6.642 Continuum Electromechanics
Fall 2008

6.642, Continuum Electromechanics, Fall 2004
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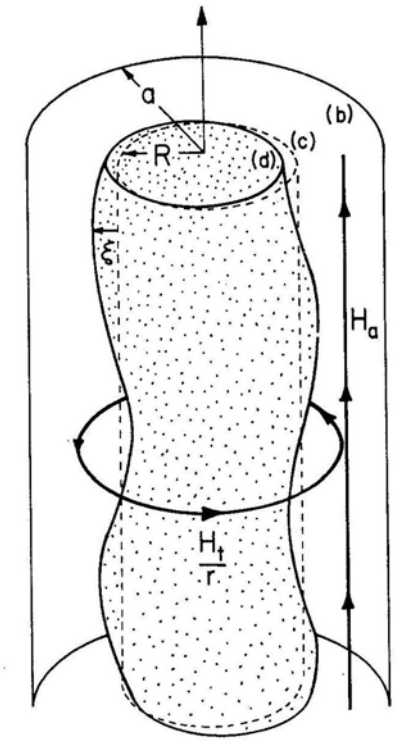
Lecture 9: Plasma Stability (z-θ pinch)
Continuum Electromechanics (Melcher) – Section 8.12



Plasma column showing equilibrium radius R and equilibrium magnetic fields.

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I. Equilibrium

$$\bar{H} = \frac{R}{r} H_t \,\bar{i}_\theta + H_a \,\bar{i}_z$$

$$\bar{K}\left(r = R\right) = - H_a \,\bar{i}_\theta + H_t \,\bar{i}_z$$

$$P_{od} - P_{oc} + \left\| T_{rr} \right\|_{r=R} - \frac{\gamma}{R} = 0$$

$$T_{rr} = \frac{1}{2}\mu\left[H_r^2 - H_\theta^2 - H_z^2\right] = -\frac{\mu_0}{2}\left[H_t^2 + H_a^2\right]$$

$$P_{od} - P_{oc} = \frac{\mu_0}{2}\left[H_t^2 + H_a^2\right] + \frac{\gamma}{R}$$

II. Perturbations, $\xi = \mathrm{Re}\,\hat{\xi}e^{j(\omega t - m\theta - kz)}$, $\bar{h} = -\nabla\psi$

$$\begin{bmatrix}\hat{\Psi}^b \\ \hat{\Psi}^c\end{bmatrix} = \begin{bmatrix}F_m(R,a) & G_m(a,R) \\ G_m(R,a) & F_m(a,R)\end{bmatrix}\begin{bmatrix}\hat{h}_r^b \\ \hat{h}_r^c\end{bmatrix}$$

$$\begin{bmatrix}\hat{P}^\alpha \\ \hat{P}^\beta\end{bmatrix} = j(\omega - kU)\rho\begin{bmatrix}F_m(\beta,\alpha) & G_m(\alpha,\beta) \\ G_m(\beta,\alpha) & F_m(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\hat{v}_r^\alpha \\ \hat{v}_r^\beta\end{bmatrix}$$

$$F_m(x,y) = \frac{1}{k}\frac{\left[I_m'(kx)K_m(ky) - K_m'(kx)I_m(ky)\right]}{\left[I_m'(ky)K_m'(kx) - I_m'(kx)K_m'(ky)\right]}$$

$$G_m(x,y) = \frac{1}{k^2x\left[I_m'(ky)K_m'(kx) - I_m'(kx)K_m'(ky)\right]}$$

III. Boundary Conditions

$$\bar{H}(R+\xi) = \frac{R}{R+\xi}H_t\,\bar{i}_\theta + H_a\,\bar{i}_z + \bar{h}(r=R)$$

$$= \underbrace{H_t\,\bar{i}_\theta + H_a\,\bar{i}_z}_{\text{Equilibrium}} \underbrace{-\frac{\xi}{R}H_t\,\bar{i}_\theta + \bar{h}(r=R)}_{\text{Perturbation}}$$

$$F = r - \xi(\theta, z, t) - R$$

$$\bar{n} = \frac{\nabla F}{|\nabla F|} = \frac{\bar{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta - \frac{\partial\xi}{\partial z}\bar{i}_z}{\left[1 + \frac{1}{R^2}\left(\frac{\partial\xi}{\partial\theta}\right)^2 + \left(\frac{\partial\xi}{\partial z}\right)^2\right]^{1/2}} \approx \bar{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta - \frac{\partial\xi}{\partial z}\bar{i}_z$$

