18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Hopefully all of us had a good spring break! Let's quickly review material from last class.

We started talking about **partition theory** last time: letting $p(n)$ be the number of partitions of n, we can write a generating function

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

Last time, we started discussing the reciprocal of this quantity: let $f(q) = (1-q)(1-q^2)\cdots$.

Theorem 1 (Euler's Pentagonal Number Theorem, conjectured 1741, proved 1750)
We have the following form for $f(q)$:

$$f(q) = \sum_{m=-\infty}^{\infty} (-1)^m q^{m(3m-1)/2}.$$

A few terms of this infinite sum are

$$1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} \cdots.$$

This is surprising because we expect that there's a lot of different possible coefficients in the infinite product. But it turns out that a lot of terms end up having complete cancellation, and a few others have ± 1s! Here are a few pentagonal numbers, by the way:

m	1	2	3	4
$\frac{m(3m-1)}{2}$	1	5	12	22

So we get these values, but also contributions from negative values of m.

Proposition 2 (Gauss)
We also have

$$(f(q))^3 = \sum_{m=-\infty}^{\infty} (-1)^m \cdot m \cdot q^{m(m+1)/2}.$$

Here the coefficients are no longer ± 1, but they're pretty simple. Also, we still have pretty sparsely populated coefficients!

Fact 3

Unfortunately, $f(q)^2$ is a total mess. There isn't quite a simple expression for $f(q)^2$, and there's a deep representation theory reason for that!

Remember that we also discussed different ways to represent our partition: we can either write it as

$$\lambda = (\lambda_1, \lambda_2, \cdots)$$

in non-increasing order, or as

$$\lambda = (1^{m_1} 2^{m_2} \cdots)$$

as multiplicities, where m_i is the number of parts of λ equal to i. Then we can expand out our product:

$$(1-q)(1-q^2)\cdots = \sum_{\substack{m_1, m_2, \cdots \\ q_i \in \{0,1\}}} (-1)^{\sum m_i} q^{m_1 + 2m_2 + 3m_3 + \cdots}$$

where m_i being 0 corresponds to picking 1 in the product, and m_i being 1 corresponds to picking q^{m_i}. This can be written in another way: since all m_is are 0 or 1, we only pick each number i at most once. So this counts partitions with distinct parts:

$$\sum_{\substack{\text{partitions } \lambda \\ \text{with distinct parts}}} (-1)^{\text{parts in } \lambda} q^{|\lambda|}.$$

This lets us write Euler's pentagonal number theorem slightly differently:

Theorem 4

Let $p_{\text{dist}}^{\text{even}}(n)$ (resp. $p_{\text{dist}}^{\text{odd}}(n)$) be the partitions of n with distinct parts and an even (resp. odd) number of parts. Then

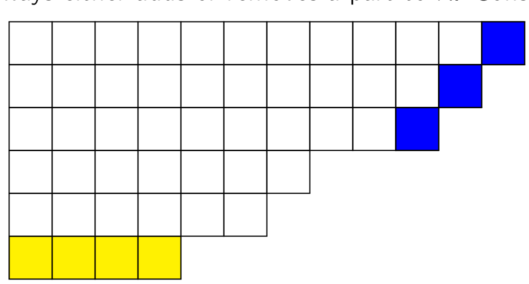
$$p_{\text{dist}}^{\text{even}}(n) - p_{\text{dist}}^{\text{odd}}(n) = \begin{cases} (-1)^n & n = \frac{m(3m-1)}{2} \\ 0 & \text{otherwise} \end{cases}.$$

To prove this, we want to somehow set up a matching between partitions with an even and odd number of parts, and (almost always) perfectly pair them! This is related to the **involution principle**!

Proof by Franklin, 1881. An involution is a function f whose square is the identity (that is, it is its own inverse). Our goal is to find an involution σ on almost all partitions of n with distinct parts, such that σ sends a partition with an odd number of parts to an even number of parts and vice versa. Rigorously, if $\mu = \sigma(\lambda)$, μ and λ should have different parities of the number of parts.

We'll try to construct σ so that it always either adds or removes a part to λ. Consider the shape

$\lambda =$



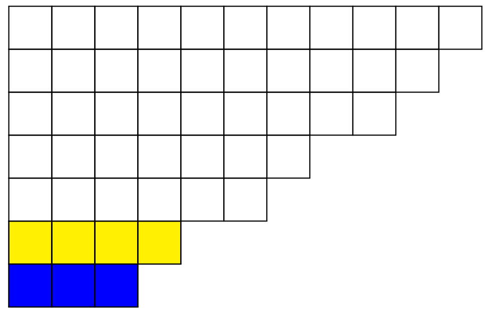
Denote the yellow part, which is the last row, as A, and denote the blue part, which is the longest diagonal segment starting from the top right corner, as B. Denote $a = |A|$ and $b = |B|$ (in this case, $a = 4, b = 3$).

