6.642 Continuum Electromechanics
Fall 2008

Lecture 4: Continuum Electromechanics (Melcher) – Sections 2.18-2.19

I. Section (2.18) Solenoidal Fields

A. Vector Potential

$$\nabla \bullet \overline{B} = 0 \;\Rightarrow\; \overline{B} = \nabla \times \overline{A}$$

$$\Phi = \int_S \overline{B} \bullet \overline{n}\, da = \int_S \left(\nabla \times \overline{A}\right) \bullet \overline{n}\, da = \oint_C \overline{A} \bullet \overline{dl}$$

$$\nabla \times \overline{H} = \nabla \times \frac{\overline{B}}{\mu} = \overline{J} \;\Rightarrow\; \nabla \times \left(\nabla \times \overline{A}\right) = \mu \overline{J}$$

$$\nabla\left(\nabla \bullet \overline{A}\right) - \nabla^2 \overline{A} = \mu \overline{J}$$

Setting Gauge: $\nabla \bullet \overline{A} = 0 \;\Rightarrow\; \nabla^2 \overline{A} = -\mu \overline{J}$ (Vector Poisson's Equation)

B. Uniqueness

$$\overline{A} \rightarrow \overline{A} + \nabla f \;\Rightarrow\; \overline{B} = \left(\nabla \times \left(\overline{A} + \nabla f\right)\right) = \nabla \times \overline{A} + \underbrace{\nabla \times \left(\nabla f\right)}_{0}$$

$$\overline{C} = \overline{A} + \overline{a}$$

$$\nabla \times \overline{C} = \nabla \times \left(\overline{A} + \overline{a}\right) = \nabla \times \overline{A} + \nabla \times \overline{a}$$

For uniqueness: $\nabla \times \overline{a} = 0$ so that $\overline{C} = \overline{A}$

$$\overline{a} = \nabla f$$

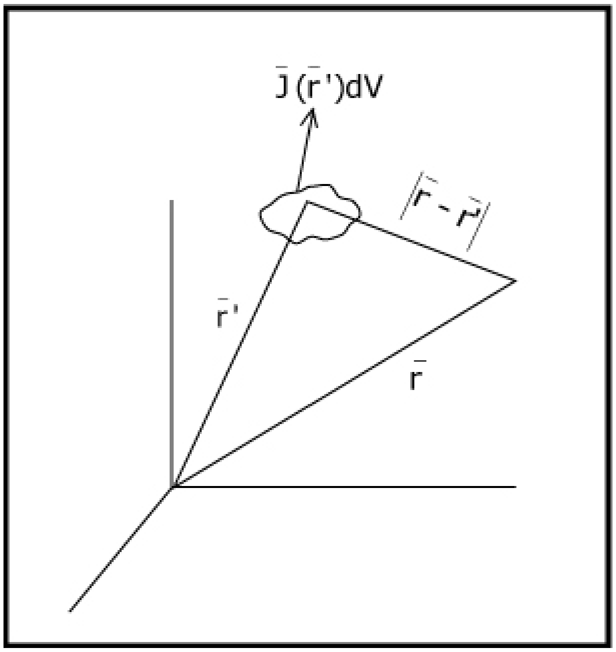
$$\nabla \bullet \overline{C} = \left(\nabla \bullet \left(\overline{A} + \overline{a}\right)\right) = \nabla \bullet \overline{A} \;\Rightarrow\; \nabla \bullet \overline{a} = 0 \;\Rightarrow\; \nabla^2 f = 0$$

$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon} \;\Rightarrow\; \Phi = \int_V \frac{\rho\, dV}{4\pi\varepsilon \left|\overline{r} - \overline{r}\,'\right|}$$

