18.01 Single Variable Calculus
Fall 2006

Lecture 24: Numerical Integration

Numerical Integration

We use numerical integration to find the definite integrals of expressions that look like:

$$\int_a^b (\text{a big mess})$$

We also resort to numerical integration when an integral has no elementary antiderivative. For instance, there is no formula for

$$\int_0^x \cos(t^2)dt \quad \text{or} \quad \int_0^3 e^{-x^2}dx$$

Numerical integration yields numbers rather than analytical expressions.

We'll talk about three techniques for numerical integration: Riemann sums, the trapezoidal rule, and Simpson's rule.

1. Riemann Sum

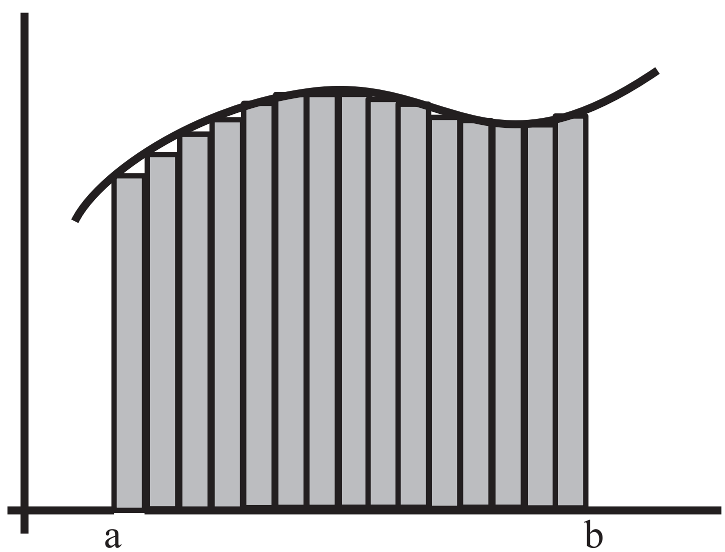

Figure 1: Riemann sum with left endpoints: $(y_0 + y_1 + \ldots + y_{n-1})\Delta x$

Here,

$$x_i - x_{i-1} = \Delta x$$

$$(\text{or, } x_i = x_{i-1} + \Delta x)$$

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b$$
$$y_0 = f(x_0),\ y_1 = f(x_1), \ldots y_n = f(x_n)$$

2. Trapezoidal Rule

The trapezoidal rule divides up the area under the function into trapezoids, rather than rectangles. The area of a trapezoid is the height times the average of the parallel bases:

$$\text{Area} = \text{height}\left(\frac{\text{base 1} + \text{base 2}}{2}\right) = \left(\frac{y_3 + y_4}{2}\right)\Delta x \quad \text{(See Figure 2)}$$

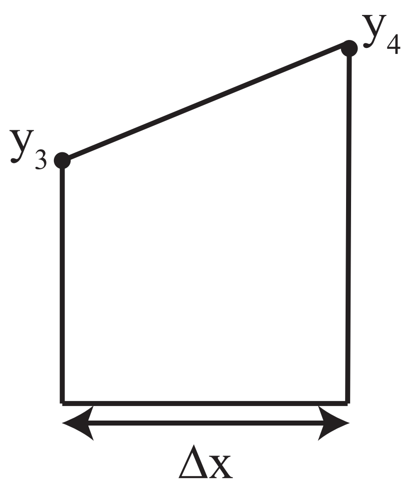

Figure 2: Area $= \left(\frac{y_3 + y_4}{2}\right)\Delta x$

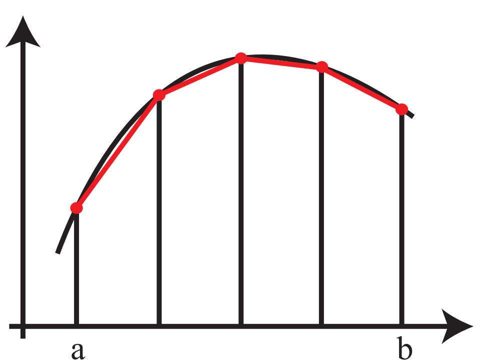

Figure 3: Trapezoidal rule = sum of areas of trapezoids.

$$\begin{aligned} \text{Total Trapezoidal Area} &= \Delta x\left(\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \frac{y_2 + y_3}{2} + \ldots + \frac{y_{n-1} + y_n}{2}\right) \\ &= \Delta x\left(\frac{y_0}{2} + y_1 + y_2 + \ldots + y_{n-1} + \frac{y_n}{2}\right) \end{aligned}$$

Note: The trapezoidal rule gives a more symmetric treatment of the two ends (a and b) than a Riemann sum does — the average of left and right Riemann sums.

3. Simpson's Rule

This approach often yields much more accurate results than the trapezoidal rule does. Here, we match quadratics (i.e. parabolas), instead of straight or slanted lines, to the graph. This approach requires an even number of intervals.

