18.212: Algebraic Combinatorics

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Spring 2019

This class is being taught by **Professor Postnikov**.

April 22, 2019

Last time, we mentioned the directed version of the matrix tree theorem. We're going to prove this theorem today! As a reminder, here's the statement: given a directed graph G on n vertices (with no loops), $A = (a_{ij})$ is the adjacency matrix which tracks the number of edges from i to j. (This is a matrix with nonnegative integer entries, particularly zero on the diagonal.)

We also have two diagonal matrices D^{in} and D^{out}: the former is the diagonal matrix with entries equal to indegrees of the vertices, and the latter is the diagonal matrix with entries equal to outdegrees.

Then we have two Laplacian matrices:

$$L^{\text{in}} = D^{\text{in}} - A, L^{\text{out}} = D^{\text{out}} - A.$$

We also defined the **cofactors** of a matrix L

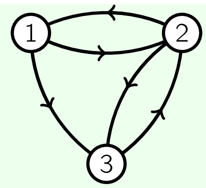
$$L^{ij} = (-1)^{i+j} \det(L \text{ without the } i\text{th row and } j\text{th column}).$$

Theorem 1 (Directed matrix tree theorem)
Fixing a root r, the number of **in-trees** rooted at r (that is, all edges point toward the root) is $(L^{\text{out}})^{rk}$, where k is an arbitrary vertex.

Meanwhile, the number of **out-trees** rooted at r (all edges growing out of the root) is $(L^{\text{in}})^{kr}$, where k is an arbitrary index.

By the way, this implies the undirected version, since in-degree and out-degrees are equal, and $L^{\text{in}} = L^{\text{out}}$ becomes the usual Laplacian matrix. Also, these two statements are obtained by reversing all directions of a graph, so we just need to prove one of them!

Example 2
Consider the graph



1 to 3, 2 edges 1 to 2, 2 to 3, 3 to 2.

Its adjacency matrix is

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and we have the Laplacian matrices

$$L^{\text{out}} = \begin{pmatrix} 3 & -2 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}, L^{\text{in}} = \begin{pmatrix} 0 & -2 & -1 \\ 0 & 3 & -1 \\ 0 & 01 & 2 \end{pmatrix}.$$

Since

$$(L^{\text{out}})^{1,1} = \det \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0,$$

there are no intrees rooted at 1 (which makes sense because no edges are entering 1 anyway). Similarly ,

$$(L^{\text{out}})^{2,1} = -\det \begin{pmatrix} -2 & -1 \\ -1 & 1 \end{pmatrix} = 3,$$

and indeed there are 3 in-trees rooted at 2 here. Finally,

$$(L^{\text{in}})^{2,1} = -\det \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix} = 5:$$

indeed there are 5 out-trees with root 1.

Let's prove this more general version: it turns out this is easier, just like in the Cayley formula proof! Again, this is doable because our induction hypothesis is stronger.

Proof. We'll do induction on the number of edges from $i \to j$ of G, where i is not the root r of our graph. In other words, this is the number of edges that don't come from the root!

Denote $\text{In}_r(G)$ to be the number of in-trees of G rooted at r. Our goal is to show that this is the cofactor $(L^{\text{out}})^{rk}$.

Our base case is where G is the graph where there are no edges out of non-root vertices: then there are no in-trees, since nothing goes into r anyway. Then L^{out} has all nonzero entries except for possibly the row corresponding to r: any cofactor $(L^{\text{out}})^{rk}$ is zero, because it must remove the rth row.

So now for the induction step: pick any edge e from i to j, where i is not the root. We construct two other graphs: G_1 is the graph G with edge e removed, and G_2 is G with all edges e' from i to j' except e (this is all edges with the same source as e).

Fact 3

We can define in-trees rooted at r to be all trees T such that the outdegree of any vertex is 1 (except the root, whose outdegree is 0).

To use the inductive step, note that

$$\text{In}_r(G) = \text{In}_r(G_1) + \text{In}_r(G_2),$$

because vertex i has to use either edge e or one of the other ones through vertex i.

Fact 4

By the way, if there's only one edge from vertex i, we can just contract it, and the proof continues to work!

Now by induction, if we look at the Laplacian matrices, the ith row of $L^{\text{out}}(G)$ has some entries $(a_1, \cdots, a_n)$. L^{out} looks almost identical, except that one edge is removed: in particular, this means in row i, we **decrease** a_i by one and **increase** a_j by one. On the other hand, $L^{\text{out}}(G_2)$ also looks identical to $L^{\text{out}}(G)$ except in row i: then we have all 0s, except a 1 in the ith column and a -1 in the jth column.

In particular, the sum of the ith rows of $L^{\text{out}}(G_1)$ and $L^{\text{out}}(G_2)$ add up to the ith row of $L^{\text{out}}(G)$. By linearity of determinants, this just means that **whenever we don't remove the *i*th row**,

$$(L^{\text{out}}(G))^{rk} = (L^{\text{out}}(G_1))^{rk} + (L^{\text{out}}(G_2))^{rk},$$

and we're done by induction, since the right hand side counts the number of in-trees for graph G_1 and G_2 separately! $\square$

But often it's better to do a combinatorial proof: for example, why do we need to use L^{out} for in-trees and vice versa? Let's see another proof based on the involution principle (which we used for Euler's pentagonal number theorem)! Remember that last time we did this, we tried to construct Young diagrams for partitions.

Slightly less general proof. Assign a weight x_{ij} to each $i \to j$: now the adjacency matrix just has entries x_{ij}, where $x_{ii} = 0$. Think of x_{ij} as multiplicity of the directed edges from i to j. Since the diagonal entry $(D^{\text{out}})_{ii}$ is supposed to count the outdegree, we define it to be

$$\sum_{j \neq i} x_{ij}.$$

Our goal is to show that the cofactors for L^{out} do count the number of in-trees! We're going to prove the directed matrix tree theorem for $k = r$ only: this will make things a bit simpler. Now our goal is to show that

$$(L^{\text{out}})^{rr} = \sum_{\text{in-trees rooted at } r} \text{weight}(T).$$

For simplicity, let's look at $n = 3, r = 3$: then our Laplacian martix is

$$\begin{pmatrix} x_{12} + x_{13} & -x_{12} & -x_{13} \\ -x_{21} & x_{21} + x_{23} & -x_{23} \\ -x_{31} & -x_{32} & x_{31} + x_{32} \end{pmatrix}.$$

Then removing the last row and last column yields a cofactor of

$$(x_{12} + x_{13})(x_{21} + x_{23}) - x_{12}x_{21} = x_{12}x_{23} + x_{13}x_{21} + x_{13}x_{23},$$

which is basically all paths we can take to make an in-tree rooted at vertex 3. We claim this works in general! We'll do this next time, but here's the main point: open up the determinant as a sum over all permutations, and we'll get collection of monomials that correspond to graphs. We can then construct a sign-reversing involution which preserves weights but changes $+$ signs to $-$ signs, so the only remaining terms are those coming from trees. $\square$

Also, as a problem in the next problem set: we can use the directed matrix tree theorem to prove lots of identities. In particular,

Proposition 5 (Abel's Identity)

We have (for any z)

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} y(y + kz)^{k-1}(x - kz)^{n-k}.$$

When $z = 0$, this is just the binomial theorem.