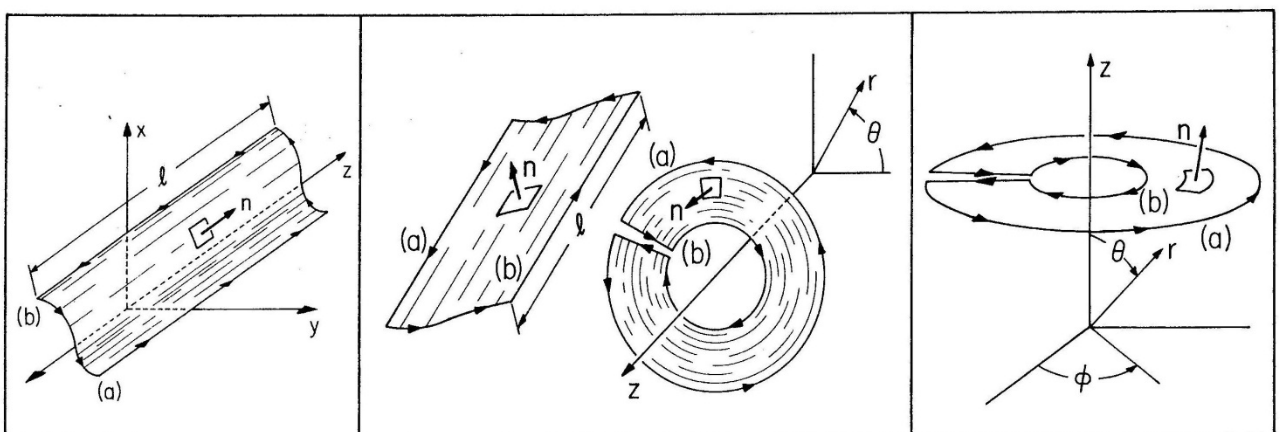
$$\nabla^2 f = 0 \;\Rightarrow\; f = 0 \;\Rightarrow\; \overline{C} = \overline{A}$$

C. Vector Poisson's Equation Solutions

$$\nabla^2 \overline{A} = -\mu \overline{J} \Rightarrow \overline{A}\left(\overline{r}\right) = \frac{\mu}{4\pi} \int_V \frac{\overline{J}\left(\overline{r}\,'\right) dV}{\left|\overline{r} - \overline{r}\,'\right|}$$



Important configurations having solenoidal field $\vec{B}$ represented by single components of vector potential $\vec{A}$.



Two-dimensional Cartesian	Polar	Axisymmetric cylindrical	Axisymmetric spherical
$\vec{A} = A(x,y)\vec{i}_z$ (a)	$\vec{A} = A(r,\theta)\vec{i}_z$ (d)	$\vec{A} = \frac{\Lambda(r,z)}{r}\vec{i}_\theta$ (g)	$\vec{A} = \frac{\Lambda(r,\theta)}{r\sin\theta}\vec{i}_\phi$ (j)
$\vec{B} = \frac{\partial A}{\partial y}\vec{i}_x - \frac{\partial A}{\partial x}\vec{i}_y$ (b)	$\vec{B} = \frac{1}{r}\frac{\partial A}{\partial\theta}\vec{i}_r - \frac{\partial A}{\partial r}\vec{i}_\theta$ (e)	$\vec{B} = -\frac{1}{r}\frac{\partial\Lambda}{\partial z}\vec{i}_r + \frac{1}{r}\frac{\partial\Lambda}{\partial r}\vec{i}_z$ (h)	$\vec{B} = \frac{1}{r\sin\theta}[\frac{1}{r}\frac{\partial\Lambda}{\partial\theta}\vec{i}_r - \frac{\partial\Lambda}{\partial r}\vec{i}_\theta]$ (k)
$\Phi_\lambda = \ell[A(a) - A(b)]$ (c)	$\Phi_\lambda = \ell[A(a) - A(b)]$ (f)	$\Phi_\lambda = 2\pi[\Lambda(a) - \Lambda(b)]$ (i)	$\Phi_\lambda = 2\pi[\Lambda(a) - \Lambda(b)]$ (l)

D. Magnetic Field Lines in 2 Dimensional Geometries

1. 2-D Cartesian

$$\overline{A} = A(x,y)\,\overline{i}_z$$

$$\overline{B} = \nabla \times \overline{A} = \overline{i}_x \frac{\partial A}{\partial y} - \overline{i}_y \frac{\partial A}{\partial x}$$

$$\frac{dy}{dx} = \frac{B_y}{B_x} = \frac{-\frac{\partial A}{\partial x}}{\frac{\partial A}{\partial y}} \Rightarrow \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy = dA = 0 \Rightarrow A(x,y) = \text{constant}$$

