# 18.404/6.840 Lecture 20

**Last time:**
- Games and Quantifiers
- Generalized Geography is PSPACE-complete
- Logspace: L and NL

**Today:** (Sipser §8.4)
- Review NL $\subseteq$ P
- Review NL $\subseteq$ SPACE$(\log^2 n)$
- NL-completeness
- NL = coNL

# Review: log space

**Model:** 2-tape TM with read-only input tape for defining sublinear space computation.

**Defn:** L = SPACE$(\log n)$
NL = NSPACE$(\log n)$

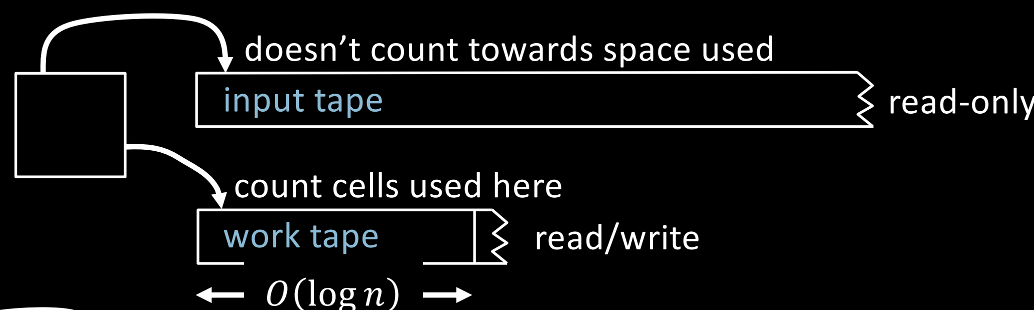
Log space can represent a constant number of pointers into the input.

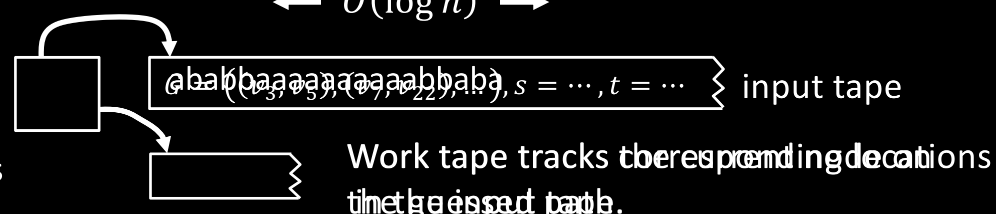
Examples

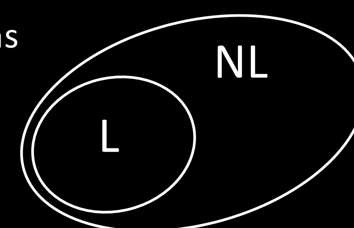
1. $\{ww^{\mathcal{R}} \mid w \in \Sigma^*\} \in$ L

2. $PATH \in$ NL

Nondeterministically select the nodes of a path connecting $s$ to $t$.




L = NL? Unsolved

# Review: L ⊆ P

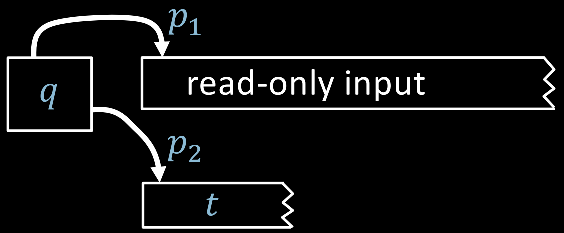
**Theorem:** L ⊆ P

Proof: Say $M$ decides $A$ in space $O(\log n)$.

**Defn:** A configuration for $M$ on $w$ is $(q, p_1, p_2, t)$ where $q$ is a state, $p_1$ and $p_2$ are the tape head positions, and $t$ is the work tape contents.

The number of such configurations is $|Q| \times n \times O(\log n) \times d^{O(\log n)} = O(n^k)$ for some $k$.

