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6.642 Continuum Electromechanics
Fall 2008

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Lecture 4: Continuum Electromechanics (Melcher) – Sections 2.18-2.19

I. Section (2.18) Solenoidal Fields

A. Vector Potential

$$\nabla \cdot \bar{\mathbf{B}} = 0 \Rightarrow \bar{\mathbf{B}} = \nabla \times \bar{\mathbf{A}}$$

$$\Phi = \int_S \bar{\mathbf{B}} \cdot \bar{\mathbf{n}} \, da = \int_S (\nabla \times \bar{\mathbf{A}}) \cdot \bar{\mathbf{n}} \, da = \oint_C \bar{\mathbf{A}} \cdot d\bar{\mathbf{l}}$$

$$\nabla \times \bar{\mathbf{H}} = \nabla \times \frac{\bar{\mathbf{B}}}{\mu} = \bar{\mathbf{J}} \Rightarrow \nabla \times (\nabla \times \bar{\mathbf{A}}) = \mu \bar{\mathbf{J}}$$

$$\nabla (\nabla \cdot \bar{\mathbf{A}}) - \nabla^2 \bar{\mathbf{A}} = \mu \bar{\mathbf{J}}$$

Setting Gauge: $\nabla \cdot \bar{\mathbf{A}} = 0 \Rightarrow \nabla^2 \bar{\mathbf{A}} = -\mu \bar{\mathbf{J}}$ (Vector Poisson's Equation)

B. Uniqueness

$$\bar{\mathbf{A}} \rightarrow \bar{\mathbf{A}} + \nabla f \Rightarrow \bar{\mathbf{B}} = (\nabla \times (\bar{\mathbf{A}} + \nabla f)) = \nabla \times \bar{\mathbf{A}} + \nabla \times (\nabla f) \overset{0}{\rightarrow}$$

$$\bar{\mathbf{C}} = \bar{\mathbf{A}} + \bar{\mathbf{a}}$$

$$\nabla \times \bar{\mathbf{C}} = \nabla \times (\bar{\mathbf{A}} + \bar{\mathbf{a}}) = \nabla \times \bar{\mathbf{A}} + \nabla \times \bar{\mathbf{a}}$$

For uniqueness: $\nabla \times \bar{\mathbf{a}} = 0$ so that $\bar{\mathbf{C}} = \bar{\mathbf{A}}$

$$\bar{\mathbf{a}} = \nabla f$$

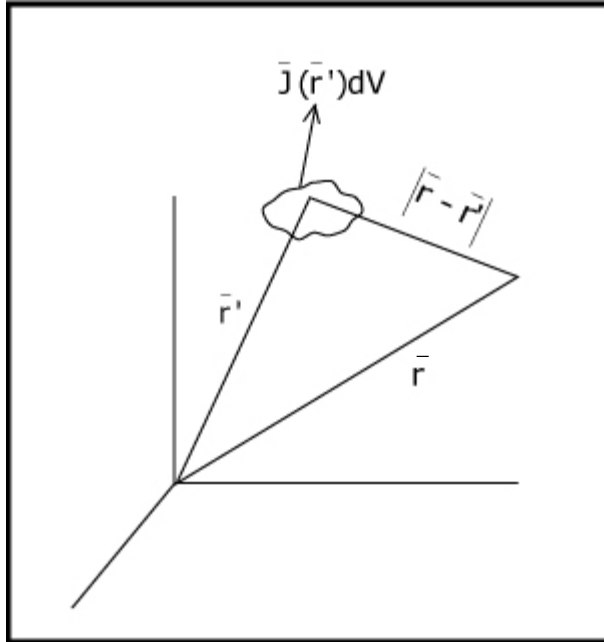
$$\nabla \cdot \bar{\mathbf{C}} = (\nabla \cdot (\bar{\mathbf{A}} + \bar{\mathbf{a}})) = \nabla \cdot \bar{\mathbf{A}} \Rightarrow \nabla \cdot \bar{\mathbf{a}} = 0 \Rightarrow \nabla^2 f = 0$$