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \left(\bar{v}\cdot\nabla\right)F = 0 = \frac{\partial F}{\partial t} + v_r\frac{\partial F}{\partial r} + \frac{v_\theta}{r}\frac{\partial F}{\partial\theta} + v_z\frac{\partial F}{\partial z}$$

$$= -\frac{\partial\xi}{\partial t} + v_r - \frac{v_\theta}{r}\frac{\partial\xi}{\partial\theta} - v_z\frac{\partial\xi}{\partial z}$$

$v_r = \frac{\partial\xi}{\partial t}$ [for stationary equilibrium]

$$\hat{v}_r = j\omega\hat{\xi}$$

$$\bar{T}_s = -\gamma\left(\nabla\bullet\bar{n}\right)\bar{n} = -\gamma\left[\frac{1}{r}\frac{\partial}{\partial r}(r\,n_r) + \frac{1}{r}\frac{\partial n_\theta}{\partial\theta} + \frac{\partial n_z}{\partial z}\right]\bar{n}$$

$$n_r = 1$$

$$n_\theta = -\frac{1}{R}\frac{\partial \xi}{\partial \theta}$$

$$n_z = -\frac{\partial \xi}{\partial z}$$

$$\overline{T}_s = -\gamma\left[\frac{1}{r} - \frac{1}{R^2}\frac{\partial^2 \xi}{\partial \theta^2} - \frac{\partial^2 \xi}{\partial z^2}\right]\left[\bar{i}_r - \frac{1}{R}\frac{\partial \xi}{\partial \theta}\bar{i}_\theta - \frac{\partial \xi}{\partial z}\bar{i}_z\right]$$

$$\overline{T}_s\left(r = R + \xi\right) = -\gamma\left[\frac{1}{R + \xi} - \frac{1}{R^2}\frac{\partial^2 \xi}{\partial \theta^2} - \frac{\partial^2 \xi}{\partial z^2}\right]\left[\bar{i}_r - \frac{1}{R}\frac{\partial \xi}{\partial \theta}\bar{i}_\theta - \frac{\partial \xi}{\partial z}\bar{i}_z\right]$$

$$T_{sr} = -\gamma\left[\frac{1}{R\left(1 + \frac{\xi}{R}\right)} - \frac{1}{R^2}\frac{\partial^2 \xi}{\partial \theta^2} - \frac{\partial^2 \xi}{\partial z^2}\right]$$

$$= -\frac{\gamma}{R} + \gamma\left[+\frac{\xi}{R^2} + \frac{1}{R^2}\frac{\partial^2 \xi}{\partial \theta^2} + \frac{\partial^2 \xi}{\partial z^2}\right]$$

$$\xi = \mathrm{Re}\left[\hat{\xi}e^{j[\omega t - m\theta - kz]}\right]$$

$$\hat{T}_{sr} = \frac{\gamma\hat{\xi}}{R^2}\left[1 - m^2 - \left(kR\right)^2\right]$$

$$\|P\| n_i = \|T_{ij}\| n_j - \gamma\nabla \bullet \overline{n}\,\overline{n}$$

$$i = r,\ n_i = n_r = 1$$

$$\|P\| = \|T_{rr}\| n_r + \|T_{r\theta}\| n_\theta + \|T_{rz}\| n_z + T_{sr}$$

$$T_{rr} = \frac{1}{2}\mu_0\left[h_r^2 - \left(H_t\left(1 - \frac{\xi}{R}\right) + h_\theta\right)^2 - \left(H_a + h_z\right)^2\right]$$

$$\approx \frac{1}{2}\mu_0\left[-H_t^2 - 2H_t\left(h_\theta - \frac{H_t\xi}{R}\right) - H_a^2 - 2H_a h_z\right]$$

$$T_{rr}^{'} = -\mu_0\left[H_t\left(h_\theta - H_t\xi/R\right) + H_a h_z\right]$$