Therefore $M$ runs in polynomial time.
Conclusion: $A \in$ P

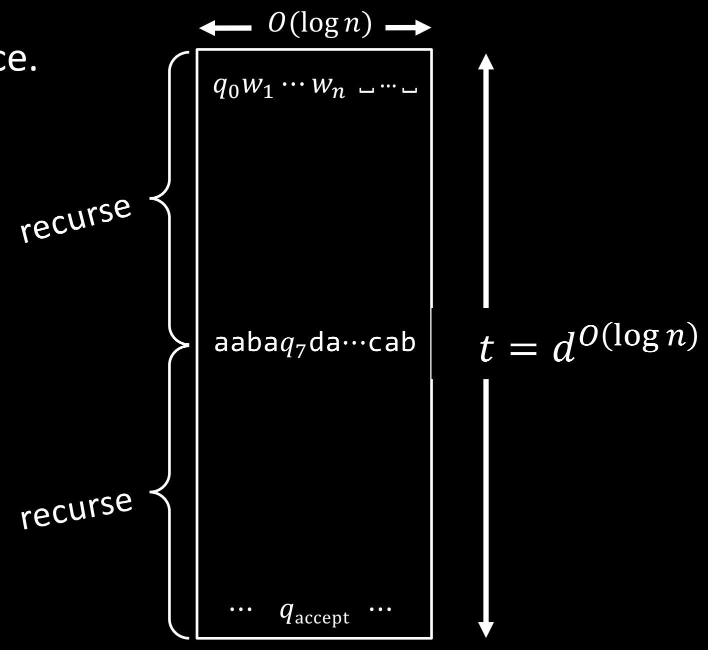


# Review: $\text{NL} \subseteq \text{SPACE}(\log^2 n)$

Theorem: $\text{NL} \subseteq \text{SPACE}(\log^2 n)$

Proof: Savitch's theorem works for log space

Each recursion level stores 1 config = $O(\log n)$ space.
Number of levels = $\log t = O(\log n)$.
Total $O(\log^2 n)$ space.



# Review: NL ⊆ P

**Theorem:** NL ⊆ P

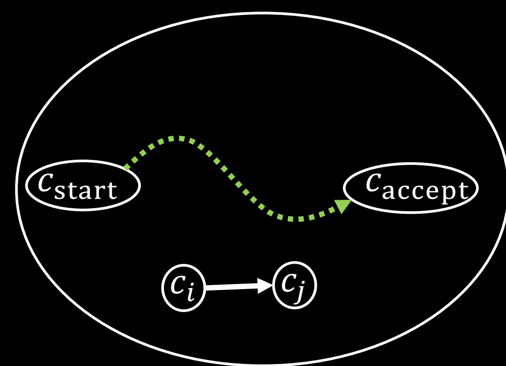
Proof: Say NTM $M$ decides $A$ in space $O(\log n)$.

**Defn:** The <u>configuration graph</u> $G_{M,w}$ for $M$ on $w$ has
  **nodes:** all configurations for $M$ on $w$
  **edges:** edge from $c_i \to c_j$ if $c_i$ can yield $c_j$ in 1 step.

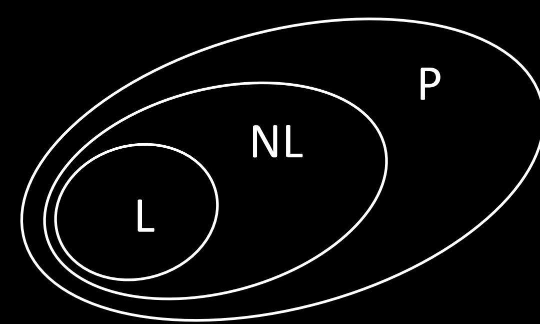
**Claim:** $M$ accepts $w$ iff the configuration graph $G_{M,w}$ has a path from $c_{\text{start}}$ to $c_{\text{accept}}$