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon} \Rightarrow \Phi = \int_V \frac{\rho dV}{4\pi\epsilon |\bar{\mathbf{r}} - \bar{\mathbf{r}}'|}$$

$$\nabla^2 f = 0 \Rightarrow f = 0 \Rightarrow \bar{\mathbf{C}} = \bar{\mathbf{A}}$$

C. Vector Poisson's Equation Solutions

$$\nabla^2 \bar{\mathbf{A}} = -\mu \bar{\mathbf{J}} \Rightarrow \bar{\mathbf{A}}(\bar{\mathbf{r}}) = \frac{\mu}{4\pi} \int_V \frac{\bar{\mathbf{J}}(\bar{\mathbf{r}}') dV}{|\bar{\mathbf{r}} - \bar{\mathbf{r}}'|}$$



Important configurations having solenoidal field \vec{B} represented by single components of vector potential \vec{A} .

<p>A 3D Cartesian coordinate system with x, y, and z axes. A rectangular region is shown in the xy-plane, extending from x=a to x=b and y=0 to y=l. A magnetic field vector \vec{B} points in the positive z-direction. A vector potential \vec{A} is shown as a dashed line along the x-axis.</p>	<p>Two diagrams. (a) A wedge-shaped region in the xy-plane with a magnetic field \vec{B} pointing in the positive z-direction and a vector potential \vec{A} pointing in the positive x-direction. (b) A circular region in the xy-plane with a magnetic field \vec{B} pointing in the positive z-direction and a vector potential \vec{A} pointing in the positive x-direction. A polar coordinate system (r, θ) is shown.</p>	<p>A 3D coordinate system with z, r, and φ axes. A cylindrical region is shown with a magnetic field \vec{B} pointing in the positive z-direction and a vector potential \vec{A} pointing in the positive φ-direction.</p>	
<p>Two-dimensional Cartesian</p>	<p>Polar</p>	<p>Axisymmetric cylindrical</p>	<p>Axisymmetric spherical</p>
<p>$\vec{A} = A(x,y)\vec{i}_z$ (a)</p>	<p>$\vec{A} = A(r,\theta)\vec{i}_z$ (d)</p>	<p>$\vec{A} = \frac{\Lambda(r,z)}{r}\vec{i}_\theta$ (g)</p>	<p>$\vec{A} = \frac{\Lambda(r,\theta)}{r \sin \theta}\vec{i}_\phi$ (j)</p>
<p>$\vec{B} = \frac{\partial A}{\partial y}\vec{i}_x - \frac{\partial A}{\partial x}\vec{i}_y$ (b)</p>	<p>$\vec{B} = \frac{1}{r}\frac{\partial A}{\partial \theta}\vec{i}_r - \frac{\partial A}{\partial r}\vec{i}_\theta$ (e)</p>	<p>$\vec{B} = -\frac{1}{r}\frac{\partial \Lambda}{\partial z}\vec{i}_r + \frac{1}{r}\frac{\partial \Lambda}{\partial r}\vec{i}_z$ (h)</p>	<p>$\vec{B} = \frac{1}{r \sin \theta} \left[\frac{1}{r}\frac{\partial \Lambda}{\partial \theta}\vec{i}_r - \frac{\partial \Lambda}{\partial r}\vec{i}_\theta \right]$ (k)</p>
<p>$\phi_\lambda = \ell[A(a) - A(b)]$ (c)</p>	<p>$\phi_\lambda = \ell[A(a) - A(b)]$ (f)</p>	<p>$\phi_\lambda = 2\pi[\Lambda(a) - \Lambda(b)]$ (i)</p>	<p>$\phi_\lambda = 2\pi[\Lambda(a) - \Lambda(b)]$ (l)</p>

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D. Magnetic Field Lines in 2 Dimensional Geometries

1. 2-D Cartesian

$$\bar{A} = A(x, y) \bar{i}_z$$

$$\bar{B} = \nabla \times \bar{A} = \bar{i}_x \frac{\partial A}{\partial y} - \bar{i}_y \frac{\partial A}{\partial x}$$

$$\frac{dy}{dx} = \frac{B_y}{B_x} = \frac{-\frac{\partial A}{\partial x}}{\frac{\partial A}{\partial y}} \Rightarrow \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy = dA = 0 \Rightarrow A(x, y) = \text{constant}$$

