Graph Theory and Additive Combinatorics

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2

Forbidding subgraphs

2.1 *Mantel's theorem: forbidding a triangle*

We begin our discussion of extremal graph theory with the following basic question.

Question 2.1. What is the maximum number of edges in an n-vertex graph that does not contain a triangle?

Bipartite graphs are always triangle-free. A complete bipartite graph, where the vertex set is split equally into two parts (or differing by one vertex, in case n is odd), has $\lfloor n^2/4 \rfloor$ edges. Mantel's theorem states that we cannot obtain a better bound:

Theorem 2.2 (Mantel)**.** *Every triangle-free graph on n vertices has at most $\lfloor n^2/4 \rfloor$ edges.*

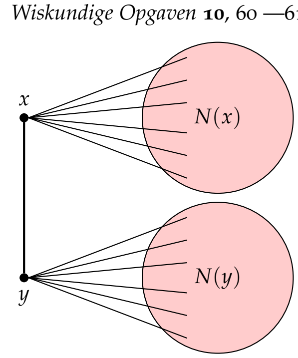
W. Mantel, "Problem 28 (Solution by H. Gouwentak, W. Mantel, J. Teixeira de Mattes, F. Schuh and W. A. Wythoff). *Wiskundige Opgaven* **10**, 60 —61, 1907.

We will give two proofs of Theorem 2.2.

Proof 1. Let $G = (V, E)$ a triangle-free graph with n vertices and m edges. Observe that for distinct $x, y \in V$ such that $xy \in E$, x and y must not share neighbors by triangle-freeness.

Therefore, $d(x) + d(y) \le n$, which implies that

$$\sum_{x \in V} d(x)^2 = \sum_{xy \in E} (d(x) + d(y)) \le mn.$$



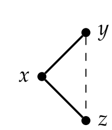
Adjacent vertices have disjoint neighborhoods in a triangle-free graph.

On the other hand, by the handshake lemma, $\sum_{x \in V} d(x) = 2m$. Now by the Cauchy–Schwarz inequality and the equation above,

$$4m^2 = \left(\sum_{x \in V} d(x)\right)^2 \le n \left(\sum_{x \in V} d(x)^2\right) \le mn^2;$$

hence $m \le n^2/4$. Since m is an integer, this gives $m \le \lfloor n^2/4 \rfloor$. $\square$

Proof 2. Let $G = (V, E)$ be as before. Since G is triangle-free, the neighborhood $N(x)$ of every vertex $x \in V$ is an independent set.



An edge within $N(x)$ creates a triangle

Let $A \subseteq V$ be a maximum independent set. Then $d(x) \leq |A|$ for all $x \in V$. Let $B = V \setminus A$. Since A contains no edges, every edge of G intersects B. Therefore,

$$e(G) \leq \sum_{x \in B} d(x) \leq |A||B| \underset{\text{AM-GM}}{\leq} \left\lfloor \left(\frac{|A|+|B|}{2} \right)^2 \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor .$$

□

Remark 2.3. For equality to occur in Mantel's theorem, in the above proof, we must have

- $e(G) = \sum_{x \in B} d(x)$, which implies that no edges are strictly in B.
- $\sum_{x \in B} d(x) = |A||B|$, which implies that every vertex in B is complete in A.
- The equality case in AM-GM must hold (or almost hold, when n is odd), hence $||A| - |B|| \leq 1$.

Thus a triangle-free graph on n vertices has exactly $\lfloor n^2/4 \rfloor$ edges if and only if it is the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$.

2.2 *Turán's theorem: forbidding a clique*

Motivated by Theorem 2.2, we turn to the following more general question.

Question 2.4. What is the maximum number of edges in a K_{r+1}-free graph on n vertices?

Extending the bipartite construction earlier, we see that an r-partite graph does not contain any copy of K_{r+1}.

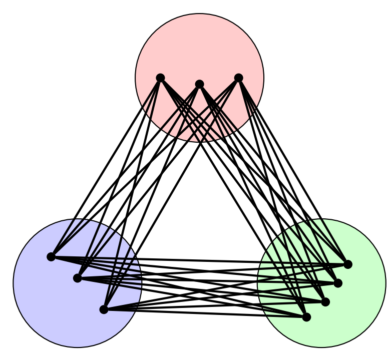
Definition 2.5. The ***Turán graph*** $T_{n,r}$ is defined to be the complete, n-vertex, r-partite graph, with part sizes either $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.

In this section, we prove that $T_{n,r}$ does, in fact, maximize the number of edges in a K_r-free graph:

Theorem 2.6 (Turán). *If G is an n-vertex K_{r+1}-free graph, then $e(G) \leq e(T_{n,r})$.*

When $r = 2$, this is simply Theorem 2.2.

We now give three proofs of Theorem 2.6. The first two are in the same spirit as the proofs of Theorem 2.2.



The Turán graph $T_{10,3}$

P. Turán, On an extremal problem in graph theory. *Math. Fiz. Lapok* **48**, 436—452, 1941.

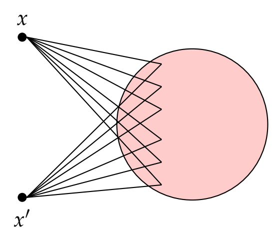
Proof 1. Fix r. We proceed by induction on n. Observe that the statement is trivial if $n \le r$, as K_n is K_{r+1}-free. Now, assume that $n > r$ and that Turán's theorem holds for all graphs on fewer than n vertices. Let G be an n-vertex, K_{r+1}-free graph with the maximum possible number of edges. Note that G must contain K_r as a subgraph, or else we could add an edge in G and still be K_{r+1}-free. Let A be the vertex set of an r-clique in G, and let $B := V \backslash A$. Since G is K_{r+1}-free, every $v \in B$ has at most $r-1$ neighbors in A. Therefore

$$\begin{aligned} e(G) &\le \binom{r}{2} + (r-1)|B| + e(B) \\ &\le \binom{r}{2} + (r-1)(n-r) + e(T_{n-r,r}) \\ &= e(T_{n,r}). \end{aligned}$$

The first inequality follows from counting the edges in A, B, and everything in between. The second inequality follows from the inductive hypothesis. The last equality follows by noting removing one vertex from each of the r parts in $T_{n,r}$ would remove a total of $\binom{r}{2} + (r-1)(n-r)$ edges. □

Proof 2 (Zykov symmetrization). As before, let G be an n-vertex, K_{r+1}-free graph with the maximum possible number of edges.

We claim that the non-edges of G form an equivalence relation; that is, if $xy, yz \notin E$, then $xz \notin E$. Symmetry and reflexivity are easy to check. To check transitivity, Assume for purpose of contradiction that there exists $x, y, z \in V$ for which $xy, yz \notin E$ but $xz \in E$.



x and its clone x'

If $d(y) < d(x)$, we may replace y with a "clone" of x. That is, we delete y and add a new vertex x' whose neighbors are precisely the as the neighbors of x (and no edge between x and x'). (See figure on the right.)

Then, the resulting graph G' is also K_{r+1}-free since x was not in any K_{r+1}. On the other hand, G' has more edges than G, contradicting maximality.

Therefore we have that $d(y) \ge d(x)$ for all $xy \notin E$. Similarly, $d(y) \ge d(z)$. Now, replace both x and z by "clones" of y. The new graph G' is K_{r+1}-free since y was not in any K_{r+1}, and

$$e(G') = e(G) - (d(x) + d(z) - 1) + 2d(y) > e(G),$$

contradicting maximality of $e(G)$. Therefore such a triple (x, y, z) cannot exist in G, and transitivity holds.

The equivalence relation shows that the complement of G is a union of cliques. Therefore G is a complete multipartite graph with at most r parts. One checks that increasing the number of parts increases the number of edges in G. Similarly, one checks that if the number of vertices in two parts differ by more than 1, moving one vertex from the larger part to the smaller part increases the number of edges in G. It follows that the graph that achieves the maximum number of edges is $T_{n,r}$. □

Our third and final proof uses a technique called the ***probabilistic method***. In this method, one introduces randomness to a deterministic problem in a clever way to obtain deterministic results.

