pricing**, the discounted price $\frac{X(n)}{A(n)}$, where A is a risk-free asset, is a martingale with respected to **risk neutral probability**. More on this next lecture.

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- This means that the three-element sequence $E[X], E[X|Y], X$ is a martingale.
- More generally if Y_i are any random variables, the sequence $E[X], E[X|Y_1], E[X|Y_1, Y_2], E[X|Y_1, Y_2, Y_3], \ldots$ is a martingale.

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Martingales as real-time subjective probability updates

- Ivan sees email from girlfriend with subject "some possibly serious news", thinks there's a 20 percent chance she'll break up with him by email's end. Revises number after each line:
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- I have something crazy to tell you, 24
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- Call me!!! I love you! Alice 0

More conditional probability martingale examples

- Example: let C be the amount of oil available for drilling under a particular piece of land. Suppose that ten geological tests are done that will ultimately determine the value of C. Let C_n be the **conditional expectation** of C *given* the outcome of the first n of these tests. Then the sequence $C_0, C_1, C_2, \dots, C_{10} = C$ is a martingale.

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- Let A_i be my best guess at the probability that a basketball team will win the game, given the outcome of the first i minutes of the game. Then (assuming some "rationality" of my personal probabilities) A_i is a martingale.