2. Polar

$$\overline{A} = A(r,\theta)\,\overline{i}_z$$

$$\overline{B} = \nabla \times \overline{A} = \overline{i}_r \frac{1}{r}\frac{\partial A}{\partial \theta} - \overline{i}_\theta \frac{\partial A}{\partial r}$$

$$\frac{B_r}{B_\theta} = \frac{dr}{rd\theta} = \frac{\frac{1}{r}\frac{\partial A}{\partial \theta}}{-\frac{\partial A}{\partial r}} \Rightarrow \frac{\partial A}{\partial r}dr + \frac{\partial A}{\partial \theta}d\theta = dA = 0 \Rightarrow A(r,\theta) = \text{constant}$$

3. Axisymmetric Cylindrical

$$\overline{A} = \frac{\Lambda(r,z)}{r}\,\overline{i}_\theta$$

$$\overline{B} = \nabla \times \overline{A} = -\frac{1}{r}\frac{\partial \Lambda}{\partial z}\overline{i}_r + \frac{1}{r}\frac{\partial \Lambda}{\partial r}\overline{i}_z$$

$$\frac{dz}{dr} = \frac{B_z}{B_r} = \frac{\frac{1}{r}\frac{\partial \Lambda}{\partial r}}{-\frac{1}{r}\frac{\partial \Lambda}{\partial z}} \Rightarrow \frac{\partial \Lambda}{\partial r}dr + \frac{\partial \Lambda}{\partial z}dz = d\Lambda = 0$$

$$\Lambda(r,z) = \text{constant}$$

4. Axisymmetric Spherical

$$\overline{A} = \frac{\Lambda(r,\theta)}{r\sin\theta}\,\overline{i}_\phi$$

$$\overline{B} = \nabla \times \overline{A} = \frac{1}{r\sin\theta}\left[\frac{1}{r}\frac{\partial \Lambda}{\partial \theta}\overline{i}_r - \frac{\partial \Lambda}{\partial r}\overline{i}_\theta\right]$$

$$\frac{dr}{r d\theta} = \frac{B_r}{B_\theta} = \frac{\frac{1}{r}\frac{\partial \Lambda}{\partial \theta}}{-\frac{\partial \Lambda}{\partial r}} \Rightarrow \frac{\partial \Lambda}{\partial r} dr + \frac{\partial \Lambda}{\partial \theta} d\theta = d\Lambda = 0$$

$$\Lambda(r, \theta) = \text{constant}$$

E. Electric Field Lines in Volume Charge Free Regions

$$\nabla \cdot \bar{E} = 0 \Rightarrow \bar{E} = \nabla \times \bar{A}$$

II. Vector Potential Transfer Relations with $\bar{J}=0$ (Section 2.19)

$\nabla^2 \bar{A} = 0$ [Vector Laplace's Equation]

A. Cartesian Coordinates $\left[\nabla^2 \bar{A} = \nabla^2 A_x \bar{i}_x + \nabla^2 A_y \bar{i}_y + \nabla^2 A_z \bar{i}_z\right]$

$$\bar{A} = \bar{i}_z \operatorname{Re}\left[\tilde{A}(x) e^{-jky}\right]$$

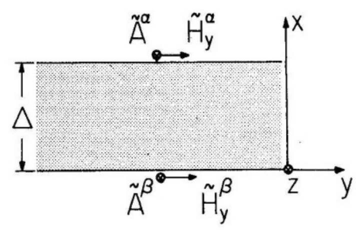
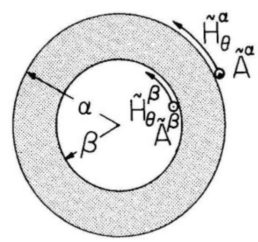
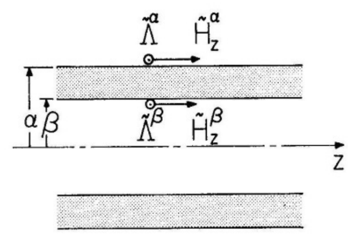
$$\tilde{A}(x) = \frac{\tilde{A}^\alpha \sinh kx - \tilde{A}^\beta \sinh k(x - \Delta)}{\sinh k\Delta}$$

$$\tilde{H}_y = -\frac{1}{\mu}\frac{\partial \tilde{A}}{\partial x} = -\frac{k}{\mu}\left[\frac{\tilde{A}^\alpha \cosh kx - \tilde{A}^\beta \cosh k(x - \Delta)}{\sinh k\Delta}\right]$$