Polynomial time algorithm $T$ for $A$:
$T$ = "On input $w$
1. Construct $G_{M,w}$. [polynomial size]
2. *Accept* if there is a path from $c_{\text{start}}$ to $c_{\text{accept}}$.
   *Reject* if not."

configuration graph $G_{M,w}$




iff $M$ accepts $w$




L = P? Unsolved

# NL-completeness

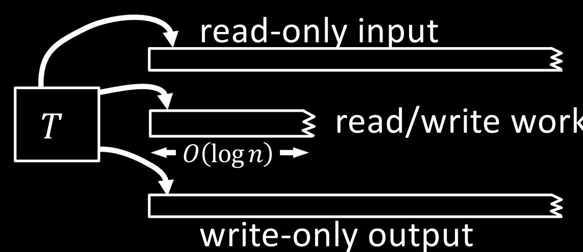
**Check-in 20.1**

If $T$ is a log-space transducer that computes $f$, then for inputs $w$ of length $n$, how long can $f(w)$ be?

(a) at most $O(\log n)$
(b) at most $O(n)$
(c) at most polynomial in $n$
(d) at most $2^{O(n)}$
(e) any length

**Defn:** A <u>log-space transducer</u> is a TM with three tapes:

1. read-only input tape of size $n$
2. read/write work tape of size $O(\log n)$
3. write-only output tape




A log-space transducer $T$ computes a function $f: \Sigma^* \to \Sigma^*$ if $T$ on input $w$ halts with $f(w)$ on its output tape for all $w$.
Say that $f$ is computable in log-space.

**Defn:** $A$ is <u>log-space reducible</u> to $B$ $(A \leq_{\mathrm{L}} B)$ if $A \leq_{\mathrm{m}} B$ by a reduction function that is computable in log-space.

**Theorem:** If $A \leq_{\mathrm{L}} B$ and $B \in \mathrm{L}$ then $A \in \mathrm{L}$

Proof: TM for $A =$ "On input $w$

1. Compute $f(w)$
2. Run decider for $B$ on $f(w)$. Output same."

BUT we don't have space to store $f(w)$.
So, (re-)compute symbols of $f(w)$ as needed.

Check-in 20.1

# $PATH$ is NL-complete

**Theorem:** $PATH$ is NL-complete

Proof: 1) $PATH \in$ NL ✓

2) For all $A \in$ NL, $A \leq_{\mathrm{L}} PATH$

Let $A \in$ NL be decided by NTM $M$ in space $O(\log n)$.

[Modify $M$ to erase work tape and move heads to left end upon accepting.]

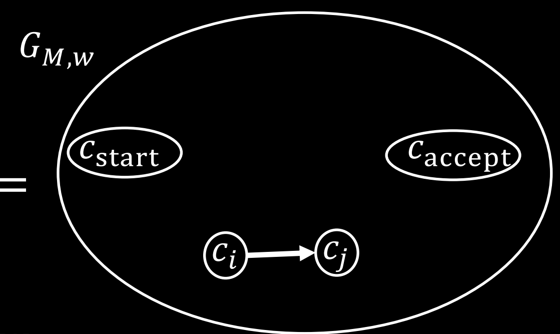
Give a log-space reduction $f$ mapping $A$ to $PATH$.

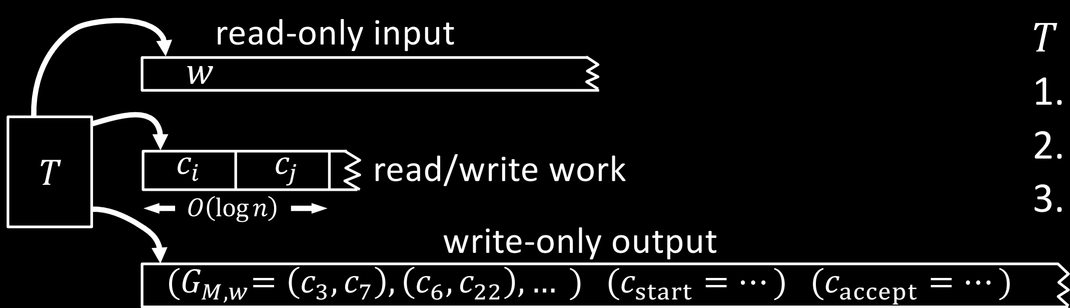
$f(w) = \langle G, s, t \rangle$

$w \in A$ iff $G$ has a path from $s$ to $t$

Here is a log-space transducer $T$ to compute $f$ in log-space.

