18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Last time, we discussed a duality between chains and antichains of a poset: the Young diagrams formed were transposes of each other. Prior to that, we had discussed the Schensted correspondence: every permutation $w \in S_n$ can be sent to pairs (P, Q) SYTs of shape λ, where the rows and columns tell you about increasing and decreasing subsequences.

Is there a relationship between these two ideas?

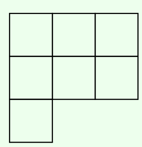
Given any $w \in S_n$, we can construct a poset $P = \{1, 2, \cdots, n\}, <_p\}$, where our relation is

$$i <_p j \iff i < j \text{ and } w_i < w_j.$$

Here the $<$ on the right hand sides are standard "less than." So $i \le_p j$ if we don't have an inversion between (i, j).

Example 1

Let's take $w = (3, 5, 2, 4, 7, 1, 6)$. The Schensted correspondence gives shape



The Hasse diagram for the associated poset then has $1 <_p 6, 3 <_p 5 <_p 6, 4 <_p 6, 5 <_p 7, 3 <_p 4 <_p 7, 2 <_p 7$. Notice that increasing subsequences correspond to chains, and decreasing subsequences correspond to antichains!

So Greene's construction gives chain "numbers" of $(3, 6-3, 7-1) = (3, 3, 1)$, and antichain numbers of $(3, 5-3, 7-5) = (3, 2, 2)$, as we expect.

Remember that we used the Schensted correspondence to prove

$$\sum_{\lambda : |\lambda| = n} (f^\lambda)^2 = n!,$$

where f^λ is the number of Standard Young Tableaux of a shape λ. Today, we're going to do a simpler proof that is more general!

Let's look at Young's lattice $\mathbb{Y}$, which is isomorphic to $J(\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0})$, the lattice of order ideals in a quadrant. Denote $\mathbb{Y}_n$ to be the set of all Young diagrams with exactly n boxes: it's the nth level.

Let $p(n) = |\mathbb{Y}_n|$, which is the number of ways we can write n as a sum (disregarding order), these are called the **partition numbers**.

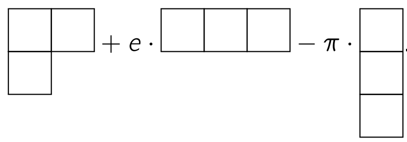
Proposition 2

A Standard Young Tableau of shape λ is a path or saturated chain in the Hasse diagram of $\mathbb{Y}$ from the empty shape to λ.

This is because we add on a square each time we go up: place a 1 in the first box you add, then place a 2 in the next one, and so on!

Meanwhile, in the Frobenius-Young identity and in the Schensted correspondence, we have pairs of Young tableaux, so we have two paths from the empty Young diagram to λ. So let's reverse the second path: **let (P, Q) be a path in $\mathbb{Y}$ from the empty shape back to itself with n up steps, followed by n down steps.** We just want to count how many ways we can do this!

Let's formalize this algebraically. Let $\mathbb{R}[\mathbb{Y}_n]$ be the linear space of formal linear combinations of Young diagrams with n boxes: for example, an element could be



So $\mathbb{R}[\mathbb{Y}_n]$ is isomorphic to $\mathbb{R}^{p(n)}$, and

$$\mathbb{R}[\mathbb{Y}] = \bigoplus_{n \geq 0} \mathbb{R}[\mathbb{Y}_n].$$

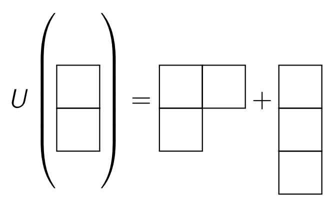
Define **up** and **down** operators: the up operator sends λ to

$$\lambda \to \sum_{\mu : \lambda \lessdot \mu} \mu.$$

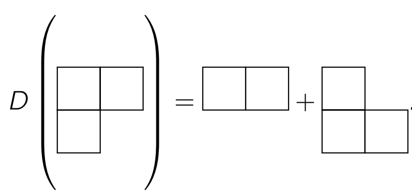
In other words, μ has to be λ with a single box added. Similarly, the down operator sends μ to

$$\mu \to \sum_{\lambda : \lambda \lessdot \mu} \lambda.$$

For example,



and



So the number of (P, Q) of shape λ is the coefficient of $\varnothing$ (the bottom element in the poset, which is not zero!) in $D^n U^n \varnothing$. We're trying to show that this is $n!$.

Proposition 3 (Key identity)

We have

$$[D, U] = DU - UD = I,$$

where I is the identity operator.

Why is this true? Consider the coefficient of μ in $(DU - UD)\lambda$. If $\lambda \neq \mu$, and we want to get from λ to μ, we have to add a box a and remove a box r, and these can be done in either order! (This is because $a \neq r$, or else λ would be equal to μ.)

Meanwhile, if $\lambda = \mu$, so we add a box and remove that same box, or we can remove a box and add that same box back. The number of boxes we can remove is the number of inner corners, but the number of boxes we can add is the number of outer corners! This is always a difference of 1, so we get a coefficient of 1 as desired.

Definition 4 (Stanley)
A **differential poset** is a ranked (infinite) poset with a unique minimally ranked element $\hat{0}$ such that we can define up and down operators in the same way as we have done for $\mathbb{Y}$:

$$[U, D] = I.$$

Combinatorially, this means that for any x, y not equal to each other on the same level, there are a ways to go up and then down from x to y, and also a ways to go down and then up from x to y. Another way is to say that

$$\#\{u : u \gtrdot x \& u \gtrdot y\} = \#\{v : v \lessdot x \& v \lessdot y\}$$

On the other hand, for any individual element x:

$$\#\{u : u \gtrdot x\} = 1 + \#\{v : v \lessdot x\}$$

Fact 5
These are called differential posets, because we can think of u as multiplying a polynomial $f(x)$ by x, and d as taking its derivative. Then

$$(du - ud)f(x) = f(x) \implies (xf(x))' - xf'(x),$$

which is true.

It turns out that this alone is enough to prove the Frobenius-Young identity!

Theorem 6 (Stanley)
For any differential poset,

$$D^n U^n \hat{0} = n! \hat{0}.$$

Proof. First, we use the following lemma, which can be proved by induction:

Lemma 7
$DU^n = nU^{n-1} + U^n D$.

So now,

$$D^n U^n (\hat{0}) = D^{n-1}(DU^n)\hat{0}$$

Applying the lemma, this is equal to

$$D^{n-1}\left(nU^{n-1} + U^n D\right)\hat{0}$$

and now $U^n D\hat{0}$ gives us nothing, since $D\hat{0} = 0$. So we just have

$$D^{n-1} nU^{n-1}\hat{0} = nD^{n-1}U^{n-1}\hat{0} = n(n-1)!\hat{0}$$

by induction!

□

One more question: Is this the only differential poset, or are there others? We will see!

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