Proof 3. Let $G = (V, E)$ be an n-vertex, K_{r+1}-free graph. Consider a uniform random ordering σ of the vertices. Let

$$X = \{v \in V : v \text{ is adjacent to all earlier vertices in } \sigma\}.$$

Observe that the set of vertices in X form a clique. Since the permutation was chosen uniformly at random, we have

$$\mathbb{P}(v \in X) = \mathbb{P}(v \text{ appears before all non-neighbors}) = \frac{1}{n - d(v)}.$$

Therefore,

$$r \geq \mathbb{E}|X| = \sum_{v \in V} \mathbb{P}(v \in X) = \sum_{v \in V} \frac{1}{n - d(v)} \overset{\text{convexity}}{\geq} \frac{n}{n - 2m/n}.$$

Rearranging gives $m \leq \left(1 - \frac{1}{r}\right)\frac{n^2}{2}$ (a bound that is already good for most purposes). Note that if n is divisible by r, then the bound immediately gives a proof of Turán's theorem. When n is not divisible by r, one needs to a bit more work and use convexity to argue that the $d(v)$ should be as close as possible. We omit the details. □

2.3 *Hypergraph Turán problem*

The short proofs given in the previous sections make problems in extremal graph theory seem deceptively simple. In reality, many generalizations of what we just discussed remain wide open.

Here we discuss one notorous open problem that is a hypergraph generalization of Mantel/Turán.

An r-uniform hypergraph consists of a vertex set V and an edge set, where every edge is now an r-element subset of V. Graphs correspond to $r = 2$.

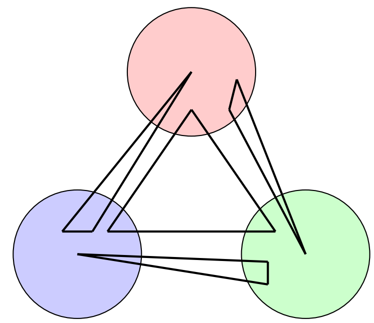
Question 2.7. What is the maximum number of triples in an n vertex 3-uniform hypergraph without a tetrahedron?

Turán proposed the following construction, which is conjectured to be optimal.

Example 2.8 (Turán)**.** Let V be a set of n vertices. Partition V into 3 (roughly) equal sets V_1, V_2, V_3. Add a triple $\{x, y, z\}$ to $e(G)$ if it satisfies one of the four following conditions:

- x, y, z are in different partitions
- $x, y \in V_1$ and $z \in V_2$
- $x, y \in V_2$ and $z \in V_3$
- $x, y \in V_3$ and $z \in V_1$

where we consider x, y, z up to permutation (See Example 2.8). One checks that the 3-uniform hypergraph constructed is tetrahedron-free, and that it has edge density $5/9$.



Turán's construction of a tetrahedron-free 3-uniform hypergraph

On the other hand, the best known upper bound is approximately 0.562 , obtained recently using the technique of flag algebras.

Keevash (2011)
Baber and Talbot (2011)
Razborov (2010)

2.4 Erdős–Stone–Simonovits theorem (statement): forbidding a general subgraph

One might also wonder what happens if K_{r+1} in Theorem 2.6 were replaced with an arbitrary graph H:

Question 2.9. Fix some graph H. If G is an n vertex graph in which H does not appear as a subgraph, what is the maximum possible number of edges in G?

Notice that we only require H to be a ***subgraph***, not necessarily an ***induced subgraph***. An induced subgraph H' of G must contain all edges present between the vertices of H', while there is no such restriction for arbitrary subgraphs.

Definition 2.10. For a graph H and $n \in \mathbb{N}$, define $\text{ex}(n, H)$ to be the maximum number of edges in an n-vertex H-free graph.

For example, Theorem 2.6 tells us that for any given r,

$$\text{ex}(n, K_{r+1}) = e(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}$$

where $o(1)$ represents some quantity that goes to zero as $n \to \infty$.

At a first glance, one might not expect a clean answer to Question 2.9. Indeed, the solution would seem to depend on various characteristics of H (for example, its diameter or maximum degree). Surprisingly, it turns out that a single parameter, the chromatic number of H, governs the growth of $\text{ex}(n, H)$.

Definition 2.11. The ***chromatic number*** of a graph G, denoted $\chi(G)$, is the minimal number of colors needed to color the vertices of G such that no two adjacent vertices have the same color.

Example 2.12. $\chi(K_{r+1}) = r + 1$ and $\chi(T_{n,r}) = r$.

Observe that if $H \subseteq G$, then $\chi(H) \leq \chi(G)$. Indeed, any proper coloring of G restricts to a proper coloring of H. From this, we gather that if $\chi(H) = r + 1$, then $T_{n,r}$ is H-free. Therefore,

$$\operatorname{ex}(n, H) \geq e(T_{n,r}) = \left(1 - \frac{1}{r} + o(1)\right)\binom{n}{2}.$$

Is this the best we can do? The answer turns out to be affirmative.

Theorem 2.13 (Erdős–Stone–Simonovits). *For all graphs H, we have*

Erdős and Stone (1946)
Erdős and Simonovits (1966)

$$\lim_{n\to\infty} \frac{\operatorname{ex}(n, H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H) - 1}.$$

We'll skip the proof for now.

Remark 2.14. Later in the book we will show how to deduce Theorem 2.13 from Theorem 2.6 using the ***Szemerédi regularity lemma***.

Example 2.15. When $H = K_3$, Theorem 2.13 tells us that

$$\lim_{n\to\infty} \frac{\operatorname{ex}(n, H)}{\binom{n}{2}} = \frac{1}{2},$$

in agreement with Theorem 2.6.

When $H = K_4$, we get

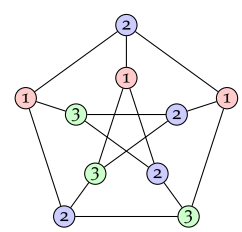
$$\lim_{n\to\infty} \frac{\operatorname{ex}(n, H)}{\binom{n}{2}} = \frac{2}{3},$$

also in agreement with Theorem 2.6.

When H is the Peterson graph, Theorem 2.13 tells us that

$$\lim_{n\to\infty} \frac{\operatorname{ex}(n, H)}{\binom{n}{2}} = \frac{1}{2},$$

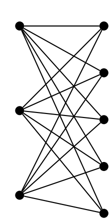
which is the same answer as for $H = K_3$! This is surprising since the Peterson graph seems much more complicated than the triangle.



The Peterson graph with a proper 3-coloring.

2.5 *Kővári–Sós–Turán theorem: forbidding a complete bipartite graph*

The Erdős–Stone–Simonovits Theorem (Theorem 2.13) gives a first-order approximation of $\operatorname{ex}(n, H)$ when $\chi(H) > 2$. Unfortunately, Theorem 2.13 does not tell us the whole story. When $\chi(H) = 2$, i.e. H is bipartite, the theorem implies that $\operatorname{ex}(n, H) = o(n^2)$, which compels us to ask if we may obtain more precise bounds. For example, if we write $\operatorname{ex}(n, H)$ as a function of n, what its growth with respect to n? This is an open problem for most bipartite graphs (for example, $K_{4,4}$) and the focus of the remainder of the chapter.



An example of a complete bipartite graph $K_{3,5}$.

Let $K_{s,t}$ be the complete bipartite graph where the two parts of the bipartite graph have s and t vertices respectively. In this section, we consider $\operatorname{ex}(n, K_{s,t})$, and seek to answer the following main question:

Question 2.16 (Zarankiewicz problem)**.** For some $r, s \geq 1$, what is the maximum number of edges in an n-vertex graph which does not contain $K_{s,t}$ as a subgraph.

Every bipartite graph H is a subgraph of some complete bipartite graph $K_{s,t}$. If $H \subseteq K_{s,t}$, then $\mathrm{ex}(n, H) \leq \mathrm{ex}(n, K_{s,t})$. Therefore, by understanding the upper bound on the extremal number of complete bipartite graphs, we obtain an upper bound on the extremal number of general bipartite graphs as well. Later, we will give improved bounds for several specific biparite graphs.

Kővári, Sós and Turán gave an upper bound on $K_{s,t}$:

Theorem 2.17 (Kővári–Sós–Turán)**.** *For every integers $1 \leq s \leq t$, there exists some constant C, such that*

Kővári, Sós, and Turán (1954)

$$\mathrm{ex}(n, K_{s,t}) \leq Cn^{2-\frac{1}{s}}.$$

There is an easy way to remember the name of this theorem: "KST", the initials of the authors, is also the letters for the complete bipartite graph $K_{s,t}$.

Proof. Let G be a $K_{s,t}$-free n-vertex graph with m edges.

First, we repeatedly remove all vertices $v \in V(G)$ where $d(v) < s-1$. Since we only remove at most $(s-2)n$ edges this way, it suffices to prove the theorem assuming that all vertices have degree at least $s-1$.

