18.404/6.840 Lecture 18

Last time:

- Space complexity
- SPACE$(f(n))$, NSPACE$(f(n))$, PSPACE, NPSPACE
- Relationship with TIME classes

Today: (Sipser §8.3)

- Review $LADDER_{\text{DFA}} \in$ PSPACE
- Savitch's Theorem: NSPACE$(f(n)) \subseteq$ SPACE$(f^2(n))$
- PSPACE-completeness
- $TQBF$ is PSPACE-complete

Review: SPACE Complexity

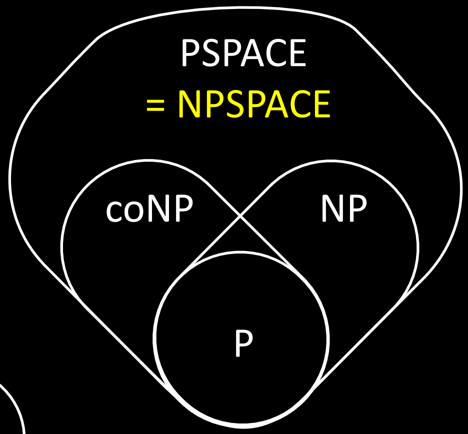
Defn: Let $f\colon \mathbb{N} \to \mathbb{N}$ where $f(n) \geq n$. Say TM M runs in space $f(n)$ if M always halts and uses at most $f(n)$ tape cells on all inputs of length n.

An NTM M runs in space $f(n)$ if all branches halt and each branch uses at most $f(n)$ tape cells on all inputs of length n.

SPACE$\big(f(n)\big) = \{B \mid$ some 1-tape TM decides B in space $O\big(f(n)\big)\}$
NSPACE$\big(f(n)\big) = \{B \mid$ some 1-tape NTM decides B in space $O\big(f(n)\big)\}$

PSPACE $= \bigcup_k$ SPACE(n^k) "polynomial space"
NPSPACE $= \bigcup_k$ NSPACE(n^k) "nondeterministic polynomial space"

Today: PSPACE = NPSPACE



Or possibly: P = NP = coNP = PSPACE

Review: $LADDER_{\text{DFA}} \in$ PSPACE

Theorem: $LADDER_{\text{DFA}} \in$ SPACE(n^2)

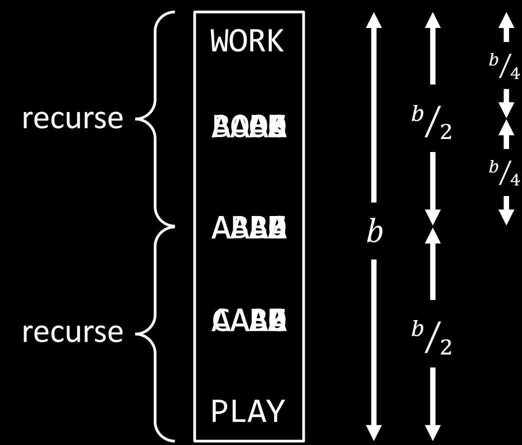
Proof: Write $u \xrightarrow{b} v$ if there's a ladder from u to v of length $\leq b$.

Here's a recursive procedure to solve the bounded DFA ladder problem:

$BOUNDED\text{-}LADDER_{\text{DFA}} = \{\langle B,u,v,b\rangle \mid B$ a DFA and $u \xrightarrow{b} v$ by a ladder in $L(B)\}$

$B\text{-}L =$ "On input $\langle B,u,v,b\rangle$ Let $m = |u| = |v|$.

1. For $b = 1$, *accept* if $u,v \in L(B)$ and differ in ≤ 1 place, else *reject*.
2. For $b > 1$, repeat for each $w \in L(B)$ of length $|u|$
3. Recursively test $u \xrightarrow{b/2} w$ and $w \xrightarrow{b/2} v$ [division rounds up]
4. *Accept* both accept.
5. *Reject* [if all fail]."