$$T_{r\theta} = \mu_0 h_r H_t$$

$T_{r\theta}n_\theta = \mu_0 h_r H_t n_\theta$ second order

$T_{rz} n_z = \mu_0 h_r H_a n_z$ second order

$$-\hat{P}_d = -\mu_0\left[H_t\left(\hat{h}_\theta - \frac{H_t\hat{\xi}}{R}\right) + H_a\hat{h}_z\right] + \frac{\gamma\hat{\xi}}{R^2}\left[1 - m^2 - (kR)^2\right]$$

$$\hat{P}_d = j\omega\rho\, F_m(0,R)\,\hat{v}_{rd} = -\omega^2\rho F_m(0,R)\,\hat{\xi}$$

$$\bar{n}\cdot\bar{h}(r = R + \xi) = 0 = \left[\bar{i}_r - \frac{1}{R}\frac{\partial\xi}{\partial\theta}\bar{i}_\theta - \frac{\partial\xi}{\partial z}\bar{i}_z\right]\cdot\left[h_r\,\bar{i}_r + (H_\theta + h_\theta)\,\bar{i}_\theta + (H_a + h_z)\,\bar{i}_z\right]$$

$$h_r - \frac{H_t}{R}\frac{\partial\xi}{\partial\theta} - H_a\frac{\partial\xi}{\partial z} = 0$$

$$\hat{h}_{rc} = \frac{H_t}{R}\left(-jm\,\hat{\xi}\right) - jk\,\hat{\xi}\,H_a$$

at r=a, $\sigma \to \infty \;\Rightarrow\; \hat{h}_{rb} = 0$

$$h_\theta = -\frac{1}{R}\frac{\partial\Psi}{\partial\theta} \Rightarrow \hat{h}_\theta = \frac{jm}{R}\hat{\Psi}$$

$$\hat{\Psi}^c = F_m(a,R)\,\hat{h}_{rc} = -jF_m(a,R)\,\hat{\xi}\left[\frac{m}{R}H_t + kH_a\right]$$

$$\hat{h}_{\theta c} = F_m(a,R)\frac{m}{R}\,\hat{\xi}\left[\frac{m}{R}H_t + kH_a\right]$$

$$\hat{h}_{zc} = jk\hat{\Psi}_c = kF_m(a,R)\,\hat{\xi}\left[\frac{m}{R}H_t + kH_a\right]$$

IV. Dispersion Relation

$$\omega^2\rho F_m(0,R)\,\hat{\xi} = -\mu_0 H_t F_m(a,R)\frac{m}{R}\,\hat{\xi}\left[\frac{m}{R}H_t + kH_a\right] + \frac{\mu_0 H_t^2}{R}\hat{\xi} - \mu_0 H_a k F_m(a,R)\,\hat{\xi}\left[\frac{m}{R}H_t + kH_a\right] + \frac{\gamma\hat{\xi}}{R^2}\left[1 - m^2 - (kR)^2\right]$$

$$\omega^2\rho F_m(0,R) = \frac{\gamma}{R^2}\left[1 - m^2 - (kR)^2\right] - \mu_0 F_m(a,R)\left[\frac{m}{R}H_t + kH_a\right]^2 + \frac{\mu_0 H_t^2}{R}$$

$$F_m(0,R) = -\frac{1}{k}\frac{I_m(kR)}{I_m'(kR)} < 0;\quad F_m(a,R) > 0$$

$$-\rho\omega^2 F_m(0,R) = \frac{\gamma}{R^2}\left[(kR)^2 + m^2 - 1\right] + \underset{\overset{\uparrow}{\text{Stabilizing}}}{\mu_0 F_m(a,R)\left[\frac{m}{R}H_t + kH_a\right]^2} - \underset{\overset{\uparrow}{\text{Destabilizing}}}{\frac{\mu_0 H_t^2}{R}}$$

V. Stability

Surface tension: stabilizing for $m \geq 1$
destabilizing for $m=0$ and $kR<1$

$H_t = 0$

$$-\rho\omega^2 F_m(0,R) = \frac{\gamma}{R^2}\left[(kR)^2 + m^2 - 1\right] + \mu_0 F_m(a,R) k^2 H_a^2$$

$H_a = 0$

$$-\rho\omega^2 F_m(0,R) = \frac{\gamma}{R^2}\left[(kR)^2 + m^2 - 1\right] - \frac{\mu_0 H_t^2}{R}\left[1 - \frac{F_m(a,R)m^2}{R}\right]$$