We denote the number of copies of $K_{s,1}$ in G as $\#K_{s,1}$. The proof establishes an upper bound and a lower bound on $\#K_{s,1}$, and then gets a bound on m by combining the upper bound and the lower bound.

Since $K_{s,1}$ is a complete bipartite graph, we can call the side with s vertices the 'left side', and the side with 1 vertices the 'right side'.

On the one hand, we can count $\#K_{s,1}$ by enumerating the 'left side'. For any subset of s vertices, the number of $K_{s,1}$ where these s vertices form the 'left side' is exactly the number of common neighbors of these s vertices. Since G is $K_{s,t}$-free, the number of common neighbors of any subset of s vertices is at most $t-1$. Thus, we establish that $\#K_{s,1} \leq \binom{n}{s}(t-1)$.

On the other hand, for each vertex $v \in V(G)$, the number of copies of $K_{s,1}$ where v is the 'right side' is exactly $\binom{d(v)}{s}$. Therefore,

Here we regard $\binom{x}{s}$ as a degree s polynomial in x, so it makes sense for x to be non-integers. The function $\binom{x}{s}$ is convex when $x \geq s-1$.

$$\#K_{s,1} = \sum_{v \in V(G)} \binom{d(v)}{s} \geq n\binom{\frac{1}{n}\sum_{v\in V(G)} d(v)}{s} = n\binom{2m/n}{s},$$

where the inequality step uses the convextiy of $x \mapsto \binom{x}{s}$.

Combining the upper bound and lower bound of $\#K_{s,1}$, we obtain that $n\binom{2m/n}{s} \leq \binom{n}{s}(t-1)$. For constant s, we can use $\binom{x}{s} = (1+o(1))\frac{n^s}{s!}$ to get $n\left(\frac{2m}{n}\right)^s \leq (1+o(1))n^s(t-1)$. The above inequality simplifies to

$$m \leq \left(\frac{1}{2} + o(1)\right)(t-1)^{1/s} n^{2-\frac{1}{s}}. \qquad \square$$

Let us discuss a geometric application of Theorem 2.17.

Question 2.18 (Unit distance problem)**.** What is the maximum number of unit distances formed by n points in $\mathbb{R}^2$?

Erdős (1946)

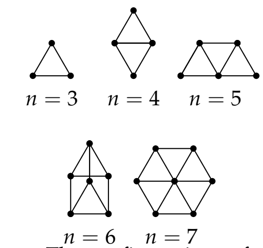


Figure 2.1: The configurations of points for small values of n with maximum number of unit distances. The edges between vertices mean that the distance is 1. These constructions are unique up to isomorphism except when $n = 6$.

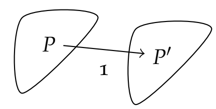
For small values of n, we precisely know the answer to the unit distance problem. The best configurations are listed in Figure 2.1.

It is possible to generalize some of these constructions to arbitrary n.

- A line graph has $(n-1)$ unit distances.

- A chain of triangles has $(2n-3)$ unit distances for $n \geq 3$.

- There is also a recursive construction. Given a configuration P with $n/2$ points that have $f(n/2)$ unit distances, we can copy P and translate it by an arbitrary unit vector to get P'. The configuration $P \cup P'$ have at least $2f(n/2) + n/2$ unit distances. We can solve the recursion to get $f(n) = \Omega(n \log n)$.



The current best lower bound on the maximum number of unit distances is given by Erdős.

Proposition 2.19. *There exists a set of n points in $\mathbb{R}^2$ that have at least $n^{1+c/\log\log n}$ unit distances for some constant c.*

Erdős (1946)

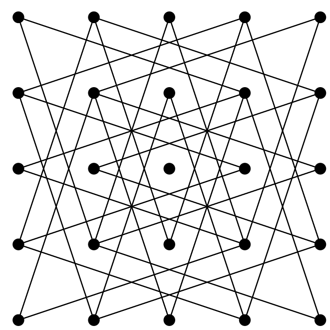


Figure 2.2: An example grid graph where $n = 25$ and $r = 10$.

Proof sketch. Consider a square grid with $\lfloor\sqrt{n}\rfloor \times \lfloor\sqrt{n}\rfloor$ vertices. We can scale the graph arbitrarily so that $\sqrt{r}$ becomes the unit distance for some integer r. We can pick r so that r can be represented as a sum of two squares in many different ways. One candidate of such r is a product of many primes that are congruent to 1 module 4. We can use some number-theoretical theorems to analyze the best r, and get the $n^{1+c/\log\log n}$ bound. □

Theorem 2.17 can be used to prove an upper bound on the number of unit distances.

Theorem 2.20. *Every set of n points in $\mathbb{R}^2$ has at most $O(n^{3/2})$ unit distances.*

Proof. Given any set of points $S \subset \mathbb{R}^2$, we can create the *unit distance graph G* as follows:

- The vertex set of G is S,
- For any point p, q where $d(p,q) = 1$, we add an edge between p and q.

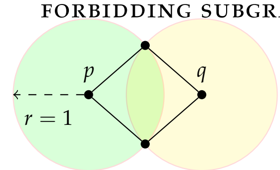


Figure 2.3: Two vertices p, q can have at most two common neighbors in the unit distance graph.

The graph G is $K_{2,3}$-free since for every pair of points p, q, there are at most 2 points that have unit distances to both of them. By applying Theorem 2.17, we obtain that $e(G) = O(n^{3/2})$. □

Remark 2.21. The best known upper bound on the number of unit distances is $O(n^{4/3})$. The proof is a nice application of the crossing number inequality which will be introduced later in this book.

Spencer, Szemerédi and Trotter (1984)

Here is another problem that is strongly related to the unit distance problem:

Question 2.22 (Distinct distance problem)**.** What is the minimum number of distinct distances formed by n points in $\mathbb{R}^2$?

Example 2.23. Consider n points on the x-axis where the i-th point has coordinate $(i, 0)$. The number of distinct distances for these points is $n - 1$.

The current best construction for minimum number of distinct distances is also the grid graph. Consider a square grid with $\lfloor \sqrt{n} \rfloor \times \lfloor \sqrt{n} \rfloor$ vertices. Possible distances between two vertices are numbers that can be expressed as a sum of the squares of two numbers that are at most $\lfloor \sqrt{n} \rfloor$. Using number-theoretical methods, we can obtain that the number of such distances: $\Theta(n/\sqrt{\log n})$.

The maximum number of unit distances is also the maximum number that each distance can occur. Therefore, we have the following relationship between distinct distances and unit distances:

$$\#\text{distinct distances} \geq \frac{\binom{n}{2}}{\max \#\text{unit distances}}.$$

If we apply Theorem 2.20 to the above inequality, we immediately get an $\Omega(n^{0.5})$ lower bound for the number of distinct distances. Many mathematicians successively improved the exponent in this lower bound over the span of seven decades. Recently, Guth and Katz gave the following celebrated theorem, which almost matches the upper bound (only off by an $O(\sqrt{\log n})$ factor).

Theorem 2.24 (Guth–Katz)**.** *Every set of n points in $\mathbb{R}^2$ has at least $cn/\log n$ distinct distances for some constant c.*

Guth and Katz (2015)

The proof of Theorem 2.24 is quite sophisticated: it uses tools ranging from polynomial method to algebraic geometry. We won't cover it in this book.

2.6 *Lower bounds: randomized constructions*

It is conjectured that the bound proven in Theorem 2.17 is tight. In other words, $\mathrm{ex}(n, K_{s,t}) = \Theta(n^{2-1/s})$. Although this still remains open for arbitrary $K_{s,t}$, it is already proven for a few small cases, and in cases where t is way larger than s. In this and the next two sections, we will show techniques for constructing H-free graphs. Here are the three main types of constructions that we will cover:

- **Randomized construction.** This method is powerful and general, but introducing randomness means that the constructions are usually *not tight*.
- **Algebraic construction.** This method uses tools in number theory or algebra to assist construction. It gives tighter results, but they are usually 'magical', and only works in a small set of cases.
- **Randomized algebraic construction.** This method is the hybrid of the two methods above and combines the advantages of both.

This section will focus on randomized constructions. We start with a general lower bound for extremal numbers.

Theorem 2.25. *For any graph H with at least 2 edges, there exists a constant $c > 0$, such that for any $n \in \mathbb{N}$, there exists an H-free graph on n vertices with at least $cn^{2-\frac{v(H)-2}{e(H)-1}}$ edges. In other words,*

$$\mathrm{ex}(n, H) \geq cn^{2-\frac{v(H)-2}{e(H)-1}}.$$

Proof. The idea is to use the *alteration method*: we can construct a graph that has few copies of H in it, and delete one edge from each copy to eliminate the occurrences of H.