$$\begin{bmatrix} \tilde{H}_y^\alpha \\ \tilde{H}_y^\beta \end{bmatrix} = \frac{k}{\mu}\begin{bmatrix} -\coth k\Delta & \frac{1}{\sinh k\Delta} \\ -\frac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix}\begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix}$$

$$B_x = \frac{\partial A_z}{\partial y} \Rightarrow \tilde{B}_x = -jk\tilde{A} \Rightarrow \begin{bmatrix} \tilde{B}_x^\alpha \\ \tilde{B}_x^\beta \end{bmatrix} = -jk\begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix}$$

Vector potential transfer relations for two-dimensional or symmetric Laplacian fields.

Two-dimensional Cartesian	Polar	Axisymmetric cylindrical
		
$\vec{A} = \vec{i}_z \operatorname{Re} \tilde{A}(x) \exp(-jky)$ $\begin{bmatrix}\tilde{H}_y^\alpha \\ \tilde{H}_y^\beta\end{bmatrix} = \frac{k}{\mu}\begin{bmatrix}-\coth(k\Delta) & \frac{1}{\sinh(k\Delta)} \\ \frac{-1}{\sinh(k\Delta)} & \coth(k\Delta)\end{bmatrix}\begin{bmatrix}\tilde{A}^\alpha \\ \tilde{A}^\beta\end{bmatrix}$ (a) $\begin{bmatrix}\tilde{A}^\alpha \\ \tilde{A}^\beta\end{bmatrix} = \frac{\mu}{k}\begin{bmatrix}-\coth(k\Delta) & \frac{1}{\sinh(k\Delta)} \\ \frac{-1}{\sinh(k\Delta)} & \coth(k\Delta)\end{bmatrix}\begin{bmatrix}\tilde{H}_y^\alpha \\ \tilde{H}_y^\beta\end{bmatrix}$ (b)	$\vec{A} = \vec{i}_z \operatorname{Re} \tilde{A}(r) \exp(-jm\theta)$ $\begin{bmatrix}\tilde{H}_\theta^\alpha \\ \tilde{H}_\theta^\beta\end{bmatrix} = \frac{1}{\mu}\begin{bmatrix}f_m(\beta,\alpha) & g_m(\alpha,\beta) \\ g_m(\beta,\alpha) & f_m(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\tilde{A}^\alpha \\ \tilde{A}^\beta\end{bmatrix}$ (c) $\begin{bmatrix}\tilde{A}^\alpha \\ \tilde{A}^\beta\end{bmatrix} = \mu\begin{bmatrix}F_m(\beta,\alpha) & G_m(\alpha,\beta) \\ G_m(\beta,\alpha) & F_m(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\tilde{H}_\theta^\alpha \\ \tilde{H}_\theta^\beta\end{bmatrix}$ (d) For f_m, g_m, F_m, G_m see Table 2.16.2, $k = 0$, $m \neq 0$	$\vec{A} = \vec{i}_\theta \operatorname{Re} \tilde{A}(r)\exp(-jkz)$; $\tilde{\Lambda} = \tilde{A}r$ $\begin{bmatrix}\tilde{H}_z^\alpha \\ \tilde{H}_z^\beta\end{bmatrix} = \frac{-k^2}{\mu}\begin{bmatrix}F_o(\beta,\alpha) & G_o(\alpha,\beta) \\ G_o(\beta,\alpha) & F_o(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\frac{\tilde{\Lambda}^\alpha}{\alpha} \\ \frac{\tilde{\Lambda}^\beta}{\beta}\end{bmatrix}$ (e) $\begin{bmatrix}\frac{\tilde{\Lambda}^\alpha}{\alpha} \\ \frac{\tilde{\Lambda}^\beta}{\beta}\end{bmatrix} = -\left(\frac{\mu}{k^2}\right)\begin{bmatrix}f_o(\beta,\alpha) & g_o(\alpha,\beta) \\ g_o(\beta,\alpha) & f_o(\alpha,\beta)\end{bmatrix}\begin{bmatrix}\tilde{H}_z^\alpha \\ \tilde{H}_z^\beta\end{bmatrix}$ (f) For F_o, G_o, f_o, g_o see Table 2.16.2, $m = 0$, $k \neq 0$
$\begin{bmatrix}\hat{A}^\alpha \\ \hat{A}^\beta\end{bmatrix} = \frac{j}{k}\begin{bmatrix}\hat{B}_x^\alpha \\ \hat{B}_x^\beta\end{bmatrix}$	$\begin{bmatrix}\hat{A}^\alpha \\ \hat{A}^\beta\end{bmatrix} = \frac{j}{m}\begin{bmatrix}\alpha\hat{B}_r^\alpha \\ \beta\hat{B}_r^\beta\end{bmatrix}$	$\begin{bmatrix}\frac{\hat{\Lambda}^\alpha}{\alpha} \\ \frac{\hat{\Lambda}^\beta}{\beta}\end{bmatrix} = \frac{-j}{k}\begin{bmatrix}\hat{B}_r^\alpha \\ \hat{B}_r^\beta\end{bmatrix}$

