Exam 1 Practice Questions I –solutions, 18.05, Spring 2014

Note: This is a set of practice problems for exam 1. The actual exam will be much shorter.

1. Sort the letters: A BB II L O P R T Y. There are 11 letters in all. We build arrangements by starting with 11 'slots' and placing the letters in these slots, e.g

$$\underline{\ A\ }\,\underline{\ B\ }\,\underline{\ I\ }\,\underline{\ B\ }\,\underline{\ I\ }\,\underline{\ L\ }\,\underline{\ O\ }\,\underline{\ P\ }\,\underline{\ R\ }\,\underline{\ T\ }\,\underline{\ Y\ }$$

Create an arrangement in stages and count the number of possibilities at each stage:

Stage 1: Choose one of the 11 slots to put the A: $\binom{11}{1}$

Stage 2: Choose two of the remaining 10 slots to put the B's: $\binom{10}{2}$

Stage 3: Choose two of the remaining 8 slots to put the B's: $\binom{8}{2}$

Stage 4: Choose one of the remaining 6 slots to put the L: $\binom{6}{1}$

Stage 5: Choose one of the remaining 5 slots to put the O: $\binom{5}{1}$

Stage 6: Choose one of the remaining 4 slots to put the P: $\binom{4}{1}$

Stage 7: Choose one of the remaining 3 slots to put the R: $\binom{3}{1}$

Stage 8: Choose one of the remaining 2 slots to put the T: $\binom{2}{1}$

Stage 9: Use the last slot for the Y: $\binom{1}{1}$

Number of arrangements:

$$\binom{11}{1}\binom{10}{2}\binom{8}{2}\binom{6}{1}\binom{5}{1}\binom{4}{1}\binom{3}{1}\binom{2}{1}\binom{1}{1} = 11 \cdot \frac{10 \cdot 9}{2} \cdot \frac{8 \cdot 7}{2} \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 9979200$$

Note: choosing 11 out of 1 is so simple we could have immediately written 11 instead of belaboring the issue by writing $\binom{11}{1}$. We wrote it this way to show one systematic way to think about problems like this.

2. Build the pairings in stages and count the ways to build each stage:

Stage 1: Choose the 4 men: $\binom{6}{4}$.

Stage 2: Choose the 4 women: $\binom{7}{4}$

We need to be careful because we don't want to build the same 4 couples in multiple ways.
Line up the 4 men M_1, M_2, M_3, M_4
Stage 3: Choose a partner from the 4 women for M_1: 4.
Stage 4: Choose a partner from the remaining 3 women for M_2: 3
Stage 5: Choose a partner from the remaining 2 women for M_3: 2
Stage 6: Pair the last women with M_4: 1

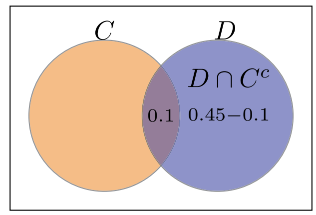
Number of possible pairings: $\binom{6}{4}\binom{7}{4}4!$.

Note: we could have done stages 3-6 in on go as: Stages 3-6: Arrange the 4 women opposite the 4 men: 4! ways.

3. We are given $P(A^c \cap B^c) = 2/3$ and asked to find $P(A \cup B)$.

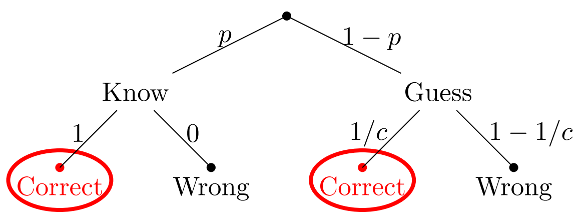
$A^c \cap B^c = (A \cup B)^c \;\Rightarrow\; P(A \cup B) = 1 - P(A^c \cap B^c) = \boxed{1/3.}$

4. D is the disjoint union of $D \cap C$ and $D \cap C^c$.



So, $P(D \cap C) + P(D \cap C^c) = P(D)$
$\Rightarrow\; P(D \cap C^c) = P(D) - P(D \cap C) = 0.45 - 0.1 = \boxed{0.35.}$
(We never use $P(C) = 0.25$.)

5. The following tree shows the setting



Let C be the event that you answer the question correctly. Let K be the event that you actually know the answer. The left circled node shows $P(K \cap C) = p$. Both circled nodes together show $P(C) = p + (1-p)/c$. So,

$$P(K|C) = \frac{P(K \cap C)}{P(C)} = \frac{p}{p + (1-p)/c}$$

Or we could use the algebraic form of Bayes theorem and the law of total probability: Let G stand for the event that you're guessing. Then we have,
$P(C|K) = 1$, $P(K) = p$, $P(C) = P(C|K)P(K) + P(C|G)P(G) = p + (1-p)/c$. So,

$$P(K|C) = \frac{P(C|K)P(K)}{P(C)} = \frac{p}{p + (1-p)/c}$$

6. Sample space =

$$\Omega = \{(1,1), (1,2), (1,3), \ldots, (6,6)\} = \{(i,j) \,|\, i,j = 1, 2, 3, 4, 5, 6\}.$$

(Each outcome is equally likely, with probability 1/36.)