Consider $G = G(n,p)$ as a random graph with n vertices where each edge appears with probability p (p to be determined). Let $\#H$ be the number of copies of H in G. Then,

The random graph $G(n,p)$ is called the ***Erdős–Rényi random graph***, which appears in many randomized constructions.

$$\mathbb{E}[\#H] = \frac{n(n-1)\cdots(n-v(H)+1)}{|\operatorname{Aut}(H)|}p^{e(H)} \leq p^{e(H)}n^{v(H)},$$

where $\operatorname{Aut}(H)$ is the automorphism group of graph H, and

$$\mathbb{E}[e(G)] = p\binom{n}{2}.$$

Let $p = \frac{1}{2}n^{-\frac{v(H)-2}{e(H)-1}}$, chosen so that

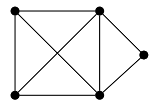
$$\mathbb{E}[\#H] \leq \frac{1}{2}\mathbb{E}[e(G)],$$

which further implies

$$\mathbb{E}[e(G) - \#H] \geq \frac{1}{2}p\binom{n}{2} \geq \frac{1}{16}n^{2-\frac{v(H)-2}{e(H)-1}}.$$

Thus, there exists a graph G, such that the value of $(e(G) - \#H)$ is at least the expectation. Remove one edge from each copy of H in G, and we get an H-free graph with enough edges. □

Remark 2.26. For example, if H is the following graph



then applying Theorem 2.25 directly gives

$$\text{ex}(n, H) \gtrsim n^{11/7}.$$

However, if we forbid H's subgraph K_4 instead (forbidding a subgraph will automatically forbid the original graph), Theorem 2.25 actually gives us a better bound:

$$\text{ex}(n, H) \geq \text{ex}(n, K_4) \gtrsim n^{8/5}.$$

For a general H, we apply Theorem 2.25 to the subgraph of H with the maximum $(e-1)/(v-2)$ value. For this purpose, define the ***2-density of*** H as

$$m_2(H) := \max_{\substack{H' \subseteq H \\ v(H') \geq 3}} \frac{e(H') - 1}{v(H') - 2}.$$

We have the following corollary.

Corollary 2.27. *For any graph H with at least two edges, there exists constant $c = c_H > 0$ such that*

$$\text{ex}(n, H) \geq cn^{2-1/m_2(H)}.$$

Example 2.28. We present some specific examples of Theorem 2.25. This lower bound, combined with the upper bound from the Kővári–Sós–Turán theorem (Theorem 2.17), gives that for every $2 \leq s \leq t$,

$$n^{2-\frac{s+t-2}{st-1}} \lesssim \text{ex}(n, K_{s,t}) \lesssim n^{2-1/s}.$$

When t is large compared to s, the exponents in the two bounds above are close to each other (but never equal).

When $t = s$, the above bounds specialize to

$$n^{2-\frac{2}{s+1}} \lesssim n^{2-\frac{s+t-2}{st-1}} \lesssim\lesssim n^{2-1/s}.$$

In particular, for $s = 2$, we obtain

$$n^{4/3} \lesssim \text{ex}(n, K_{2,2}) \lesssim n^{3/2}.$$

It turns out what the upper bound is close to tight, as we show next a different, algebraic, construction of a $K_{2,2}$-free graph.

2.7 Lower bounds: algebraic constructions

In this section, we use algebraic constructions to find $K_{s,t}$-free graphs, for various values of (s,t), that match the upper bound in the Kővári–Sós–Turán theorem (Theorem 2.17) up to a constant factor.

The simplest example of such an algebraic construction is the following construction of $K_{2,2}$-free graphs with many edges.

Theorem 2.29 (Erdős–Rényi–Sós)**.**

Erdős, Rényi and Sós (1966)

$$\operatorname{ex}(n, K_{2,2}) \geq \left(\frac{1}{2} - o(1)\right) n^{3/2}.$$

Proof. Suppose $n = p^2 - 1$ where p is a prime. Consider the following graph G (called *polarity graph*):

- $V(G) = \mathbb{F}_p^2 \setminus \{(0,0)\}$,
- $E(G) = \{(x,y) \sim (a,b) | ax + by = 1 \text{ in } \mathbb{F}_p\}$.

For any two distinct vertices $(a,b) \neq (a',b') \in V(G)$, there is at most one solution (common neighbour) $(x,y) \in V(G)$ satisfying both $ax + by = 1$ and $a'x + b'y = 1$. Therefore, G is $K_{2,2}$-free.

Moreover, every vertex has degree p or $p-1$, so the total number of edges

$$e(G) = \left(\frac{1}{2} - o(1)\right) p^3 = \left(\frac{1}{2} - o(1)\right) n^{3/2},$$

which concludes our proof.

If n does not have the form $p^2 - 1$ for some prime, then we let p be the largest prime such that $p^2 - 1 \leq n$. Then $p = (1 - o(1)n$ and constructing the same graph G_{p^2-1} with $n - p^2 + 1$ isolated vertices. □

Why is it called a polarity graph? It may be helpful to first think about the partite version of the construction, where one vertex set is the set of points of of a (projective) plane over $\mathbb{F}_p$, and the other vertex set is the set of lines in the same plane, and one has an edge between point p and line ℓ if $p \in \ell$. This graph is C_4-free since no two lines intersect in two distinct points.

The construction in the proof of Theorem 2.29 has one vertex set that identifies points with lines. This duality pairing between points and lines is known in projective geometry a polarity.

Most vertices have degree p because the equation $ax + by = 1$ has exactly p solutions (x,y). Sometimes we have to subtract 1 because one of the solutions might be (a,b) itself, which forms a self-loop.

Here we use that the smallest prime greater than n has size $n + o(n)$. The best result of this form says that there exists a prime in the interval $[n - n^{0.525}, n]$ for every sufficiently large n.

Baker, Harman and Pintz (2001)

A natural question to ask here is whether the construction above can be generalized. The next construction gives us a construction for $K_{3,3}$-free graphs.

Theorem 2.30 (Brown)**.**

Brown (1966)

It is known that the constant $1/2$ in Theorem 2.30 is the best constant possible.

$$\operatorname{ex}(n, K_{3,3}) \geq \left(\frac{1}{2} - o(1)\right) n^{5/3}$$

Proof sketch. Let $n = p^3$ where p is a prime. Consider the following graph G:

- $V(G) = \mathbb{F}_p^3$
- $E(G) = \{(x,y,z) \sim (a,b,c) | (a-x)^2 + (b-y)^2 + (c-z)^2 = u \text{ in } \mathbb{F}_p\}$, where u is some carefully-chosen fixed nonzero element in $\mathbb{F}_p$

One needs to check that it is possible to choose u so that the above graph is $K_{3,3}$. We omit the proof but give some intuition. Had we used points in $\mathbb{R}^3$ instead of $\mathbb{F}_p^3$, the $K_{3,3}$-freeness is equivalent to the statement that three unit spheres have at most two common points. This statement about unit spheres in $\mathbb{R}^3$, and it can be proved rigorously by some algebraic manipulation. One would carry out a similar algebraic manipulation over $\mathbb{F}_p$ to verify that the graph above is $K_{3,3}$-free.

Moreover, each vertex has degree around p^2 since the distribution of $(a-x)^2 + (b-y)^2 + (c-z)^2$ is almost uniform across $\mathbb{F}_p$ as (x, y, z) varies randomly over $\mathbb{F}_p^3$, and so we expect roughly a $1/p$ fraction of (x, y, z) to have $(a-x)^2 + (b-y)^2 + (c-z)^2 = u$. Again we omit the details. □

Although the case of $K_{2,2}$ and $K_{3,3}$ are fully solved, the corresponding problem for $K_{4,4}$ is a central open problem in extremal graph theory.

Open problem 2.31. What is the order of growth of $\text{ex}(n, K_{4,4})$? Is it $\Theta(n^{7/4})$, matching the upper bound in Theorem 2.17?

We have obtained the Kővári–Sós–Turán bound up to a constant factor for $K_{2,2}$ and $K_{3,3}$. Now we present a construction that matches the Kővári–Sós–Turán bound for $K_{s,t}$ whenever t is sufficiently large compared to s.