B. Polar Coordinates

$$\nabla^2 \overline{A} = \nabla^2\left(A(r,\theta)\,\bar{i}_z\right) = 0 \Rightarrow \nabla^2\left(A(r,\theta)\right) = 0$$

$$A(r,\theta) = \operatorname{Re}\left[\tilde{A}(r)\,e^{-jm\theta}\right]$$

$$\tilde{A}(r) = \frac{\tilde{A}^\alpha\left[\left(\frac{\beta}{r}\right)^m - \left(\frac{r}{\beta}\right)^m\right] + \tilde{A}^\beta\left[\left(\frac{r}{\alpha}\right)^m - \left(\frac{\alpha}{r}\right)^m\right]}{\left[\left(\frac{\beta}{\alpha}\right)^m - \left(\frac{\alpha}{\beta}\right)^m\right]}$$

$$H_\theta = -\frac{1}{\mu}\frac{\partial A}{\partial r}$$

$$H_r = \frac{1}{\mu r}\frac{\partial A}{\partial \theta} \Rightarrow \tilde{H}_r = \frac{-jm}{\mu r}\tilde{A}$$

C. Axisymmetric Cylindrical

$$\nabla^2 \bar{A} = \nabla^2\left(A(r,z)\,\bar{i}_\theta\right) = 0 \Rightarrow \nabla^2 A_\theta - \frac{A_\theta}{r^2} = 0$$

$$\nabla^2\left(A(r,\theta,z)\,\bar{i}_\theta\right) = \left[\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rA_\theta)\right] + \frac{1}{r^2}\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial \theta} + \frac{\partial^2 A_\theta}{\partial z^2}\right]\bar{i}_\theta - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}\bar{i}_r$$

(Vector Laplacian)

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$ (Scalar Laplacian)

$$A(r,z) = \mathrm{Re}\left[\tilde{A}(r,t)\,e^{-jkz}\right]$$

$$\frac{d^2\tilde{A}}{dr^2} + \frac{1}{r}\frac{d\tilde{A}}{\partial r} - \left(k^2 + \frac{1}{r^2}\right)\tilde{A} = 0 \Rightarrow \tilde{A} = B_1 I_1(kr) + B_2 K_1(kr)$$
$$= C_1 J_1(jkr) + C_2 H_1(jkr)$$

$$\nabla^2\Phi = 0;\ \Phi(r,\theta,z,t) = \mathrm{Re}\left[\tilde{\Phi}(r,t)\,e^{-j(m\theta+kz)}\right]$$

$$\frac{d^2\tilde{\Phi}}{dr^2} + \frac{1}{r}\frac{d\tilde{\Phi}}{dr} - \left(k^2 + \frac{m^2}{r^2}\right)\tilde{\Phi} = 0 \Rightarrow \tilde{\Phi} = A_1 I_m(kr) + A_2 K_m(kr)$$

$$\tilde{A} = \frac{\left\{\tilde{A}^\alpha\left[H_1(jk\beta)J_1(jkr) - J_1(jk\beta)H_1(jkr)\right] + \tilde{A}^\beta\left[J_1(jk\alpha)H_1(jkr) - H_1(jk\alpha)J_1(jkr)\right]\right\}}{\left[J_1(jk\alpha)H_1(jk\beta) - H_1(jk\alpha)J_1(jk\beta)\right]}$$