18.212: Algebraic Combinatorics

Andrew Lin

Spring 2019

This class is being taught by **Professor Postnikov**.

March 20, 2019

Recall that a **differential poset** is ranked with a minimal element $\hat{0}$, and we define **up** and **down** operators such that $[D, U] = DU - UD = I$, where I is the identity operator. Combinatorially, this is equivalent to the following two conditions:

- For any $x \neq y$ in the same level, if there are a elements that cover both x and y, there are a elements covered by x and y.
- For any x, there is one more element covering x than elements covered by x.

Fact 1
We found an example last time: Young's lattice is a differential poset.

Are there other differential posets?

We know we must start with a unique minimal element $\hat{0}$. There must be exactly 1 element above $\hat{0}$, and then 2 elements above that. Now, for the two elements x and y we just constructed, we need one element to cover both x and y, and then each of x and y need to add another extra edge.

Let's see how we could construct the next ($n + 1$th) level inductively: to satisfy the first condition of a differential poset, reflect over the previous level's edges. In other words, take the bipartite graph between the nth and $n - 1$th levels, and flip it over the nth level. (This is not the only way, but it works.)

Now add in an extra edge to a new vertex for each one to satisfy the second condition. in other words, add an edge above each element x on the nth level.

What's the rank number - that is, how many vertices r_i do we have? The numbers are $1, 1, 2, 3, 5, 8, \cdots$: these are Fibonacci numbers by construction, since we add one vertex from each of the $n - 1$th level, and then add another vertex from the nth level.

Theorem 2
This poset is a lattice! So any two elements have a well-defined meet and join.

This is called the **Fibonacci lattice.** Let's compare this to Young's lattice: the rank numbers there are $1, 1, 2, 3, 5, 7$, so from that point on, the numbers will be different. However, they have very similar properties! If we find the number of walks from $\hat{0}$ up n steps and then down n steps, it turns out it is $n!$ as well (since we only care about the identity $DU - UD = I$.

Theorem 3

For any differential poset,

$$D^n U^n \hat{0} = n!\hat{0}.$$

This follows from induction. But where is the $n!$ actually coming from? If we look at

$$DD \cdots DDUU \cdots UU\hat{0}.$$

We can think of them as particles and antiparticles. Each pair can collide or go past each other! Each D must collide with one of the Us, because $D\hat{0}$ gives zero contribution. So each D is matched with a U, and that means we really have a permutation of n Ds!

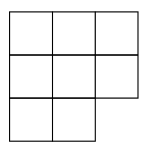
In general, we can pick any word with n U's and n D's:

$$DDUDUU\hat{0} = c\hat{0}.$$

What is this number? This counts the number of paths that go up, up, down, up, down, down. In this case, we have a total of 4 paths, but is there a way to find c?

We have to match all Ds with Us, where each D matches with a U to its right, and our goal is to find the total number of matchings.

Equivalently, we can describe this by a rook placement! First trace out a path from the bottom left to top right of an $n \times n$ board by going right when we see a D and going up when we see a U, and draw the Young diagram: for example, $DDUDUU$ gives



and any rook placement gives us a valid annihilation! Specifically, we get a Young diagram of form $\nu = (\nu_1, \cdots, \nu_n)$, where the ith row is the number of Ds that appear before the $n+1-i$th U in our sequence. Then we just match the corresponding Us and Ds in our rooks.

Theorem 4

The constant c in the equation

$$W\hat{0} = c\hat{0},$$

where W is a word of n Ds and n Us, is the number of placement of n non-attacking rooks in the shape ν. Specifically, this is

$$\nu_n(\nu_{n-1} - 1)(\nu_{n-2} - 2) \cdots (\nu_1 - n + 1).$$

Proof. Just start by placing a rook on the bottom row! There's ν_n choices. Then we can place a rook on the next row in any of the remaining columns in $\nu_{n-1} - 1$ ways, and so on. □

So let's go back to the unimodality of Gaussian coefficients. Recall the Gaussian coefficients a_n in the equation

$$\begin{bmatrix} k+l \\ l \end{bmatrix}_q = \sum_{n=0}^{kl} a_n q^n.$$

We found by the Young diagram complement idea that these are symmetric. The following was first formulated by Cayley in 1856 and proved by Sylvester in 1878:

Theorem 5 (Unimodality of the Gaussian coefficients)

We have $a_0 \leq a_1 \leq \cdots \leq a_{\lfloor \frac{kl}{2} \rfloor} \geq \cdots \geq a_{kl}$.

Proof. Let P be the product poset $[k] \times [l]$. Then the lattice of order ideals of P (as we know) is the lattice of Young diagrams that fit inside a k by l rectangle

$$J(P) \sim L(k, l).$$

Denote $J(P)_n$ to be all Young diagrams in $J(P)$ on the nth level (with n boxes). Our goal is to show that the number of elements in $J(P)_{n+1} \geq J(P)_n$ as long as $n < \frac{kl}{2}$.

For $\lambda \subset J(P)_n$, let us denote Add(λ) to be all boxes $x \in P$ such that $\lambda \cup \{x\} \gtrdot \lambda$. Similarly, define Remove(λ) to be all boxes $y \in P$ such that $\lambda \setminus \{y\} \lessdot \lambda$. Remember that in Young's lattice, the number of addable boxes is always one more than the number of removable boxes: this is no longer true because we can't add boxes outside of our $k \times l$ rectangle.

Here's a key fact:

Lemma 6

Fix n, k, l. Suppose we have a function $w : P \to \mathbb{R}_{>0}$ such that for any $\lambda \in J(P)_n$, we have

$$\sum_{x \in \text{Add}(\lambda)} w(x) > \sum_{y \in \text{Remove}(\lambda)} w(y).$$

Then the number of elements in $J(P)_{n+1}$ is at least the number of elements in $J(P)_n$.

Here's an example of a weight function:

12	12	10	6
10	12	12	10
6	10	12	12

We can check that this works for all $n < 6$. Let's prove the lemma, but we'll need some more general up and down operators: consider the **weighted** up and down operators

$$U : \lambda \to \sum_{\substack{\mu = \lambda \cup \{x\}, \\ x \in \text{Add}(\lambda)}} \sqrt{w(x)} \cdot \mu$$

and

$$D : \lambda \to \sum_{\substack{\mu = \lambda \setminus \{y\}, \\ y \in \text{Remove}(\lambda)}} \sqrt{w(y)} \cdot \mu.$$

We can think of these as linear operators: it is clear that $D = U^T$ are transpose matrices.

Claim 6.1. $H = [D, U] = DU - UD$ is a diagonal matrix. It sends H from λ to

$$\left(\sum_{x \in \text{Add}(\lambda)} w(x) - \sum_{y \in \text{Remove}(\lambda)} w(y) \right) \lambda.$$

It's diagonal because (as before), adding a box and removing another box can be done in either order. On the other hand, if we want to get back to ourself, we can add any box and remove it, which gives $\sqrt{w(x)}^2 = w(x)$, or remove any box and add it back, which gives $w(y)$.

What can we do with this? Here's a property of linear algebra: since $D = U^T$,

$$DU = UD + H = UU^T + H.$$

UU^T is positive semi-definite, and H is a positive definite matrix, so this means DU must be positive definite! Therefore, DU has positive determinant. We'll see the rest of the details in the next lecture! $\square$

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