Theorem 2.32 (Alon, Kollár, Rónyai, Szabó). *If $t \geq (s-1)! + 1$ then*

$$ex(n, K_{s,t}) = \Theta(n^{2-\frac{1}{s}}).$$

Kollár, Rónyai, and Szabó (1996)
Alon, Rónyai, and Szabó (1999)

We begin by proving a weaker version for $t \geq s! + 1$. This will be similar in spirit and later we will make an adjustment to achieve the desired bound. Take a prime p and $n = p^s$ with $s \geq 2$. Consider the norm map $N \colon \mathbb{F}_{p^s} \to \mathbb{F}_p$ defined by

$$N(x) = x \cdot x^p \cdot x^{p^2} \cdots x^{p^{s-1}} = x^{\frac{p^s-1}{p-1}}.$$

Notice that we said the image of N lies in $\mathbb{F}_p$ rather than $\mathbb{F}_{p^s}$. We can easily check this is indeed the case as $N(x)^p = N(x)$.

Define the graph $\mathsf{NormGraph}_{p,s} = (V, E)$ with

$$V = \mathbb{F}_{p^s} \text{ and } E = \{\{a, b\} | a \neq b, N(a+b) = 1\}.$$

Proposition 2.33. *In $\mathsf{NormGraph}_{p,s}$ defined as above, letting $n = p^s$ be the number of vertices,*

$$|E| \geq \frac{1}{2} n^{2-\frac{1}{s}}.$$

Proof. Since $\mathbb{F}_{p^s}^\times$ is a cyclic group of order $p^s - 1$ we know that

$$|\{x \in \mathbb{F}_{p^s} | N(x) = 1\}| = \frac{p^s - 1}{p - 1}.$$

Thus for every vertex x (the minus one accounts for vertices with $N(x+x)=1$)

$$\deg(x) \geq \frac{p^s-1}{p-1} - 1 \geq p^{s-1} = n^{1-\frac{1}{s}}.$$

This gives us the desired lower bound on the number of edges. □

Proposition 2.34. *$\mathsf{NormGraph}_{p,s}$ is $K_{s,s!+1}$-free.*

We wish to upper bound the number of common neighbors to a set of s vertices. We quote without proof the following result, which can be proved using algebraic geometry.

Theorem 2.35. *Let $\mathbb{F}$ be any field and $a_{ij}, b_i \in \mathbb{F}$ such that $a_{ij} \neq a_{i'j}$ for all $i \neq i'$. Then the system of equations*

Kollár, Rónyai, and Szabó (1996)

$$\begin{aligned}(x_1 - a_{11})(x_2 - a_{12}) \cdots (x_s - a_{1s}) &= b_1 \\ (x_1 - a_{21})(x_2 - a_{22}) \cdots (x_s - a_{2s}) &= b_2 \\ &\vdots \\ (x_1 - a_{s1})(x_2 - a_{s2}) \cdots (x_s - a_{ss}) &= b_s\end{aligned}$$

has at most $s!$ solutions in $\mathbb{F}^s$.

Remark 2.36. Consider the special case when all the b_i are 0. In this case, since the a_{ij} are distinct for a fixed j, we are picking an i_j for which $x_j = a_{i_j j}$. Since all the i_j are distinct, this is equivalent to picking a permutation on $[s]$. Therefore there are exactly $s!$ solutions.

We can now prove Proposition 2.34.

Proof of Proposition 2.34. Consider distinct $y_1, y_2, \ldots, y_s \in \mathbb{F}_{p^s}$. We wish to bound the number of common neighbors x. We can use the fact that in a field with characteristic p we have $(x+y)^p = x^p + y^p$ to obtain

$$\begin{aligned}1 = N(x+y_i) &= (x+y_i)(x+y_i)^p \ldots (x+y_i)^{p^{s-1}} \\ &= (x+y_i)(x^p + y_i^p) \ldots (x^{p^{s-1}} + y_i^{p^{s-1}})\end{aligned}$$

for all $1 \leq i \leq s$. By Theorem 2.35 these s equations have at most $s!$ solutions in x. Notice we do in fact satisfy the hypothesis since $y_i^p = y_j^p$ if and only if $y_i = y_j$ in our field. □

Now we introduce the adjustment to achieve the bound $t \geq (s-1)! + 1$ in Theorem 2.32. We define the graph $\mathsf{ProjNormGraph}_{p,s} = (V, E)$ with $V = \mathbb{F}_{p^{s-1}} \times \mathbb{F}_p^\times$ for $s \geq 3$. Here $n = (p-1)p^{s-1}$. Define the edge relation as $(X, x) \sim (Y, y)$ if and only if

$$N(X+Y) = xy.$$

Proposition 2.37. *In* ProjNormGraph$_{p,s}$ *defined as above, letting* $n = (p-1)p^{s-1}$ *denote the number of vertices,*

$$|E| = \left(\frac{1}{2} - o(1)\right) n^{2-\frac{1}{s}}.$$

Proof. It follows from that every vertex (X, x) has degree $p^{s-1} - 1 = (1 - o(1))n^{1-1/s}$ since its neighbors are $(Y, N(X+Y)/x)$ as Y ranges over elements of $\mathbb{F}_{p^{s-1}}$ over than $-X$. □

Now that we know we have a sufficient amount of edges we just need our graph to be $K_{s,(s-1)!+1}$-free.

Proposition 2.38. ProjNormGraph$_{p,s}$ *is* $K_{s,(s-1)!+1}$*-free.*

Proof. Once again we fix distinct $(Y_i, y_i) \in V$ for $1 \leq i \leq s$ and we wish to find all common neighbors (X, x). Then

$$N(X + Y_i) = xy_i.$$

Assume this system has at least one solution. Then if $Y_i = Y_j$ with $i \neq j$ we must have that $y_i = y_j$. Therefore all the Y_i are distinct. For each $i < s$ we can take $N(X + Y_i) = xy_i$ and divide by $N(X + Y_s) = xy_s$ to obtain

$$N\left(\frac{X + Y_i}{X + Y_s}\right) = \frac{y_i}{y_s}.$$

Dividing both sides by $N(Y_i - Y_s)$ we obtain

$$N\left(\frac{1}{X + Y_s} + \frac{1}{Y_i - Y_s}\right) = \frac{y_i}{N(Y_i - Y_s)y_s}$$

for all $1 \leq i \leq s - 1$. Now applying Theorem 2.35 there are at most $(s-1)!$ choices for X, which also determines $x = N(X + Y_1)/y_1$. Thus there are at most $(s-1)!$ common neighbors. □

Now we are ready to prove Theorem 2.32.

Proof of Theorem 2.32. By Proposition 2.37 and Proposition 2.38 we know that ProjNormGraph$_{p,s}$ is $K_{s,(s-1)!+1}$-free and therefore $K_{s,t}$-free and has $\left(\frac{1}{2} - o(1)\right) n^{2-\frac{1}{s}}$ edges as desired. □

2.8 *Lower bounds: randomized algebraic constructions*

So far we have seen both constructions using random graphs and algebraic constructions. In this section we present an alternative construction of $K_{s,t}$-free graphs due to Bukh with $\Theta(n^{2-\frac{1}{s}})$ edges provided $t > t_0(s)$ for some function t_0. This is an algebraic construction with some randomness added to it.

Bukh (2015)

First fix $s \geq 4$ and take a prime power q. Let $d = s^2 - s + 2$ and $f \in \mathbb{F}_q[x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_s]$ be a polynomial chosen uniformly at random among all polynomials with degree at most d in each of $X = (x_1, x_2, \ldots, x_s)$ and $Y = (y_1, y_2, \ldots, y_s)$. Take G bipartite with vertex parts $n = L = R = \mathbb{F}_q^s$ and define the edge relation as $(X, Y) \in L \times R$ when $f(X, Y) = 0$.

Lemma 2.39. *For all $u, v \in \mathbb{F}_q^s$ and f chosen randomly as above*

$$\mathbb{P}[f(u, v) = 0] = \frac{1}{q}.$$

Proof. Notice that if g is a uniformly random constant in $\mathbb{F}_q$, then $f(u, v)$ and $f(u, v) + g$ are identically distributed. Hence each of the q possibilities are equally likely to the probability is $1/q$. □

Now the expected number of edges is the order we want as $\mathbb{E}[e(G)] = \frac{n^2}{q}$. All that we need is for the number of copies of $K_{s,t}$ to be relatively low. In order to do so, we must answer the following question. For a set of vertices in L of size s, how many common neighbors can it have?