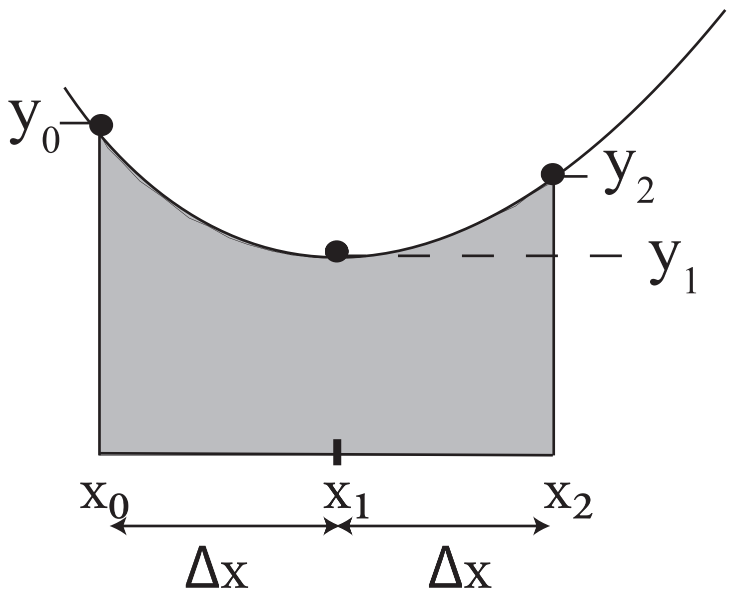

Figure 4: Area under a parabola.

$$\text{Area under parabola} = (\text{base})(\text{weighted average height}) = (2\Delta x)\left(\frac{y_0 + 4y_1 + y_2}{6}\right)$$

Simpson's rule for n intervals (n must be even!)

$$\text{Area} = (2\Delta x)\left(\frac{1}{6}\right)\left[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + (y_4 + 4y_5 + y_6) + \cdots + (y_{n-2} + 4y_{n-1} + y_n)\right]$$

Notice the following pattern in the coefficients:

$$\begin{array}{ccccccc} 1 & 4 & 1 & & & & \\ & & 1 & 4 & 1 & & \\ & & & & 1 & 4 & 1 \\ 1 & 4 & 2 & 4 & 2 & 4 & 1 \end{array}$$

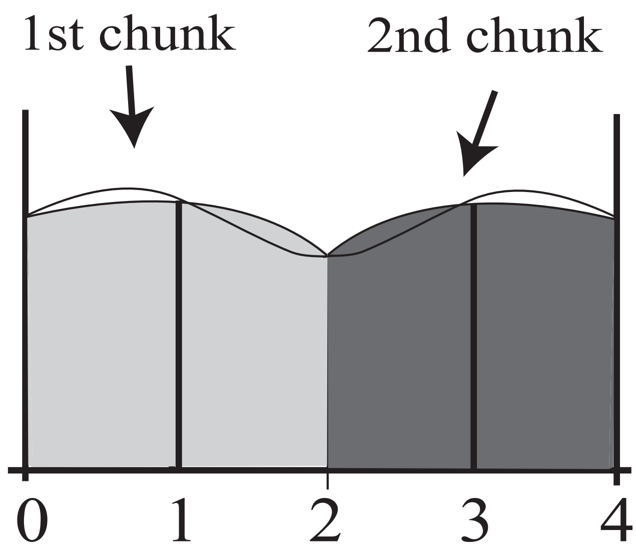

Figure 5: Area given by Simpson's rule for four intervals

Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 4y_{n-3} + 2y_{n-2} + 4y_{n-1} + y_n)$$

The pattern of coefficients in parentheses is:

$$\begin{array}{ccccccccc} & & 1 & 4 & 1 & & & = & \text{sum } 6 \\ & 1 & 4 & 2 & 4 & 1 & & = & \text{sum } 12 \\ 1 & 4 & 2 & 4 & 2 & 4 & 1 & = & \text{sum } 18 \end{array}$$

To double check – plug in $f(x) = 1$ (n even!).

$$\frac{\Delta x}{3}(1 + 4 + 2 + 4 + 2 + \cdots + 2 + 4 + 1) = \frac{\Delta x}{3}\left(1 + 1 + 4\left(\frac{n}{2}\right) + 2\left(\frac{n}{2} - 1\right)\right) = n\Delta x \quad (n \text{ even})$$

Example 1. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using two methods (trapezoidal and Simpson's) of numerical integration.

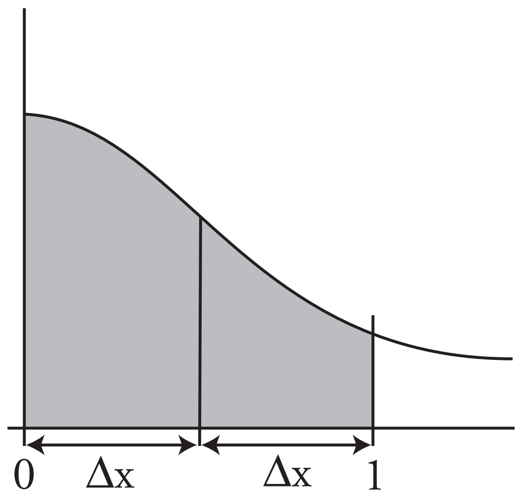

Figure 6: Area under $\frac{1}{(1+x^2)}$ above $[0, 1]$.

x	$1/(1+x^2)$
0	1
$\frac{1}{2}$	$\frac{4}{5}$
1	$\frac{1}{2}$

By the trapezoidal rule:

$$\Delta x\left(\frac{1}{2}y_0 + y_1 + \frac{1}{2}y_2\right) = \frac{1}{2}\left(\frac{1}{2}(1) + \frac{4}{5} + \frac{1}{2}\left(\frac{1}{2}\right)\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{4}{5} + \frac{1}{4}\right) = 0.775$$

By Simpson's rule:

$$\frac{\Delta x}{3}\left(y_0 + 4y_1 + y_2\right) = \frac{1/2}{3}\left(1 + 4\left(\frac{4}{5} + \frac{1}{2}\right)\right) = 0.78333...$$

Exact answer:

$$\int_0^1 \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4} - 0 = \frac{\pi}{4} \approx 0.785$$

Roughly speaking, the error, $|\text{ Simpson's} - \text{Exact }|,$ has order of magnitude $(\Delta x)^4$.