Test $\langle B,u,v\rangle \in LADDER_{\text{DFA}}$ with $B\text{-}L$ procedure on input $\langle B,u,v,t\rangle$ for $t = |\Sigma|^m$

Space analysis:

Each recursive level uses space $O(n)$ (to record w).

Recursion depth is $\log t = O(m) = O(n)$.

Total space used is $O(n^2)$.

PSPACE = NPSPACE

Savitch's Theorem: For $f(n) \geq n$, $\text{NSPACE}\big(f(n)\big) \subseteq \text{SPACE}\big(f^2(n)\big)$

Proof: Convert NTM N to equivalent TM M, only squaring the space used.

For configurations c_i and c_j of N, write $c_i \xrightarrow{b} c_j$ if can get from c_i to c_j in $\leq b$ steps.

Give recursive algorithm to test $c_i \xrightarrow{b} c_j$:

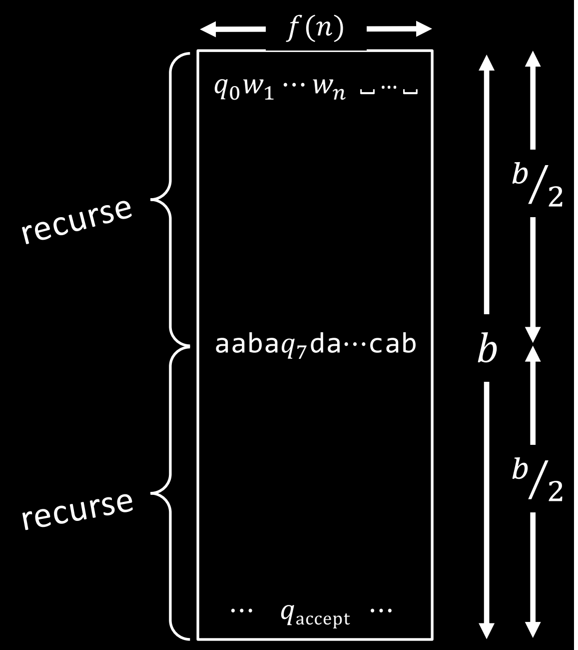
M = "On input c_i, c_j, b [goal is to check $c_i \xrightarrow{b} c_j$]

1. If $b = 1$, check directly by using N's program and answer accordingly.
2. If $b > 1$, repeat for all configurations c_{mid} that use $f(n)$ space.
3. Recursively test $c_i \xrightarrow{b/2} c_{\text{mid}}$ and $c_{\text{mid}} \xrightarrow{b/2} c_j$
4. If both are true, *accept*. If not, continue.
5. *Reject* if haven't yet accepted."

Test if N accepts w by testing $c_{\text{start}} \xrightarrow{t} c_{\text{accept}}$ where t = number of configurations $= |Q| \times f(n) \times d^{f(n)}$

Each recursion level stores 1 config = $O\big(f(n)\big)$ space.

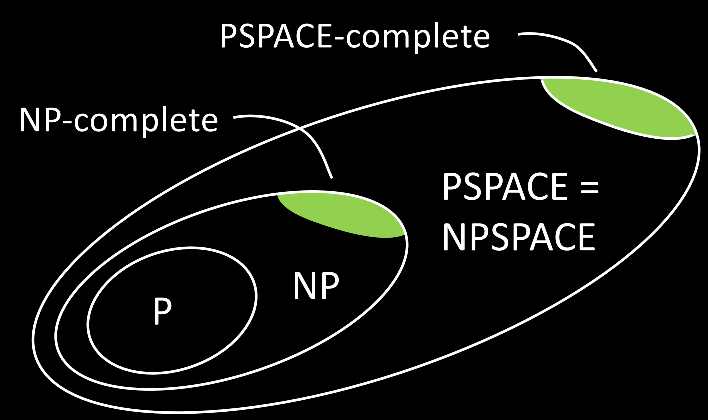
Number of levels = $\log t = O\big(f(n)\big)$. Total $O\big(f^2(n)\big)$ space.