2. Polar

$$\bar{A} = A(r, \theta) \bar{i}_z$$

$$\bar{B} = \nabla \times \bar{A} = \bar{i}_r \frac{1}{r} \frac{\partial A}{\partial \theta} - \bar{i}_\theta \frac{\partial A}{\partial r}$$

$$\frac{B_r}{B_\theta} = \frac{dr}{rd\theta} = \frac{\frac{1}{r} \frac{\partial A}{\partial \theta}}{-\frac{\partial A}{\partial r}} \Rightarrow \frac{\partial A}{\partial r} dr + \frac{\partial A}{\partial \theta} d\theta = dA = 0 \Rightarrow A(r, \theta) = \text{constant}$$

3. Axisymmetric Cylindrical

$$\bar{A} = \frac{\Lambda(r, z)}{r} \bar{i}_\theta$$

$$\bar{B} = \nabla \times \bar{A} = -\frac{1}{r} \frac{\partial \Lambda}{\partial z} \bar{i}_r + \frac{1}{r} \frac{\partial \Lambda}{\partial r} \bar{i}_z$$

$$\frac{dz}{dr} = \frac{B_z}{B_r} = \frac{\frac{1}{r} \frac{\partial \Lambda}{\partial r}}{-\frac{1}{r} \frac{\partial \Lambda}{\partial z}} \Rightarrow \frac{\partial \Lambda}{\partial r} dr + \frac{\partial \Lambda}{\partial z} dz = d\Lambda = 0$$

$$\Lambda(r, z) = \text{constant}$$

4. Axisymmetric Spherical

$$\bar{A} = \frac{\Lambda(r, \theta)}{r \sin \theta} \bar{i}_\phi$$

$$\bar{B} = \nabla \times \bar{A} = \frac{1}{r \sin \theta} \left[\frac{1}{r} \frac{\partial \Lambda}{\partial \theta} \bar{i}_r - \frac{\partial \Lambda}{\partial r} \bar{i}_\theta \right]$$

$$\frac{dr}{r d\theta} = \frac{B_r}{B_\theta} = \frac{\frac{1}{r} \frac{\partial \Lambda}{\partial \theta}}{-\frac{\partial \Lambda}{\partial r}} \Rightarrow \frac{\partial \Lambda}{\partial r} dr + \frac{\partial \Lambda}{\partial \theta} d\theta = d\Lambda = 0$$

$$\Lambda(r, \theta) = \text{constant}$$

E. Electric Field Lines in Volume Charge Free Regions

$$\nabla \cdot \vec{E} = 0 \Rightarrow \vec{E} = \nabla \times \vec{A}$$

II. Vector Potential Transfer Relations with $\vec{j} = 0$ (Section 2.19)

$$\nabla^2 \vec{A} = 0 \text{ [Vector Laplace's Equation]}$$

A. Cartesian Coordinates $\left[\nabla^2 \vec{A} = \nabla^2 A_x \vec{i}_x + \nabla^2 A_y \vec{i}_y + \nabla^2 A_z \vec{i}_z \right]$

$$\vec{A} = \vec{i}_z \operatorname{Re} \left[\tilde{A}(x) e^{-jk y} \right]$$

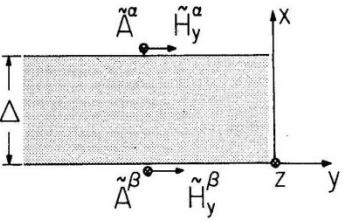
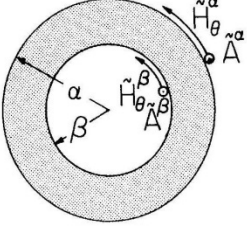
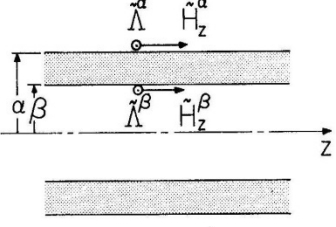
$$\tilde{A}(x) = \frac{\tilde{A}^\alpha \sinh kx - \tilde{A}^\beta \sinh k(x - \Delta)}{\sinh k\Delta}$$