Fact 5

Remember that there's nothing beyond B, because our partition has distinct parts.

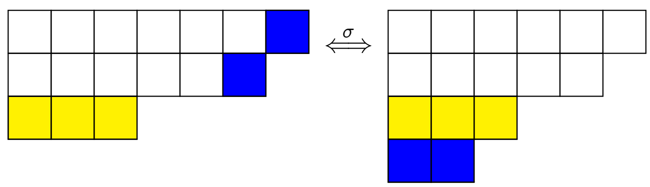
Here's how we'll construct σ:

- If $a > b$, like in this case, we remove the diagonal segment B and adding it as a new row with b boxes:

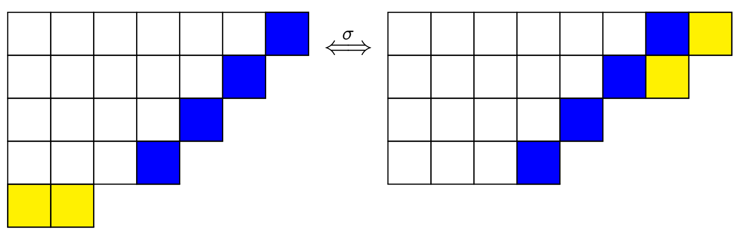
$\lambda' =$ 

- If $a \le b$, remove the last row and add a new diagonal segment! For example, take λ' and adjust it back to λ.

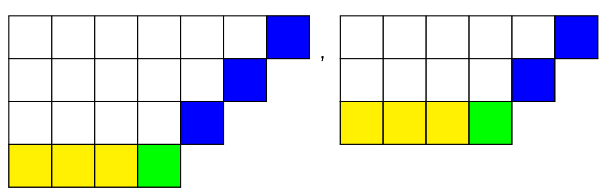
Here's a few more examples: for $\lambda = (7, 6, 3)$,



and for $\lambda = (7, 6, 5, 4, 2)$,



But we have to be a little bit careful: we're supposed to have some special cases! We have a problem if the yellow and blue segments, A and B, overlap. Our operation actually still works except when $a = b$ or $a = b + 1$! For example, consider $a = b = 4$ or $a = 4$, $b = 3$:



So we can call these the "pentagonal cases!" These turn out to be exactly the $\frac{m(3m-1)}{2}$ terms, as desired. $\square$

Theorem 6 (Jacobi's Triple Product Identity, 1829)

The infinite product

$$\prod_{n\ge 1}(1-q^{2n})(1+q^{2n-1}z)(1+q^{2n-1}z^{-1}) = \sum_{r=-\infty}^{\infty} q^{r^2}z^r.$$

This identity has many special cases!

Corollary 7

If we take $z = -x^{1/2}$ and $q = x^{3/2}$, the left hand side becomes the left side of Euler's theorem, and this yields Euler's pentagonal formula.

Also, if we take $z = -x$, $q = x^{1/2}$, we get Gauss' formula for $(f(q))^3$.

Finally, if we plug in $z = -1$, we find that

$$\prod_{m \geq 1} \frac{1 - q^m}{1 + q^m} = \sum_{r=-\infty}^{\infty} (-1)^r q^{r^2}.$$

Proof sketch. Substitute $q \to q^{1/2}$ (so the first term has all even powers) and $z \to qz$. Then the triple product identity is equivalently written as (moving the $\prod(1 - q^{2n})$ to the other side)

$$\prod_{n \geq 1} (1 + zq^n)(1 + z^{-1}q^{n-1}) = \left(\sum_{r=-\infty}^{\infty} z^r q^{r(r+1)/2} \right) \cdot \frac{1}{\prod_{n \geq 1}(1 - q^n)}.$$

Let's try to interpret this combinatorially! The first term on the left side counts partitions with distinct parts, with z keeping track of the number of parts. So the coefficient of z^a for the first part of the left hand side is the generating function

$$\sum_{\substack{\mu \text{ partition with} \\ a \text{ distinct parts}}} q^{|\lambda|},$$

and the coefficient of z^{-b} for the second part is

$$\sum_{\substack{\nu \text{ partition with} \\ b \text{ distinct parts}}} q^{|\lambda| - b},$$

since there's a q^{n-1} in the product.

So somehow we want to relate μ and ν to all partitions λ! We'll go over this next time. □