$f(w) =$




$T =$ "on input $w$

1. For all pairs $c_i, c_j$ of configurations of $M$ on $w$.
2. Output those pairs which are legal moves for $M$.
3. Output $c_{\text{start}}$ and $c_{\text{accept}}$."

# $\overline{2SAT}$ is NL-complete

**Theorem:** $\overline{2SAT}$ is NL-complete

Proof: 1) Show $\overline{2SAT} \in$ NL  good exercise

2) Show $PATH \leq_L \overline{2SAT}$

Give log-space reduction f from $PATH$ to $\overline{2SAT}$.

$f(\langle G, s, t\rangle) = \langle \phi \rangle$

For each node $u$ in $G$ put a variable $x_u$ in $\phi$.
For each edge $(u, v)$ in $G$, put a clause $(x_u \rightarrow x_v)$ in $\phi$ [equivalent to $(\overline{x_u} \vee x_v)$].
In addition put the clauses $(x_s \vee x_s)$ and $(x_t \rightarrow \overline{x_s})$ in $\phi$.

Show $G$ has an path from $s$ to $t$ iff $\phi$ is unsatisfiable.
($\rightarrow$) Follow implications to get a contradiction.
($\leftarrow$) If $G$ has no path from $s$ to $t$, then assign all $x_u$ TRUE where $u$ is reachable from $s$, and all other variables FALSE. That gives a satisfying assignment to $\phi$.

Straightforward to show $f$ is computable in log-space.

# NL = coNL (part 1/4)

**Theorem** (Immerman-Szelepcsényi): NL = coNL

Proof: Show $\overline{PATH} \in$ NL

**Defn:** NTM $M$ computes function $f{:}\,\Sigma^* \to \Sigma^*$ if for all $w$

1) All branches of $M$ on $w$ halt with $f(w)$ on the tape or reject.
2) Some branch of $M$ on $w$ does not reject.

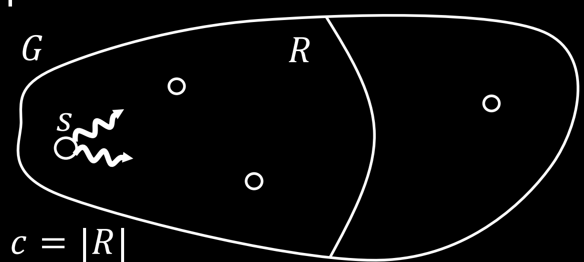
Let $path(G,s,t) = \begin{cases} \text{YES, if } G \text{ has a path from } s \text{ to } t \\ \text{NO, if not} \end{cases}$

Let $R = R(G,s) = \{u \mid path(G,s,u) = \text{YES}\}$

Let $c = c(G,s) = |R|$

$R$ = Reachable nodes

$c$ = # reachable




## Check-in 20.2

Consider the statements:

(1) $\overline{PATH} \in$ NL, and

(2) Some NL-machine computes the $path$ function.

What implications can we prove *easily*?