PSPACE-completeness

Defn: B is PSPACE-complete if
1) $B \in$ PSPACE
2) For all $A \in$ PSPACE, $A \leq_{\text{P}} B$

If B is PSPACE-complete and $B \in$ P then P $=$ PSPACE.



Think of complete problems as the "hardest" in their associated class.

Check-in 18.1

Knowing that $TQBF$ is PSPACE-complete, what can we conclude if $TQBF \in$ NP? Check all that apply.

(a) P = PSPACE

(b) NP = PSPACE

(c) P = NP

(d) NP = coNP

Check-in 18.1

$TQBF$ is PSPACE-complete

Recall: $TQBF = \{\langle\phi\rangle \mid \phi$ is a QBF that is TRUE$\}$

Examples: $\phi_1 = \forall x\, \exists y\, [(x \vee y) \wedge (\overline{x} \vee \overline{y})] \in TQBF$ [TRUE]
$\phi_2 = \exists y\, \forall x\, [(x \vee y) \wedge (\overline{x} \vee \overline{y})] \notin TQBF$ [FALSE]

Theorem: $TQBF$ is PSPACE-complete

Proof: 1) $TQBF \in$ PSPACE ✓
2) For all $A \in$ PSPACE, $A \leq_{\mathrm{P}} TQBF$

Let $A \in$ PSPACE be decided by TM M in space n^k.
Give a polynomial-time reduction f mapping A to $TQBF$.

$f: \Sigma^* \rightarrow$ QBFs

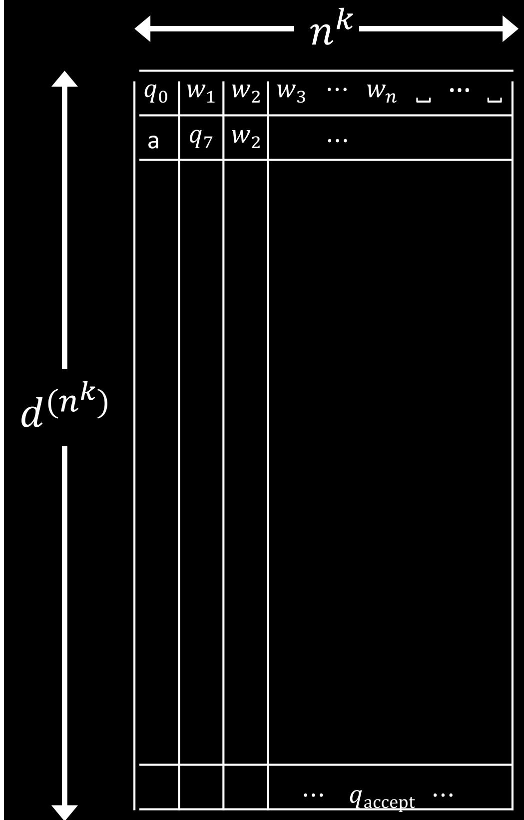
$f(w) = \langle\phi_{M,w}\rangle$

$w \in A$ iff $\phi_{M,w}$ is TRUE

Plan: Design $\phi_{M,w}$ to "say" M accepts w. $\phi_{M,w}$ simulates M on w.

Constructing $\phi_{M,w}$: 1st try

Tableau for M on w



Recall: A tableau for M on w represents a computation history for M on w when M accepts w.
Rows of that tableau are configurations.

