18.212: Algebraic Combinatorics

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This class is being taught by **Professor Postnikov**.

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Remember from last lecture: if we have a poset P, then $J(P)$ is the poset of order ideals in P, ordered by inclusion. (Order ideals are closed downward.)

Lemma 1

$J(P)$ is a distributive lattice.

Proof. If we have two order ideals I and J, we can define the meet operation $I \wedge J = I \cap J$: it can be checked that the intersection of two order ideals is an order ideal. Similarly, we define $I \vee J = I \cup J$, and everything works.

Since the usual union and intersection satisfy the distributive laws, this is also a distributive lattice. □

Remember from last time that the converse is also true:

Theorem 2 (Fundamental Theorem for Finite Distributive Lattices)

Every finite distributive lattice is of the form $J(P)$ for some poset P. In particular, $P \to J(P)$ is a bijection between finite posets and finite distributive laws.

Main sketch of the proof. We already know how to get from $P \to J(P)$: our goal is to reconstruct a poset from a finite distributive lattice L.

Definition 3

An element $x \in L$ is **join-irreducible** if it is not the minimal element of L [which exists] and we cannot express it as $x = y \vee z$ for $y, z < x$ ($\leq$ and not equal).

For example, if we take the lattice above, everything except the bottom and top element is join-irreducible.

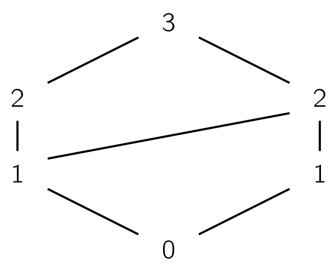
But it turns out we can just construct P to be the subposet of L of all join-irreducible elements! It's an exercise to show that $L \cong J(P)$. □

Definition 4

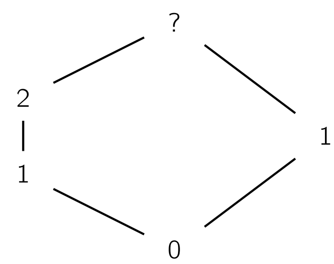
A poset P is **ranked** if there is a $\rho : P \to \{0, 1, 2, \cdots\}$ such that $\rho(x) = 0$ for any minimal elements x of P and

$\rho(y) = \rho(x) + 1$ if $x \lessdot y$.

So far, all the posets that we've been discussing have been ranked: the idea is that all elements live on different "levels." In the diagram below, the left poset is ranked, but the right poset is not.



ranked



not ranked

Proposition 5

Any finite distributive lattice is ranked with the **modularity property**: $\rho(x \vee y) + \rho(x \wedge y) = \rho(x) + \rho(y)$.

Proof. If $L = J(P)$ is a finite distributive lattice, and $I \in L$ is an order ideal of P, then the rank is just $\rho(I) = |I|$. So our goal is to show that $I \lessdot J$, they only differ in one element: this is true because $I < J$ means I is strictly contained in J. ☐

Definition 6

Let P be a finite ranked poset. Define r_i to be the number of elements in P with rank i: call this a **rank number** of P. These rank numbers form a vector $(r_0, r_1, \cdots, r_N)$, where N is the maximal rank of any element, and P is **rank-symmetric** if $r_i = r_{N-i}$ for all i. If $r_0 \le r_1 \le \cdots \le r_j \ge r_{j+1} \ge \cdots \ge r_N$, then P is **unimodal**.

In the most recent example on the left, we have rank numbers $(1, 2, 2, 1)$, so the poset is both unimodal and rank-symmetric.

Definition 7

Let P and Q be two posets. Define their product $P \times Q$ to be the poset whose elements are pairs $(p, q), p \in P, q \in Q$. The order relation

$$(p, q) \le (p', q') \iff p \le p', q \le q'.$$

Notice that not all pairs are comparable.

Example 8

Let $[n]$ denote the poset whose Hasse diagram is just an increasing chain of n elements. Clearly this is rank-symmetric and unimodular.

Example 9

What does $[m] \times [n]$ look like? We have a grid, but we have to rotate it by 45 degrees. Then (m, n) has rank $m + n$.

The rank numbers look like $(1, 2, \cdots, k-1, k, \cdots, k, k-1, \cdots, 2, 1)$, where k is the minimum of m and n. This is also rank-symmetric and unimodular.

What happens if we apply J to our posets? $J([n])$ is just $[n+1]$, but do we get something more interesting if we look at

$$J([m] \times [n])?$$

Well, every order ideal corresponds to a Young diagram if we replace elements with boxes! So order ideals are exactly the Young diagrams that fit inside our grid, meaning that

$$L(m, n) = J([m] \times [n])$$

is the poset of Young diagrams that fit in an $m \times n$ rectangle. But we know the rank numbers r_i for $J([m] \times [n])$ must be the same as the number of Young diagrams with i squares! So these are the Gaussian coefficients

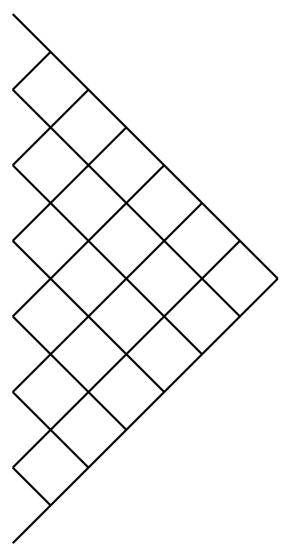
$$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = r_0 + r_1 q + \cdots + r_{mn} q^{mn}.$$

This leads to the following (after some work):

Theorem 10

$L(m, n)$ is rank-symmetric and unimodular.

Let's go back to looking at some examples! What is $J([2] \times [n])$? We want the Young diagrams that fit inside a $2 \times n$ rectangle. This Hasse diagram looks like a triangle:



But let's look at the order ideals of this triangular Hasse diagram: what is

$$J(J([2] \times [n]))?$$

Notice that the order ideals here correspond to shifted Young diagrams! So $J(J([2] \times [n]))$ is the poset of shifted Young diagrams, ordered by inclusion, which fit inside a shifted Young diagram with $n, n-1, \cdots, 1$ boxes in the first n rows.

It's an exercise to see that this is also rank-symmetric and unimodular!

Theorem 11 (Sperner's theorem, 1928)

Let $S_1, \cdots, S_M$ be different subsets of $\{1, 2, \cdots, n\}$, such that for all i, j, $S_i \not\subseteq S_j$. Then $M \leq \binom{n}{n/2}$.

How is this related to posets? A **chain** C in a poset P is a set of elements such that any two elements are "compatible:" one is contained in the other, so we have a total ordering of C. An **antichain** A in P is the opposite: no two elements are compatible.

Definition 12

Let P be a finite ranked poset with rank numbers $r_0, r_1, \cdots, r_N$. P is **Sperner** if M, the maximal size of an antichain in P, is $\max(r_0, \cdots, r_N)$.

It's clear that M should be at least the maximum of $r_0, \cdots, r_N$: just take all elements with some fixed rank. If this is an equality, we have a Sperner poset.

Sperner's theorem says that the Boolean lattice is Sperner: we're just looking at the central binomial coefficient, which is the maximal rank number of the Boolean lattice! We'll do this proof next time, but here's an example of a non-Sperner poset, because the rank numbers are $3, 3$ but there is an antichain of length 4: