18.01 Single Variable Calculus
 Fall 2006

Lecture 21: Applications to Logarithms and Geometry

Application of FTC 2 to Logarithms

The integral definition of functions like $C(x)$, $S(x)$ of Fresnel makes them nearly as easy to use as elementary functions. It is possible to draw their graphs and tabulate values. You are asked to carry out an example or two of this on your problem set. To get used to using definite integrals and FTC2, we will discuss in detail the simplest integral that gives rise to a relatively new function, namely the logarithm.

Recall that

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

except when $n = -1$. It follows that the antiderivative of $1/x$ is not a power, but something else. So let us define a function $L(x)$ by

$$L(x) = \int_1^x \frac{dt}{t}$$

(This function turns out to be the logarithm. But recall that our approach to the logarithm was fairly involved. We first analyzed a^x, and then defined the number e, and finally defined the logarithm as the inverse function to e^x. The direct approach using this integral formula will be easier.)

All the basic properties of $L(x)$ follow directly from its definition. Note that $L(x)$ is defined for $0 < x < \infty$. (We will not extend the definition past $x = 0$ because $1/t$ is infinite at $t = 0$.) Next, the fundamental theorem of calculus (FTC2) implies

$$L'(x) = \frac{1}{x}$$

Also, because we have started the integration with lower limit 1, we see that

$$L(1) = \int_1^1 \frac{dt}{t} = 0$$

Thus L is increasing and crosses the x-axis at $x = 1$: $L(x) < 0$ for $0 < x < 1$ and $L(x) > 0$ for $x > 1$. Differentiating a second time,

$$L''(x) = -1/x^2$$

It follows that L is concave down.

The key property of $L(x)$ (showing that it is, indeed, a logarithm) is that it converts multiplication into addition:

Claim 1. $L(ab) = L(a) + L(b)$

Proof: By definition of $L(ab)$ and $L(a)$,

$$L(ab) = \int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} = L(a) + \int_a^{ab} \frac{dt}{t}$$

To handle $\int_a^{ab} \dfrac{dt}{t}$, make the substitution $t = au$. Then

$$dt = adu; \quad a < t < ab \implies 1 < u < b$$

Therefore,

$$\int_a^{ab} \frac{dt}{t} = \int_{u=1}^{u=b} \frac{adu}{au} = \int_1^b \frac{du}{u} = L(b)$$

This confirms $L(ab) = L(a) + L(b)$.

Two more properties, the end values, complete the general picture of the graph.

Claim 2. $L(x) \to \infty$ as $x \to \infty$.

Proof: It suffices to show that $L(2^n) \to \infty$ as $n \to \infty$, because the fact that L is increasing fills in all the values in between the powers of 2.

$$
\begin{aligned}
L(2^n) &= L(2 \cdot 2^{n-1}) = L(2) + L(2^{n-1}) \\
&= L(2) + L(2) + L(2^{n-2}) = L(2) + L(2) + \cdots + L(2) \quad (n \text{ times})
\end{aligned}
$$

Consequently, $L(2^n) = nL(2) \to \infty$ as $n \to \infty$. (In more familiar notation, $\ln 2^n = n \ln 2$.)

Claim 3. $L(x) \to -\infty$ as $x \to 0^+$.

Proof: $0 = L(1) = L\left(x \cdot \dfrac{1}{x}\right) = L(x) + L(1/x) \implies L(x) = -L(1/x)$. As $x \to 0^+$, $1/x \to +\infty$, so Claim 2 implies $L(1/x) \to \infty$. Hence

$$L(x) = -L(1/x) \to -\infty, \quad \text{as } x \to 0^+$$

Thus $L(x)$, defined on $0 < x < \infty$ increases from $-\infty$ to ∞, crossing the x-axis at $x = 1$. It is concave down and its graph can be drawn as in Fig. 1.

This provides an alternative to our previous approach to the exponential and log functions. Starting from $L(x)$, we can *define* the log function by $\ln x = L(x)$, *define* e as the number such that $L(e) = 1$, *define* e^x as the inverse function of $L(x)$, and *define* $a^x = e^{xL(a)}$.
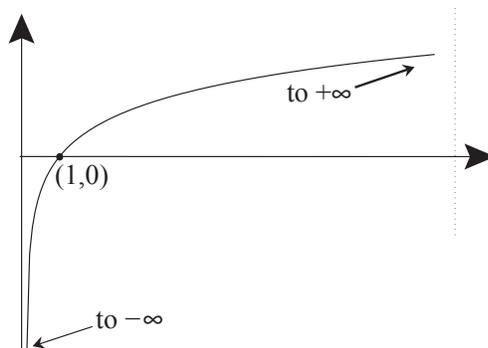
Figure 1: Graph of $y = \ln(x)$.