Lemma 2.40. *Suppose $r, s \leq min(\sqrt{q}, d)$ and $U, V \subset \mathbb{F}_q^s$ with $|U| = s$ and $|V| = r$. Furthermore let $f \in \mathbb{F}_q[x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_s]$ be a polynomial chosen uniformly at random among all polynomials with degree at most d in each of $X = (x_1, x_2, \ldots, x_s)$ and $Y = (y_1, y_2, \ldots, y_s)$. Then*

$$\mathbb{P}[f(u, v) = 0 \text{ for all } u \in U, v \in V] = q^{-sr}.$$

Proof. First let us consider the special case where the first coordinates of points in U and V are all distinct. Define

$$g(X_1, Y_1) = \sum_{\substack{0 \leq i \leq s-1 \\ 0 \leq j \leq r-1}} a_{ij} X_1^i Y_1^j$$

with a_{ij} each uniform iid random variables over $\mathbb{F}_q$. We know that f and $f + g$ have the same distribution, so it suffices to show for all $b_{uv} \in \mathbb{F}_q$ where $u \in U$ and $v \in V$ there exists a_{ij} for which $g(u, v) = b_{uv}$ for all $u \in U, v \in V$. The idea is to apply Lagrange Interpolation twice. First for all $u \in U$ we can find a single variable polynomial $g_u(Y_1)$ with degree at most $r - 1$ such that $g_u(v) = b_{uv}$ for all $v \in V$. Then we can view $g(X_1, Y_1)$ as a polynomial in Y_1 with coefficients being polynomials in X_1, i.e.,

$$g(X_1, Y_1) = \sum_{0 \leq j \leq r-1} a_j(X_1) Y_1^j.$$

Applying the Lagrange interpolation theorem for a second time we can find polynomials $a_0, a_1, \ldots, a_{r-1}$ such that for all $u \in U$, $g(u, Y_1) = g_u(Y_1)$ as polynomials in Y_1.

Now suppose the first coordinates are not necessarily distinct. It suffices to find linear maps $T, S\colon \mathbb{F}_q^s \to \mathbb{F}_q^s$ such that TU and SV have all their first coordinates different. Let us prove that such a map T exists. If we find a linear map $T_1\colon \mathbb{F}_q^s \to \mathbb{F}_q$ that sends the elements of U to distinct elements, then we can extend T_1 to T by using T_1 for the first coordinate. To find T_1 pick T_1 uniformly among all linear maps. Then for every pair in U the probability of collision is $\frac{1}{q}$. So by union bounding we have the probability of success is at least $1 - \binom{|U|}{2}\frac{1}{q} > 0$, so such a map T exists. Similarly S exists. □

Fix $U \subset \mathbb{F}_q^s$ with $|U| = s$. We wish to upper bound the number of instances of U having many common neighbors. In order to do this, we will use the method of moments. Let $I(v)$ represent the indicator variable which is 1 exactly when v is a common neighbor of U and set X to be the number of common neighbors of U. Then using Lemma 2.40,

$$\mathbb{E}[X^d] = \mathbb{E}[(\sum_{v\in\mathbb{F}_q^s} I(v))^d] = \sum_{v_1,\dots,v_d\in\mathbb{F}_q^s} \mathbb{E}[I(v_1)\cdots I(v_d)]$$
$$= \sum_{r\le d}\binom{q^s}{r} q^{-rs} M_r \le \sum_{r\le d} M_r = M,$$

where M_r is defined as the number of surjections from $[d]$ to $[r]$ and $M = \sum_{r\le d} M_r$. Using Markov's inequality we get

$$\mathbb{P}(X \ge \lambda) \le \frac{\mathbb{E}[X^d]}{\lambda^d} \le \frac{M}{\lambda^d}.$$

Now even if the expectation of X is low, we cannot be certain that the probability of X being large is low. For example if we took the random graph with $p = n^{-\frac{1}{s}}$ then X will have low expectation but a long, smooth-decaying tail and therefore it is likely that X will be large for some U.

It turns out what algebraic geometry prevents the number of common neighbors X from taking arbitrary values. The common neighbors are determined by the zeros of a set of polynomial equations, and hence form an algebraic variety. The intuition is that either we are in a "zero-dimensional" case where X is very small or a "positive dimensional" case where X is at least on the order of q.

Lemma 2.41. *For all s, d there exists a constant C such that if $f_1(Y), \dots, f_s(Y)$ are polynomials on $\mathbb{F}_q^s$ of degree at most d then* Bukh (2015)

$$\{y \in \mathbb{F}_q^s | f_1(y) = \dots f_s(y) = 0\}$$

has size either at most C at least $q - C\sqrt{q}$.

The lemma can be deduced from the following important result from algebraic geometry known as the Lang–Weil bound, which says that the number of points of an r-dimensional algebraic variety in $\mathbb{F}_q^s$ is roughly q^r, as long as certain irreducibility hypotheses are satisfied.

Theorem 2.42 (Lang–Weil bound)**.** *If* $V = \{y \in \overline{\mathbb{F}}_q^s | g_1(y) = g_2(y) = \ldots = g_m(y)\}$ *is irreducible and* g_i *has degree at most* d*, then*

Lang and Weil (1954)

$$|V \cap \mathbb{F}_q^s| = q^{\dim V}(1 + O_{s,m,d}(q^{-\frac{1}{2}})).$$

Now we can use our bound from Markov's Inequality along with Lemma 2.41. Let the s polynomials $f_1(Y), \ldots, f_s(Y)$ in Lemma 2.41 be the s polynomials $f(u, Y)$ as u ranges over the s elements of U. Then for large enough q there exists a constant C from Lemma 2.41 such that having $X > C$ would imply $X \geq q - C\sqrt{q} > q/2$, so that

$$\mathbb{P}(X > C) = \mathbb{P}\left(X > \frac{q}{2}\right) \leq \frac{M}{(q/2)^d}.$$

Thus the number of subsets of L or R with size s and more than C common neighbors is at most

$$2\binom{n}{s}\frac{M}{(q/2)^d} = O(q^{s-2})$$

in expectation. Take G and remove a vertex from every such subset to create G'. First we have that G' is $K_{s,C+1}$-free. Then

$$\mathbb{E}[e(G')] \geq \frac{n^2}{q} - O(nq^{s-2}) = (1 - o(1))\frac{n^2}{q} = (1 - o(1))n^{2-\frac{1}{s}}$$

and $v(G') \leq 2n$. So there exists an instance of G' that obtains the desired bound.

2.9 *Forbidding a sparse bipartite graph*

For any bipartite graph H, it is always contained in $K_{s,t}$ for some s, t. Therefore by Theorem 2.17,

$$\text{ex}(n, H) \leq \text{ex}(n, K_{s,t}) \lesssim n^{2-\frac{1}{s}}.$$

The first inequality is not tight in general when H is some sparse bipartite graph. In this section, we will see some techniques that give a better upper bound on $\text{ex}(n, H)$ for sparse bipartite graphs H.

The first result we are going to see is an upper bound on $\text{ex}(n, H)$ when H is bipartite and the degrees of vertices in one part are bounded above.

Theorem 2.43. *Let H be a bipartite graph whose vertex set is $A \cup B$ such that every vertex in A has degree at most r. Then there exists a constant $C = C_H$ such that*

$$\mathrm{ex}(n, H) \le Cn^{2-\frac{1}{r}}$$

Füredi (1991)
Alon, Krivelevich and Sudakov (2003)

Remark 2.44. Theorem 2.32 shows that the exponent $2 - \frac{1}{r}$ is the best possible as function of r since we can take $H = K_{r,t}$ for some $t \le (r-1)! + 1$.

To show this result, we introduce the following powerful probabilistic technique called dependent random choice. The main idea of this lemma is the following: if G has many edges, then there exists a large subset U of $V(G)$ such that all small subsets of vertices in U have many common neighbors.

Lemma 2.45 (Dependent random choice)**.** *Let $u, n, r, m, t \in \mathbb{N}, \alpha > 0$ be numbers that satisfy the inequality*

Alon, Krivelevich and Sudakov (2003)

$$n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \ge u.$$

Then every graph G with n vertices and at least $\alpha n^2/2$ edges contains a subset U of vertices with size at least u such that every r-element subset S of U has at least m common neighbors.