M runs in space n^k, its tableau has:

- n^k columns (max size of a configuration)
- $d^{(n^k)}$ rows (max number of steps)

Constructing $\phi_{M,w}$. Try Cook-Levin method.
Then $\phi_{M,w}$ will be as big as tableau.
But that is exponential: $n^k \times d^{(n^k)}$.
Too big! ☹

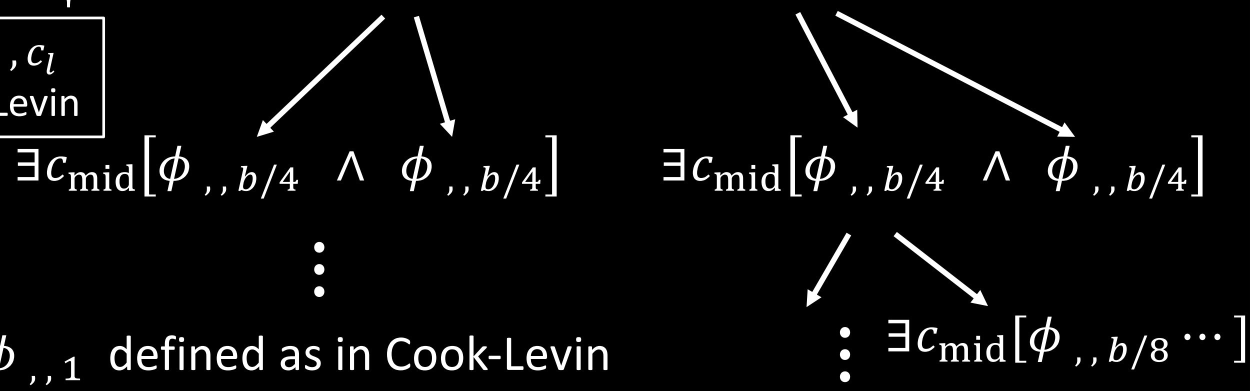
Constructing $\phi_{M,w}$: 2nd try

hide →

For configs c_i and c_j construct $\phi_{c_i, c_j, b}$ which "says" $c_i \xrightarrow{b} c_j$ recursively.

$$\phi_{c_i, c_j, b} = \underbrace{\exists c_{\text{mid}}}_{} \left[\phi_{c_i, c_{\text{mid}}, b/2} \ \land \ \phi_{c_{\text{mid}}, c_j, b/2} \right]$$

$\exists x_1, x_2, \cdots, c_l$
as in Cook-Levin



Check-in 18.2

Why shouldn't we be surprised that this construction fails?

(a) We can't define a QBF by using recursion.

(b) It doesn't use ∀ anywhere.

(c) We know that $TQBF \notin$ P.

$$\phi_{M,w} = \phi_{c_{\text{start}}, c_{\text{accept}}, t}$$
$$t = d^{(n^k)}$$

Size analysis:
Each recursive level doubles number of QBFs.
Number of levels is $\log d^{(n^k)} = O(n^k)$.
→ Size is exponential. ☹

Check-in 18.2

Constructing $\phi_{M,w}$: 3rd try

$$\phi_{c_i,\, c_j,\, b} = \exists c_{\text{mid}} \left[\phi_{c_i,\, c_{\text{mid}},\, b/2} \;\wedge\; \phi_{c_{\text{mid}},\, c_j,\, b/2} \right]$$

$$\forall (c_g, c_h) \in \{ (c_i, c_{\text{mid}}), (c_{\text{mid}}, c_j) \} \left[\phi_{c_g,\, c_h,\, b/2} \right]$$

$\forall (x \in S)\ [\psi]$
is equivalent to
$\forall x\ [(x \in S) \longrightarrow \psi]$

⋮

$\phi_{\,,\,,1}$ defined as in Cook-Levin

$\phi_{M,w} = \phi_{c_{\text{start}},\, c_{\text{accept}},\, t}$
$t = d^{(n^k)}$

Size analysis:
Each recursive level adds $O(n^k)$ to the QBF.
Number of levels is $\log d^{(n^k)} = O(n^k)$.
→ Size is $O(n^k \times n^k) = O(n^{2k})$ ☺

Check-in 18.3

Would this construction still work if M were nondeterministic?

(a) Yes.

(b) No.

Check-in 18.3

Quick review of today

1. $LADDER_{\text{DFA}} \in$ PSPACE
2. Savitch's Theorem: $\text{NSPACE}\big(f(n)\big) \subseteq \text{SPACE}\big(f^2(n)\big)$
3. $TQBF$ is PSPACE-complete