(a) (1) → (2) only

(b) (2) → (1) only

(c) Both implications

(d) Neither implication



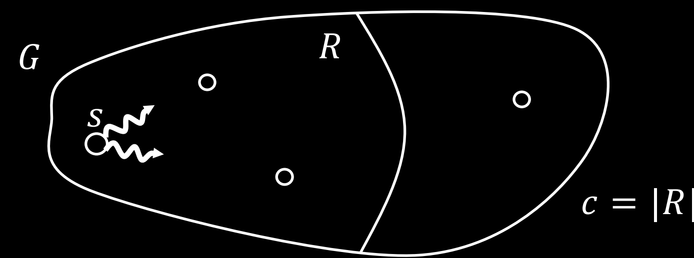
Check-in 20.2

# NL = coNL (part 2/4) – key idea

**Theorem:** If some NL-machine computes $c$, then some NL-machine computes $path$.

Proof: "On input $\langle G, s, t\rangle$

1. Compute $c$
2. $k \leftarrow 0$
3. For each node $u$
4. Nondeterministically go to (p) or (n)
   (p) Nondeterministically pick a path from $s$ to $u$ of length $\leq m$.
   If fail, then *reject*.
   If $u = t$, then output YES, else set $k \leftarrow k + 1$.
   (n) Skip $u$ and continue.
5. If $k \neq c$ then *reject*.
6. Output NO." [found all $c$ reachable nodes and none were $t$}



# NL = coNL (part 3/4)

Let $path_d(G,s,t) = \begin{cases} \text{YES, if } G \text{ has a path } s \text{ to } t \text{ of length} \leq d \\ \text{NO, if not} \end{cases}$

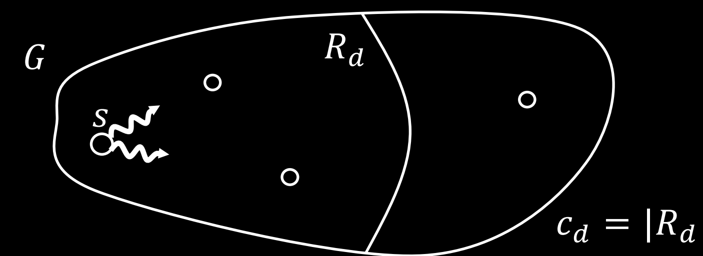
Let $R_d = R_d(G,s) = \{u \mid path_d(G,s,u) = \text{YES}\}$

Let $c_d = c_d(G,s) = |R_d|$

**Theorem:** If some NL-machine computes $c_d$, then some NL-machine computes $path_d$.

Proof: "On input $\langle G,s,t\rangle$

1. Compute $c_d$
2. $k \leftarrow 0$
3. For each node $u$
4. Nondeterministically go to (p) or (n)
   - (p) Nondeterministically pick a path from $s$ to $u$ of length $\leq d$.
     If fail, then *reject*.
     If $u = t$, then output YES, else set $k \leftarrow k+1$.
   - (n) Skip $u$ and continue.
5. If $k \neq c_d$ then *reject*.
6. Output NO" [found all $c_d$ reachable nodes and none were $t$}



# NL = coNL (part 4/4)

**Theorem:** If some NL-machine computes $c_d$, then some NL-machine computes $path_{d+1}$.

Proof: "On input $\langle G, s, t\rangle$

1. Compute $c$
2. $k \leftarrow 0$
3. For each node $u$
4. Nondeterministically go to (p) or (n)
   - (p) Nondeterministically pick a path from $s$ to $u$ of length $\leq d$.
     If fail, then *reject*.
     If $u$ has an edge to $t$, then output YES, else set $k \leftarrow k + 1$.
   - (n) Skip $u$ and continue.
5. If $k \neq c_d$ then *reject*.
6. Output NO." [found all $c_d$ reachable nodes and none had an edge to $t$}

**Corollary:** Some NL-machine computes $c_{d+1}$ from $c_d$.

## Check-in 20.3

Can we now show $2SAT$ is NL-complete?

(a) No.

(b) Yes.

Yes: $\overline{PATH} \leq_{\mathrm{L}} PATH$ & $PATH \leq_{\mathrm{L}} \overline{2SAT}$

So $\overline{PATH} \leq_{\mathrm{L}} \overline{2SAT}$ thus $PATH \leq_{\mathrm{L}} 2SAT$



Check-in 20.3

# Quick review of today

1. Log-space reducibility
2. L = NL? question
3. $PATH$ is NL-complete
4. $\overline{2SAT}$ is NL-complete
5. NL = coNL