Proof. Let T be a list of t vertices chosen uniformly at random from $V(G)$ with replacement (allowing repetition). Let A be the common neighborhood of T. The expected value of $|A|$ is

$$\begin{aligned}
\mathbb{E}|A| &= \sum_{v \in V} \mathbb{P}(v \in A) \\
&= \sum_{v \in V} \mathbb{P}(T \subseteq N(v)) \\
&= \sum_{v \in V} \left(\frac{d(v)}{n}\right)^t \\
&\ge n \left(\frac{1}{n} \sum_{v \in V} \frac{d(v)}{n}\right)^t \qquad \text{(convexity)} \\
&\ge n\alpha^t.
\end{aligned}$$

For every r-element subset S of V, the event of A containing S occurs if and only if T is contained in the common neighborhood of S, which occurs with probability

$$\left(\frac{\#\text{common neighbors of } S}{n}\right)^t.$$

Call a set S *bad* if it has less than m common neighbors. Then each bad r-element subset $S \subset V$ is contained in A with probability less than $(m/n)^t$. Therefore by linearity of expectation,

$$\mathbb{E}[\text{the number bad } r\text{-element subset of } A] < \binom{n}{r}\left(\frac{m}{n}\right)^t.$$

To make sure that there are no bad subsets, we can get rid of one element in each bad subset. The number of remaining elements is at least $|A| - (\#\text{bad } r\text{-element subset of } A)$, whose expected value is at least

$$n\alpha^t - \binom{n}{r}\left(\frac{m}{n}\right)^t \geq u.$$

Consequently, there exists a T such that there are at least u elements in A remaining after getting rid of all bad r-element subsets. The set U of the remaining u elements satisfies the desired properties. □

Setting the parameters of Lemma 2.45 to what we need for proving Theorem 2.43, we get the following corollary.

Corollary 2.46. *For any bipartite graph H with vertex set $A \cup B$ where each vertex in A has degree at most r, there exists C such that the following holds: Every graph with at least $Cn^{2-\frac{1}{r}}$ edges contains a vertex subset U with $|U| = |B|$ such that every r-element subset in U has at least $|A| + |B|$ common neighbors.*

Proof. By Lemma 2.45 with $u = |B|$, $m = |A| + |B|$, and $t = r$, it suffices to check that there exists C so that

$$n\left(2Cn^{-\frac{1}{r}}\right)^r - \binom{n}{r}\left(\frac{|A|+|B|}{n}\right)^r \geq |B|.$$

The first term evaluates to $(2C)^r$, and the second term is $O_H(1)$. Therefore we can choose C large enough to make this inequality hold. □

Now we are ready to show Theorem 2.43.

Proof of Theorem 2.43. Let G be a graph with n vertices and at least $Cn^{2-\frac{1}{r}}$ edges, where C is chosen as in Corollary 2.46. First embed B into $V(G)$ using U from Corollary 2.46. The plan is to extend this embedding furthermore to $A \cup B \hookrightarrow V(G)$. To do this, assume that we have an embedding $\phi : A' \cup B \hookrightarrow V(G)$ already where $A' \subseteq A$, and we want to extend ϕ to an arbitrary $v \in A \backslash A'$. We have to make sure that $\phi(v)$ is a common neighbor of $\phi(N(v))$ in G. Note that by assumption, $|\phi(N(v))| = |N(v)| \leq r$, and so by the choice of B, the set $\phi(N(v))$ has at least $|A| + |B|$ common neighbors. $\phi(v)$ can then be any of those common neighbors, with an exception that $\phi(v)$ cannot be the same as $\phi(u)$ for any other $u \in A' \cup B$. This eliminates $|A'| + |B| \leq |A| + |B| - 1$ possibilities for $\phi(v)$. Since there are at least $|A| + |B|$ vertices to choose from, we can just extend ϕ by setting $\phi(v)$

to be one of the remaining choices. With this process, we can extend the embedding to $A \cup B \hookrightarrow V(G)$, which shows that there is a copy of H in G. □

This is a general result that can be applied to all bipartite graphs. However, for some specific bipartite graph H, there could be room for improvement. For example, from this technique, the bound we get for C_6 is the same as C_4, which is $O(n^{3/2})$. This is nonetheless not tight.

Theorem 2.47 (Even cycles)**.** *For all integer $k \geq 2$, there exists a constant C so that*

$$\mathrm{ex}(n, C_{2k}) \leq Cn^{1+\frac{1}{k}}.$$

Bondy and Simonovits (1974)

Remark 2.48. It is known that $\mathrm{ex}(n, C_{2k}) = \Theta\left(n^{1+1/k}\right)$ for $k = 2, 3, 5$. However, it is open whether the same holds for other values of k.

Benson (1966)

Instead of this theorem, we will show a weaker result:

Theorem 2.49. *For any integer $k \geq 2$, there exists a constant C so that every graph G with n vertices and at least $Cn^{1+1/k}$ edges contains an even cycle of length at most $2k$.*

To show this theorem, we will first "clean up" the graph so that the minimum degree of the graph is large enough, and also the graph is bipartite. The following two lemmas will allow us to focus on a subgraph of G that satisfies those nice properties.

Lemma 2.50. *Let $t \in \mathbb{R}$ and G a graph with average degree $2t$. Then G contains a subgraph with minimum degree greater than r.*

Proof. We have $e(G) = v(G)t$. Removing a vertex of degree at most t cannot decrease the average degree. We can keep removing vertices of degree at most t until every vertex has degree more than t. This algorithm must terminate before reaching the empty subgraph since every graph with at most $2t$ vertices has average degree less than $2t$. The remaining subgraph when the algorithm terminates is then a subgraph whose minimum degree is more than t. □

Lemma 2.51. *Every G has a bipartite subgraph with at least $e(G)/2$ edges.*

Proof. Color every vertex with one of two colors uniformly at random. Then the expected value of non-monochromatic edges is $e(G)/2$. Hence there exists a coloring that has at least $e(G)/2$ non-monochromatic edges. □

Proof of Theorem 2.49. Suppose that G contains no even cycles of length at most $2k$. By Lemma 2.50 and Lemma 2.51 there exists a bipartite subgraph G' with minimum degree at least $\delta := Cn^{1/k}/2$.

Let $A_0 = \{u\}$ where u is an arbitrary vertex in $V(G')$. Let $A_{i+1} = N_{G'}(A_i) \setminus A_{i-1}$. Then A_i is the set of vertices that are distance exactly i away from the starting vertex u since G' is bipartite.

Now for every two different vertices v, v' in A_{i-1} for some $1 \leq i \leq k$, if they have a common neighbor w in A_i, then there are two different shortest paths from u to w. The union two distinct paths (even if they overlap) contains an even-cycle of length at most $2i \leq 2k$, which is a contradiction. Therefore the common neighbors of any two vertices in A_{i-1} can only lie in A_{i-2}, which implies that $|A_i| \geq (\delta - 1)|A_{i-1}|$. Hence $|A_k| \geq (\delta - 1)^k \geq (Cn^{1/k} - 1)^k$. If C is chosen large enough then we get $|A_k| > n$, which is a contradiction. □

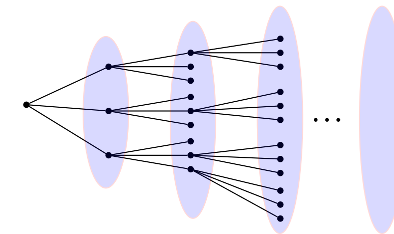


Figure 2.4: Diagram for Proof of Theorem 2.49

If H is a bipartite graph with vertex set $A \cup B$ and each vertex in A has degree at most 2, then $\mathrm{ex}(n, H) = O(n^{3/2})$. The exponent $3/2$ is optimal since $\mathrm{ex}(n, K_{2,2}) = \Theta(n^{3/2})$ and hence the same holds whenever H contains $K_{2,2}$. It turns out that this exponent can be improved whenever H does not contain any copy of $K_{2,2}$.

Colon and Lee (2019+)

Theorem 2.52. *Let H be a bipartite graph with vertex bipartition $A \cup B$ such that each vertex in A has degree at most 2, and H does not contain $K_{2,2}$. Then there exist $c, C > 0$ dependent on H such that*

$$\mathrm{ex}(n, H) \leq Cn^{\frac{3}{2} - c}.$$

To prove this theorem, we show an equivalent statement formulated using the notion of subdivisons. For a graph H, the 1-subdivision $H^{\text{1-sub}}$ of H is obtained by adding an extra vertex in the middle of every edge in H. Notice that every H in the setting of Theorem 2.52 is a subgraph of some $K_t^{\text{1-sub}}$. Therefore we can consider the following alternative formulation of Theorem 2.52.