Application of FTCs to Geometry (Volumes and Areas)

1. Areas between two curves
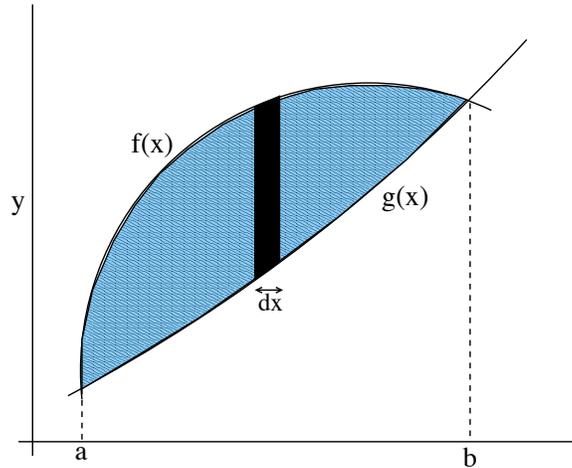
Figure 2: Finding the area between two curves.

Refer to Figure 2. Find the crossing points a and b. The area, A, between the curves is

$$A = \int_a^b \left(f(x) - g(x) \right) dx$$

Example 1. Find the area in the region between $x = y^2$ and $y = x - 2$.
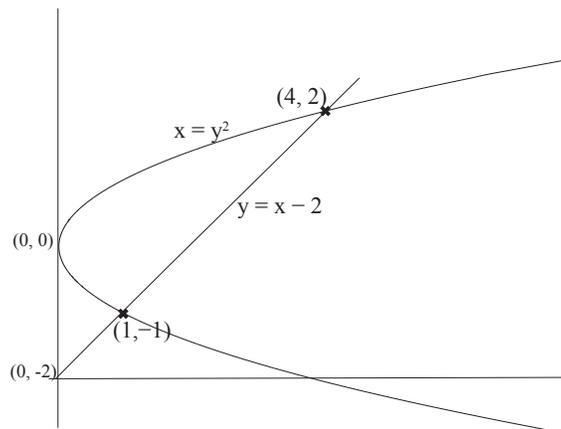
Figure 3: The intersection of $x = y^2$ and $y = x - 2$.

First, graph these functions and find the crossing points (see Figure 3).

$$
\begin{aligned}
y + 2 &= x = y^2 \\
y^2 - y - 2 &= 0 \\
(y - 2)(y + 1) &= 0
\end{aligned}
$$

Crossing points at $y = -1, 2$. Plug these back in to find the associated x values, $x = 1$ and $x = 4$. Thus the curves meet at $(1, -1)$ and $(4, 2)$ (see Figure 3).

There are two ways of finding the area between these two curves, a hard way and an easy way.

Hard Way: Vertical Slices
If we slice the region between the two curves vertically, we need to consider two different regions.
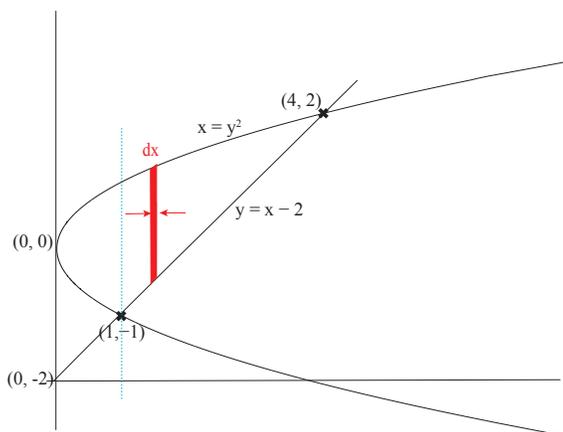
Figure 4: The intersection of $x = y^2$ and $y = x - 2$.

Where $x > 1$, the region's lower bound is the straight line. For $x < 1$, however, the region's lower bound is the lower half of the sideways parabola. We find the area, A, between the two curves by integrating the difference between the top curve and the bottom curve in each region:

$$
A = \int_0^1 \left\{ \sqrt{x} - (-\sqrt{x}) \right\} dx + \int_1^4 \left\{ \sqrt{x} - (x - 2) \right\} dx = \int (y_{top} - y_{bottom}) \, dx
$$

Easy Way: Horizontal Slices
Here, instead of subtracting the bottom curve from the top curve, we subtract the right curve from the left one.