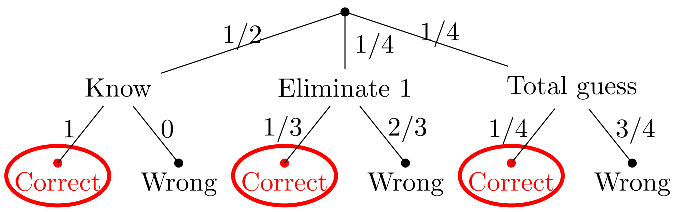
$A = \{(1,2), (2,1)\}$,
$B = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$
$C = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,1), (3,1), (4,1), (5,1), (6,1)\}$

(a) $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{2/36}{11/36} = \frac{2}{11}$..

(a) $P(B|C) = \dfrac{P(B \cap C)}{P(C)} = \dfrac{2/36}{11/36} = \dfrac{2}{11}..$

(c) $P(A) = 2/36 \neq P(A|C)$, so they are not independent. Similarly, $P(B) = 6/36 \neq P(B|C)$, so they are not independent.

7. We show the probabilities in a tree:



For a given problem let C be the event the student gets the problem correct and K the event the student knows the answer.

The question asks for $P(K|C)$.

We'll compute this using Bayes' rule:

$$P(K|C) \;=\; \frac{P(C|K)\,P(K)}{P(C)} \;=\; \frac{1 \cdot 1/2}{1/2 + 1/12 + 1/16} \;=\; \frac{24}{31} \;\approx\; 0.774 \;=\; 77.4\%$$

8. We have $P(A \cup B) = 1 - 0.42 = 0.58$ and we know because of the inclusion-exclusion principle that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Thus,

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.3 - 0.58 = 0.12 = (0.4)(0.3) = P(A)P(B)$$

So A and B are independent.

9. We will make use of the formula $\text{Var}(Y) = E(Y^2) - E(Y)^2$. First we compute

$$E[X] = \int_0^1 x \cdot 2x dx = \frac{2}{3}$$

$$E[X^2] = \int_0^1 x^2 \cdot 2x dx = \frac{1}{2}$$

$$E[X^4] = \int_0^1 x^4 \cdot 2x dx = \frac{1}{3}.$$

Thus,

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

and

$$\text{Var}(X^2) = E[X^4] - \left(E[X^2]\right)^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

10. Make a table

X:	0	1
prob:	(1-p)	p
X^2	0	1.

From the table, $\boxed{E(X) = 0 \cdot (1-p) + 1 \cdot p = p.}$

Since X and X^2 have the same table $E(X^2) = E(X) = p$.

Therefore, $\boxed{\text{Var}(X) = p - p^2 = p(1-p).}$

11. Let X be the number of people who get their own hat.

Following the hint: let X_j represent whether person j gets their own hat. That is, $X_j = 1$ if person j gets their hat and 0 if not.

We have, $X = \sum_{j=1}^{100} X_j$, so $E(X) = \sum_{j=1}^{100} E(X_j)$.

Since person j is equally likely to get any hat, we have $P(X_j = 1) = 1/100$. Thus, $X_j \sim \text{Bernoulli}(1/100) \;\Rightarrow\; E(X_j) = 1/100 \;\Rightarrow\; \boxed{E(X) = 1.}$

12. For $y = 0, 2, 4, \ldots, 2n$,

$$P(Y = y) = P(X = \frac{y}{2}) = \binom{n}{y/2}\left(\frac{1}{2}\right)^n.$$

13. The CDF for R is

$$F_R(r) = P(R \le r) = \int_0^r 2\mathrm{e}^{-2u}\,du = -\mathrm{e}^{-2u}\Big|_0^r = 1 - \mathrm{e}^{-2r}.$$

Next, we find the CDF of T. T takes values in $(0, \infty)$.