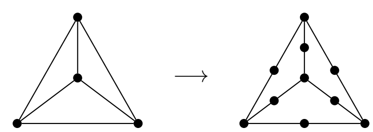


Figure 2.5: 1-subdivision of K_4

Theorem 2.53. *For all $t \geq 3$, there exists $c_t > 0$ such that*

$$\mathrm{ex}(n, K_t^{\text{1-sub}}) = O(n^{\frac{3}{2} - c_t}).$$

Janzer (2018)

Now we present a proof of Theorem 2.53 by Janzer. As in Theorem 2.49, it is helpful to pass the entire argument to a subgraph where we have a better control of the degrees of the vertices. To do so, we are going to use the following lemma (proof omitted) to find an *almost regular* subgraph.

Colon and Lee (2019+)

Lemma 2.54. *For all $0 < \alpha < 1$, there exist constants $\beta, k > 0$ such that for all $C > 0$, n sufficiently large, every n-vertex graph G with $\geq Cn^{1+\alpha}$ edges has a subgraph G' such that*

(a) $v(G') \geq n^{\beta}$,

(b) $e(G') \geq \frac{1}{10} C v(G')^{1+\alpha}$,

(c) $\max \deg(G') \leq K \min \deg(G')$,

(d) *G' is bipartite with two parts of sizes differing by factor ≤ 2.*

From now on, we treat t as a constant. For any two vertices $u, v \in A$, we say that the pair uv is *light* if the number of common neighbors of u and v is at least 1 and less than $\binom{t}{2}$; moreover, we say that the pair uv is *heavy* if the number of common neighbors of u and v is at least $\binom{t}{2}$. Note that pairs $u, v \in A$ without any common neighbors are neither light nor heavy. The following lemma gives a lower bound on the number of light pairs.

Lemma 2.55. *Let G be a $K_t^{1\text{-sub}}$-free bipartite graph with bipartition $U \cup B$, $d(x) \geq \delta$ for all $x \in U$, and $|U| \geq 4|B|t/\delta$. Then there exists $u \in U$ in $\Omega(\delta^2|U|/|B|)$ light pairs in U.*

Proof. Let S be the set of $\{(\{u,v\}, x) | u, v \in U, x \in B\}$ where $\{u, v\}$ is an unordered pair of vertices in U and x is a common neighbor of $\{u, v\}$. We can count this by choosing $x \in B$ first:

$$|S| = \sum_{x \in B} \binom{d(x)}{2} \geq |B| \binom{e(G)/|B|}{2} \geq \frac{|B|}{4} \left(\frac{\delta |U|}{|B|} \right)^2 = \frac{\delta^2 |U|^2}{4|B|}.$$

Notice that the low-degree vertices in B contributes very little since

$$\sum_{\substack{x \in B \\ d(x) < 2t}} \binom{d(x)}{2} \leq 2t^2 |B| \leq \frac{\delta^2 |U|^2}{8|B|}.$$

Therefore

$$\sum_{\substack{x \in B \\ d(x) \geq 2t}} \binom{d(x)}{2} \geq \frac{\delta^2 |U|^2}{8|B|}.$$

Note that if there are t mutually heavy vertices in U, then we can choose a common neighbor u_{ij} for every pair $\{v_i, v_j\}$ with $i < j$. Since there are at least $\binom{t}{2}$ such neighbors for each pair $\{v_i, v_j\}$, one can make choices so that all u_{ij} are distinct. This then produces a $K_t^{1\text{-sub}}$ subgraph, which is a contradiction. Therefore there do not exist t mutually heavy vertices in U, and by Turán's Theorem, the number of heavy pairs in $N(x)$ for $x \in B$ is at most $e(T_{d(x), t-1})$. Since any two vertices in $N(x)$ have at least one common neighbor x, they either form a light pair or a heavy pair. This shows that there are at least

$\binom{d(x)}{2} - e(T_{d(x),t-1})$ light pairs among $N(x)$. If $d(x) \geq 2t$, then

$$\begin{aligned}&\binom{d(x)}{2} - e(T_{d(x),t-1})\\ &\geq \binom{d(x)}{2} - \binom{t-1}{2}\left(\frac{d(x)}{t-1}\right)^2\\ &= \frac{1}{2(t-1)}d(x)^2 - \frac{1}{2}d(x)\\ &\gtrsim d(x)^2.\end{aligned}$$

If we sum over $x \in B$, then each light pair is only going to be counted for at most $\binom{t}{2}$ times according to the definition. This is constant since we view t as a constant. Therefore

$$\#\text{light pairs in } U \gtrsim \sum_{x\in B} d(x)^2 \gtrsim |S| \gtrsim \frac{\delta^2|U|^2}{|B|},$$

and by pigeon hole principle there exists a vertex $u \in U$ that is in $\Omega(\delta^2|U|/|B|)$ light pairs. □

With these lemmas, we are ready to prove Theorem 2.53.

Proof of Theorem 2.53. Let G be any $K_t^{\text{1-sub}}$-free graph. First pick G' by Lemma 2.54 with $\alpha = (t-2)/(2t-3)$, and say that the two parts are A and B. Set δ to be the minimum degree of G'. We will prove that $\delta \leq Cv(G')^{(t-2)/(2t-3)}$ for some sufficiently large constant C by contradiction. Suppose that $\delta > Cv(G')^{(t-2)/(2t-3)}$. Our plan is to pick $v_1, v_2, \ldots, v_t$ such that v_iv_j are light for all $i < j$, and no three of $v_1, \ldots, v_t$ have common neighbors. This will give us a $K_t^{\text{1-sub}}$ and hence a contradiction.

We will do so by repeatedly using Lemma 2.55 and induction on a stronger hypothesis: For each $1 \leq i \leq t$, there exists $A = U_1 \supseteq U_2 \supseteq \cdots \supseteq U_i$ and $v_j \in U_j$ such that

(a) v_j is in at least $\Theta(\delta^2|U_j|/v(G'))$ light pairs in U_j for all $1 \leq j \leq i-1$,

(b) v_j is light to all vertices in U_{j+1} for all $1 \leq j \leq i-1$.

(c) no three of $v_1, \ldots, v_i$ have common neighbors,

(d) $|U_{j+1}| \gtrsim \delta^2|U_j|/v(G')$ for all $1 \leq j \leq i-1$,

This statement clearly holds when $i = 1$ by choosing v_1 to be the vertex found by Lemma 2.55. Now suppose that we have constructed $A = U_1 \supseteq \cdots \supseteq U_{i-1}$ with $v_j \in U_j$ for all $j = 1, \ldots, i-1$. To construct U_i, let U_i' be the set of vertices that form light pairs with v_{i-1}. Then $|U_i'| \gtrsim \delta^2|U_{i-1}|/v(G')$ by the inductive hypothesis (a). Now we get rid of all the vertices in U_i' that violate (c) to get U_i. It suffices to look

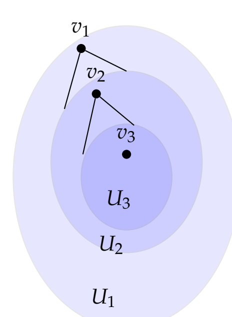


Figure 2.6: Repeatedly applying Lemma 2.55 to obtain v_i's and U_i's

at each pair $v_j v_k$, look at their common neighbors u and delete all the neighbors of u from U_i'. There are $\binom{i-1}{2}$ choices $v_j v_k$, and they have at most $\binom{t}{2}$ common neighbors since they form a light pair, and each such neighbors has degree at most $K\delta$. Therefore the number of vertices removed is at most

$$\binom{i-1}{2}\binom{t}{2}K\delta = O(\delta)$$

since t and K are constants. Therefore after this alteration, (d) will still hold as long as $|U_i'| = \Omega(\delta)$ and C is chosen sufficiently large. This is true since

$$|U_i'| \gtrsim \left(\frac{\delta^2}{V(G')}\right)^{i-1} |A| \gtrsim \delta^{2t-2} V(G')^{t-2} = \Theta(\delta)$$

given that $i \leq t$. Therefore (d) holds for i, and we just need to choose a vertex v_i from Lemma 2.55 in U_i and (a), (b), (c) follow directly. Therefore by induction, this also holds for $i = t$. Now by (b) and (c), there exists a copy of $K_t^{\text{1-sub}}$ in G', which is a contradiction.

The above argument shows that $\delta \leq Cv(G')^{(t-2)/(2t-3)}$, and so the maximum degree is at most $KCv(G')^{(t-2)/(2t-3)}$. Hence $e(G') \leq KCv(G')^{1+\alpha}$, and by the choice of G', we know that $e(G) \leq 10KCn^{1+\alpha}$, as desired. □