$$\tilde{H}_y = -\frac{1}{\mu} \frac{\partial \tilde{A}}{\partial x} = -\frac{k}{\mu} \left[\frac{\tilde{A}^\alpha \cosh kx - \tilde{A}^\beta \cosh k(x - \Delta)}{\sinh k\Delta} \right]$$

$$\begin{bmatrix} \tilde{H}_y^\alpha \\ \tilde{H}_y^\beta \end{bmatrix} = \frac{k}{\mu} \begin{bmatrix} -\coth k\Delta & \frac{1}{\sinh k\Delta} \\ -\frac{1}{\sinh k\Delta} & \coth k\Delta \end{bmatrix} \begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix}$$

$$B_x = \frac{\partial A_z}{\partial y} \Rightarrow \tilde{B}_x = -jk\tilde{A} \Rightarrow \begin{bmatrix} \tilde{B}_x^\alpha \\ \tilde{B}_x^\beta \end{bmatrix} = -jk \begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix}$$

Vector potential transfer relations for two-dimensional or symmetric Laplacian fields.

Two-dimensional Cartesian	Polar	Axisymmetric cylindrical
		
$\vec{A} = \hat{i}_z \operatorname{Re} \tilde{A}(x) \exp(-jk y)$ $\begin{bmatrix} \tilde{H}_y^\alpha \\ \tilde{H}_y^\beta \end{bmatrix} = \frac{1}{\mu} k \begin{bmatrix} -\coth(k\Delta) \frac{1}{\sinh(k\Delta)} \\ \frac{-1}{\sinh(k\Delta)} \coth(k\Delta) \end{bmatrix} \begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} \quad (a)$ $\begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} = \frac{\mu}{k} \begin{bmatrix} -\coth(k\Delta) \frac{1}{\sinh(k\Delta)} \\ \frac{-1}{\sinh(k\Delta)} \coth(k\Delta) \end{bmatrix} \begin{bmatrix} \tilde{H}_y^\alpha \\ \tilde{H}_y^\beta \end{bmatrix} \quad (b)$	$\vec{A} = \hat{i}_z \operatorname{Re} \tilde{A}(r) \exp(-jm\theta)$ $\begin{bmatrix} \tilde{H}_\theta^\alpha \\ \tilde{H}_\theta^\beta \end{bmatrix} = \frac{1}{\mu} \begin{bmatrix} f_m(\beta, \alpha) g_m(\alpha, \beta) \\ g_m(\beta, \alpha) f_m(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} \quad (c)$ $\begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} = \mu \begin{bmatrix} F_m(\beta, \alpha) G_m(\alpha, \beta) \\ G_m(\beta, \alpha) F_m(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{H}_\theta^\alpha \\ \tilde{H}_\theta^\beta \end{bmatrix} \quad (d)$ <p>For f_m, g_m, F_m, G_m see Table 2.16.2, $k = 0, m \neq 0$</p>	$\vec{A} = \hat{i}_\theta \operatorname{Re} \tilde{A}(r) \exp(-jkz); \quad \vec{A} = \tilde{A} r$ $\begin{bmatrix} \tilde{H}_z^\alpha \\ \tilde{H}_z^\beta \end{bmatrix} = \frac{-k^2}{\mu} \begin{bmatrix} F_o(\beta, \alpha) G_o(\alpha, \beta) \\ G_o(\beta, \alpha) F_o(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} \quad (e)$ $\begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} = -\left(\frac{\mu}{k^2}\right) \begin{bmatrix} f_o(\beta, \alpha) g_o(\alpha, \beta) \\ g_o(\beta, \alpha) f_o(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{H}_z^\alpha \\ \tilde{H}_z^\beta \end{bmatrix} \quad (f)$ <p>For F_o, G_o, f_o, g_o see Table 2.16.2, $m = 0, k \neq 0$</p>
$\begin{bmatrix} \hat{A}^\alpha \\ \hat{A}^\beta \end{bmatrix} = \frac{1}{k} \begin{bmatrix} \hat{B}_x^\alpha \\ \hat{B}_x^\beta \end{bmatrix}$	$\begin{bmatrix} \hat{A}^\alpha \\ \hat{A}^\beta \end{bmatrix} = \frac{1}{m} \begin{bmatrix} \alpha \hat{B}_r^\alpha \\ \beta \hat{B}_r^\beta \end{bmatrix}$	$\begin{bmatrix} \hat{A}^\alpha \\ \hat{A}^\beta \end{bmatrix} = \frac{-j}{k} \begin{bmatrix} \hat{B}_r^\alpha \\ \hat{B}_r^\beta \end{bmatrix}$

