18.212: Algebraic Combinatorics

Andrew Lin

Spring 2019

This class is being taught by **Professor Postnikov**.

April 17, 2019

We're going to talk some more about the matrix tree theorem today. Recall that if we have a graph G with n vertices, we define the Laplacian matrix $L = (L_{ij})$ whose entries are

$$L_{ij} = \begin{cases} \deg_G(i) & i = j \\ -\text{number of edges from } i \text{ to } j & i \neq j \end{cases}$$

This can also be written as

$$L = D - A,$$

where D is the diagonal matrix of degrees and A is the adjacency matrix. Last time, we discussed the **reduced Laplacian** $\tilde{L} = L^i$, which is the Laplacian matrix without row and column i. Then the matrix-tree theorem says that the number of spanning trees in G is just the determinant of $\tilde{L}$.

There are many proofs of this: we'll go through one today, and we'll generalize the theorem later on. First, though, let's look at some more applications!

Definition 1

Define the **direct product** of two graphs

$$G = (V_1, E_1), H = (V_2, E_2) \implies G \times H = (V_1 \times V_2, E_3)$$

where the edges are of the form

$$E_3 = \{(i, j), (i', j') | i = i', (j, j') \in E_2 \text{ or } j = j', (i, i') \in E_1\}.$$

For example, the product of two line graphs is a grid graph.

Lemma 2

Let $A(G)$ be the adjacency matrix of G, and let's say it has eigenvalues $\alpha_1, \cdots, \alpha_m$. Similarly, let's say $A(H)$ is the adjacency matrix of H with eigenvalues $\beta_1, \cdots, \beta_n$. Then the adjacency matrix $A(G \times H)$ has eigenvalues $\alpha_i + \beta_j$, where $1 \leq i \leq m, 1 \leq j \leq n$.

This is an exercise in linear algebra. The idea is to notice that the adjacency matrix $A(G \times H)$ is a **tensor product**

$$A(G \times H) = A(G) \otimes I + I \otimes A(H).$$

It turns out this lemma doesn't help us very much with the matrix tree theorem, except when our graphs are regular!

Proposition 3

Let G be a d_1-regular graph and H be a d_2-regular graph. Then $G \times H$ is $d_1 + d_2$-regular, and its eigenvalues are $d_1 + d_2 - \alpha_i - \beta_j$. Then we apply the matrix tree theorem: multiply all the eigenvalues except for 0 and divide by the number of vertices.

An important application of this is the hypercube graph!

Definition 4

The **hypercube graph** H_d is a 1-skeleton of a d-dimensional cube. In other words, it's the product of d copies of a 2-vertex chain.

For example, H_3 has 8 vertices (those of a cube). Note that H_d is always a regular graph, so we can apply the proposition above! The adjacency matrix of a chain is

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \implies \lambda = \pm 1.$$

Repeatedly applying the lemma about the eigenvalues of $A(G \times H)$, the adjacency matrix of the hypercube graph $A(H_d)$ has 2^d total eigenvalues of the form

$$\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_d, \varepsilon_i \in \{-1, 1\}.$$

The smallest eigenvalue is $-d$, and the largest eigenvalue is d: specifically, there is a multiplicity $\binom{d}{k}$ of the eigenvalue $-d + 2k$. Since H_d is d-regular, the Laplacian matrix has eigenvalues $2k$ of multiplicity $\binom{d}{k}$.

So now we just apply the matrix tree theorem!

Corollary 5

The number of spanning trees of H_d is

$$\frac{1}{2^d} \prod_{k=1}^{d} (2k)^{\binom{d}{k}} = \boxed{2^{2^d - d - 1} \prod_{k=1}^{d} k^{\binom{d}{k}}}.$$

(by pulling the powers of 2 out).

For example, taking $d = 3$, the number of spanning trees is

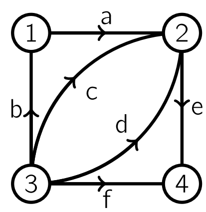
$$2^{8-3-1} \cdot 1^3 \cdot 2^3 \cdot 3^1 = 384.$$

Some time ago, Richard Stanley proposed the question of finding a combinatorial proof to this. This is pretty hard, but it was recently solved around 2012 by Bernardi. It's still open to find an explicit bijection!

By the way, if we use a grid graph (so we have a d-dimensional box), our graph is no longer regular, but if we make it a torus, we can get a similar product formula.

In the rest of this lecture, we'll start on a proof of the matrix tree theorem! Recall that we defined the incidence matrix (which is not a square matrix); we'll want to make a modification of it.

Proof. Direct our edges of G however we'd like. For example,



Let's say that G has n vertices and m edges.

Define the **oriented incidence matrix** of G to be a $n \times m$ matrix $B = (B_{ie})$ where $i \in V$ and $e \in E$:

$$b_{ie} = \begin{cases} 1 & i \text{ source of edge } e \\ -1 & i \text{ target of edge } e \\ 0 & \text{otherwise.} \end{cases}$$

For example, here we have

$$B = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

Lemma 6

The Laplacian of graph G is actually just

$$L = BB^T.$$

This is because $(BB^T)_{ij}$ is the dot product of the ith and jth rows of B. If $i = j$, the dot product will just count the number of nonzero entries, which is exactly the degree of that corresponding vertex! On the other hand, when $i \neq j$, we get a -1 each time an edge goes from i to j or vice versa. That's why the orientation is arbitrary!

So if $\tilde{B}$ is the oriented incidence matrix without row i,

$$\tilde{L} = \tilde{B}\tilde{B}^T,$$

by plugging into the definition. Well, we can use the following fact about determinants of non-square matrices:

Theorem 7 (Cauchy-Binet formula)

Let A be a $k \times m$ matrix and B be an $m \times k$ matrix, where $k \le m$. Then the determinant of AB is

$$\sum_{\substack{S \subseteq \{1,2,\cdots,m\} \\ |S| = k}} \det(A^S)\det(B^S),$$

where A^S is the $k \times k$ submatrix of A with columns in S and B^S is the $k \times k$ submatrix of B with rows in S.

The best way now is to think of the following product in block form:

$$\begin{pmatrix} I_k & A \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & 0 \\ -I_m & B \end{pmatrix} = \begin{pmatrix} 0 & AB \\ -I_m & B \end{pmatrix}.$$

Now we can just take determinants of the square matrices! The determinant of the first one is 1 (because it is an upper triangular matrix with diagonal entries 1), and the determinant of the right matrix is $\pm \det(AB)$. Now we claim that the determinant of the middle matrix is exactly the right hand side of Cauchy-Binet (up to a $\pm$ sign which we'll figure out)! The main idea here is by example: apply the usual product over all permutations of rows. For example, consider

$k = 2$, $m = 4$:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} & 0 & 0 \\ -1 & 0 & 0 & 0 & b_{11} & b_{12} \\ 0 & -1 & 0 & 0 & b_{21} & b_{22} \\ 0 & 0 & -1 & 0 & b_{31} & b_{32} \\ 0 & 0 & 0 & -1 & b_{41} & b_{42} \end{pmatrix}.$$

The determinant of this matrix must take some permutation: we want to place rooks on this shape so that we have exactly 1 rook on each row and each column. In particular, we should have one rook in each of the last two columns, and then we must pick the top and bottom -1, and then we finish by picking some square 2×2 matrix in the top 2 rows. Also all the signs magically work out. $\square$

We're almost done, and we'll continue this proof next time!