$$
A = \int (x_{\text{left}} - x_{\text{right}}) \, dy = \int_{y=-1}^{y=2} \left[(y + 2) - y^2 \right] dx = \left(\frac{y^2}{2} + 2y + \frac{-y^3}{3} \right) \Big|_{-1}^{2} = \frac{4}{2} + 4 - \frac{8}{3} - (\frac{1}{2} - 2 + \frac{1}{3}) = \frac{9}{2}
$$
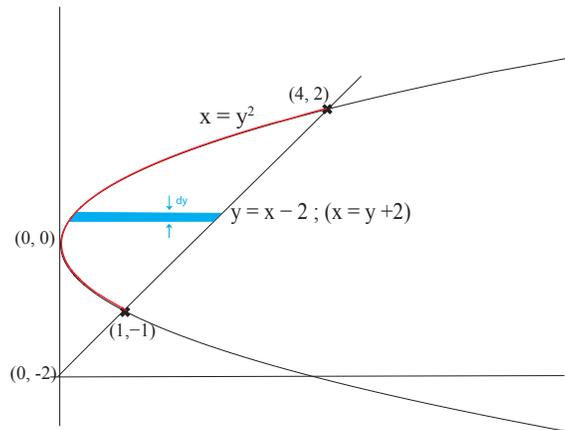
Figure 5: The intersection of $x = y^2$ and $y = x - 2$.

2. Volumes of solids of revolution

Rotate $f(x)$ about the x-axis, coming out of the page, to get:
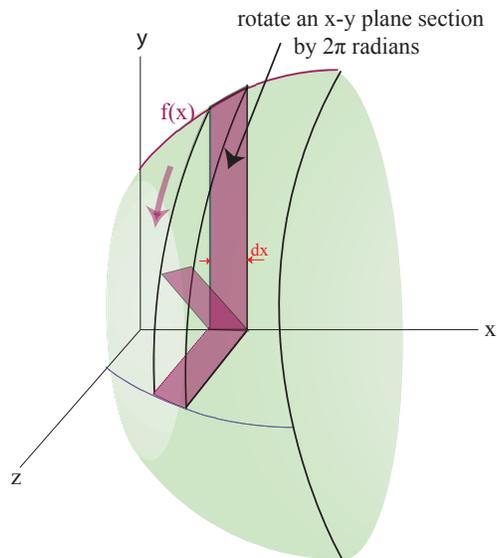
Figure 6: A solid of revolution: the purple slice is rotated by $\pi/4$ and $\pi/2$.

We want to figure out the volume of a "slice" of that solid. We can approximate each slice as a *disk* with width dx, radius y, and a cross-sectional area of πy^2. The volume of one slice is then:

$$dV = \pi y^2 dx \quad \text{(for a solid of revolution around the } x\text{-axis)}$$

Integrate with respect to x to find the total volume of the solid of revolution.

Example 2. Find the volume of a ball of radius a.
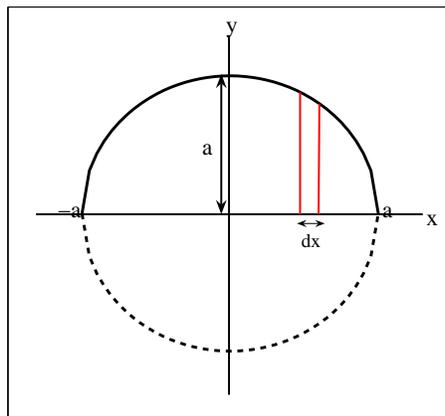
Figure 7: A ball of radius a

The equation for the upper half of the circle is

$$y = \sqrt{a^2 - x^2}.$$

If we spin the upper part of the curve about the x-axis, we get a ball of radius a. Notice that x ranges from $-a$ to $+a$. Putting all this together, we find

$$V = \int \pi y^2 dx = \int_{x=-a}^{x=a} \pi(a^2 - x^2)dx = \left(\pi a^2 x - \frac{\pi x^3}{3}\right)\Big|_{-a}^{a} = \frac{2}{3}\pi a^3 - \left(-\frac{2}{3}\pi a^3\right) = \frac{4}{3}\pi a^3$$

One can often exploit symmetry to further simplify these types of problems. In the problem above, for example, notice that the curve is symmetric about the y-axis. Therefore,

$$V = \int_{-a}^{a} \pi(a^2 - x^2)dx = 2\int_0^a \pi(a^2 - x^2)dx = 2\left(\pi a^2 x - \frac{x^3}{3}\right)\Big|_0^a$$

(The savings is that zero is an easier lower limit to work with than $-a$.) We get the same answer:

$$V = 2\left(\pi a^2 x - \frac{x^3}{3}\right)\Big|_0^a = 2\left(\pi a^3 - \frac{\pi}{3}a^3\right) = \frac{4}{3}\pi a^3$$