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B. Polar Coordinates

$$\nabla^2 \bar{A} = \nabla^2 \left(A(r, \theta) \hat{i}_z \right) = 0 \Rightarrow \nabla^2 (A(r, \theta)) = 0$$

$$A(r, \theta) = \operatorname{Re} \left[\tilde{A}(r) e^{-jm\theta} \right]$$

$$\tilde{A}(r) = \frac{\tilde{A}^\alpha \left[\left(\frac{\beta}{r} \right)^m - \left(\frac{r}{\beta} \right)^m \right] + \tilde{A}^\beta \left[\left(\frac{r}{\alpha} \right)^m - \left(\frac{\alpha}{r} \right)^m \right]}{\left[\left(\frac{\beta}{\alpha} \right)^m - \left(\frac{\alpha}{\beta} \right)^m \right]}$$

$$H_\theta = -\frac{1}{\mu} \frac{\partial A}{\partial r}$$

$$H_r = \frac{1}{\mu r} \frac{\partial A}{\partial \theta} \Rightarrow \tilde{H}_r = \frac{-jm}{\mu r} \tilde{A}$$

C. Axisymmetric Cylindrical

$$\nabla^2 \bar{A} = \nabla^2 \left(A(r, z) \bar{i}_\theta \right) = 0 \Rightarrow \nabla^2 A_\theta - \frac{A_\theta}{r^2} = 0$$

$$\nabla^2 \left(A(r, \theta, z) \bar{i}_\theta \right) = \left[\frac{\partial}{\partial r} \left[\frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) \right] + \frac{1}{r^2} \frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} + \frac{\partial^2 A_\theta}{\partial z^2} \right] \bar{i}_\theta - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} \bar{i}_r$$

(Vector Laplacian)

$$\nabla^2 \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} + \frac{\partial^2 \Phi}{\partial z^2} \quad (\text{Scalar Laplacian})$$

$$A(r, z) = \text{Re} \left[\tilde{A}(r, t) e^{-jkz} \right]$$

$$\frac{d^2 \tilde{A}}{dr^2} + \frac{1}{r} \frac{d\tilde{A}}{dr} - \left(k^2 + \frac{1}{r^2} \right) \tilde{A} = 0 \Rightarrow \tilde{A} = B_1 I_1(kr) + B_2 K_1(kr)$$

$$= C_1 J_1(jkr) + C_2 H_1(jkr)$$

$$\nabla^2 \Phi = 0; \quad \Phi(r, \theta, z, t) = \text{Re} \left[\tilde{\Phi}(r, t) e^{-j(m\theta + kz)} \right]$$

$$\frac{d^2 \tilde{\Phi}}{dr^2} + \frac{1}{r} \frac{d\tilde{\Phi}}{dr} - \left(k^2 + \frac{m^2}{r^2} \right) \tilde{\Phi} = 0 \Rightarrow \tilde{\Phi} = A_1 I_m(kr) + A_2 K_m(kr)$$

$$\tilde{A} = \frac{\left\{ \tilde{A}^\alpha \left[H_1(jk\beta) J_1(jkr) - J_1(jk\beta) H_1(jkr) \right] + \tilde{A}^\beta \left[J_1(jk\alpha) H_1(jkr) - H_1(jk\alpha) J_1(jkr) \right] \right\}}{\left[J_1(jk\alpha) H_1(jk\beta) - H_1(jk\alpha) J_1(jk\beta) \right]}$$