For $0 < t$,

$$F_T(t) = P(T \le t) = P(1/R < t) = P(1/t > R) = 1 - F_R(1/t) = \mathrm{e}^{-2/t}.$$

We differentiate to get $f_T(t) = \dfrac{d}{dt}\left(\mathrm{e}^{-2/t}\right) = \dfrac{2}{t^2}\mathrm{e}^{-2/t}$

14. The jumps in the distribution function are at 0, 2, 4. The value of $p(a)$ at a jump is the height of the jump:

a	0	2	4
$p(a)$	1/5	1/5	3/5

15. We compute

$$P(X \ge 5) = 1 - P(X < 5) = 1 - \int_0^5 \lambda\mathrm{e}^{-\lambda x}dx = 1 - (1 - \mathrm{e}^{-5\lambda}) = \mathrm{e}^{-5\lambda}.$$

(b) We want $P(X \ge 15 | X \ge 10)$. First observe that $P(X \ge 15, X \ge 10) = P(X \ge 15)$. From similar computations in (a), we know

$$P(X \ge 15) = \mathrm{e}^{-15\lambda} \qquad\qquad P(X \ge 10) = \mathrm{e}^{-10\lambda}.$$

From the definition of conditional probability,

$$P(X \geq 15 | X \geq 10) = \frac{P(X \geq 15, X \geq 10)}{P(X \geq 10)} = \frac{P(X \geq 15)}{P(X \geq 10)} = \mathrm{e}^{-5\lambda}$$

Note: This is an illustration of the memorylessness property of the exponential distribution.

16. Transforming Normal Distributions

(a) Note, Y follows what is called a *log-normal distribution.*

$F_Y(a) = P(Y \leq a) = P(\mathrm{e}^Z \leq a) = P(Z \leq \ln(a)) = \boxed{\Phi(\ln(a)).}$

Differentiating using the chain rule:

$$f_y(a) = \frac{d}{da} F_Y(a) = \frac{d}{da}\Phi(\ln(a)) = \frac{1}{a}\phi(\ln(a)) = \boxed{\frac{1}{\sqrt{2\pi}\, a}\mathrm{e}^{-(\ln(a))^2/2}.}$$

(b) (i) The 0.33 quantile for Z is the value $q_{0.33}$ such that $P(Z \leq q_{0.33}) = 0.33$. That is, we want

$$\Phi(q_{0.33}) = 0.33 \Leftrightarrow \boxed{q_{0.33} = \Phi^{-1}(0.33)}.$$

(ii) We want to find $q_{0.9}$ where

$$F_Y(q_{0.9}) = 0.9 \Leftrightarrow \Phi(\ln(q_{0.9})) = 0.9 \Leftrightarrow \boxed{q_{0.9} = \mathrm{e}^{\Phi^{-1}(0.9)}}.$$

(iii) As in (ii) $q_{0.5} = \mathrm{e}^{\Phi^{-1}(0.5)} = \mathrm{e}^0 = \boxed{1}$.

17. **(a)** Total probability must be 1, so

$$1 = \int_0^3 \int_0^3 f(x, y)\, dy\, dx = \int_0^3 \int_0^3 c(x^2 y + x^2 y^2)\, dy\, dx = c \cdot \frac{243}{2},$$

(Here we skipped showing the arithmetic of the integration) Therefore, $c = \dfrac{2}{243}$.

(b)

$$\begin{aligned} P(1 \leq X \leq 2, 0 \leq Y \leq 1) &= \int_1^2 \int_0^1 f(x, y)\, dy\, dx \\ &= \int_1^2 \int_0^1 c(x^2 y + x^2 y^2)\, dy\, dx \\ &= c \cdot \frac{35}{18} \\ &= \frac{70}{4374} \approx 0.016 \end{aligned}$$

(c) For $0 \leq a \leq 1$ and $0 \leq b \leq 1$. we have

$$F(a, b) = \int_0^a \int_0^b f(x, y) dy\, dx = c\left(\frac{a^3 b^2}{6} + \frac{a^3 b^3}{9}\right)$$

(d) Since $y = 3$ is the maximum value for Y, we have

$$F_X(a) = F(a,3) = c\left(\frac{9a^3}{6} + 3a^3\right) = \frac{9}{2}c\,a^3 = \frac{a^3}{27}$$

(e) For $0 \le x \le 3$, we have, by integrating over the entire range for y,

$$f_X(x) = \int_0^3 f(x,y)\,dy = cx^2\left(\frac{3^2}{2} + \frac{3^3}{3}\right) = c\frac{27}{2}x^2 = \frac{1}{9}x^2.$$

This is consistent with (c) because $\frac{d}{dx}(x^3/27) = x^2/9$.

(f) Since $f(x,y)$ separates into a product as a function of x times a function of y we know X and Y are independent.

18. (**Central Limit Theorem**) Let $T = X_1 + X_2 + \ldots + X_{81}$. The central limit theorem says that

$$T \approx \mathrm{N}(81 * 5, 81 * 4) = \mathrm{N}(405, 18^2)$$

Standardizing we have

$$\begin{aligned} P(T > 369) &= P\left(\frac{T - 405}{18} > \frac{369 - 405}{18}\right) \\ &\approx P(Z > -2) \\ &\approx 0.975 \end{aligned}$$

The value of 0.975 comes from the rule-of-thumb that $P(|Z| < 2) \approx 0.95$. A more exact value (using R) is $P(Z > -2) \approx 0.9772$.