6.302 Feedback Systems

Recitation 7: Root Locus

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To start with, let's make sure we're clear on exactly what we mean by the words "root locus plot." Webster can help us with this:

ROOT: "A number that reduces and equation to an identity when it is substituted for one variable."

The equation that we care about is $1 + L(s) = 0$, and the identity that it reduces to is $0 = 0$ when s is chosen to be a root. Roots of this equation are the closed-loop poles of the feedback system.

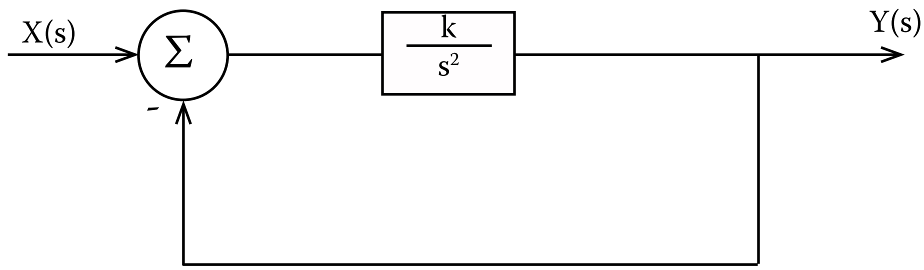
LOCUS: "The set of all points whose location is determined by stated conditions."

The "stated conditions" here are that $1 + kL_0(s) = 0$ for some value of k, and the "points" whose locations matter to us are points in the s-plane.

Put them together:

ROOT LOCUS: The set of all points in the s-plane that satisfy the equation $1 + kL_0(s) = 0$ for some value of k.

Can begin to construct such a plot for a simple system:



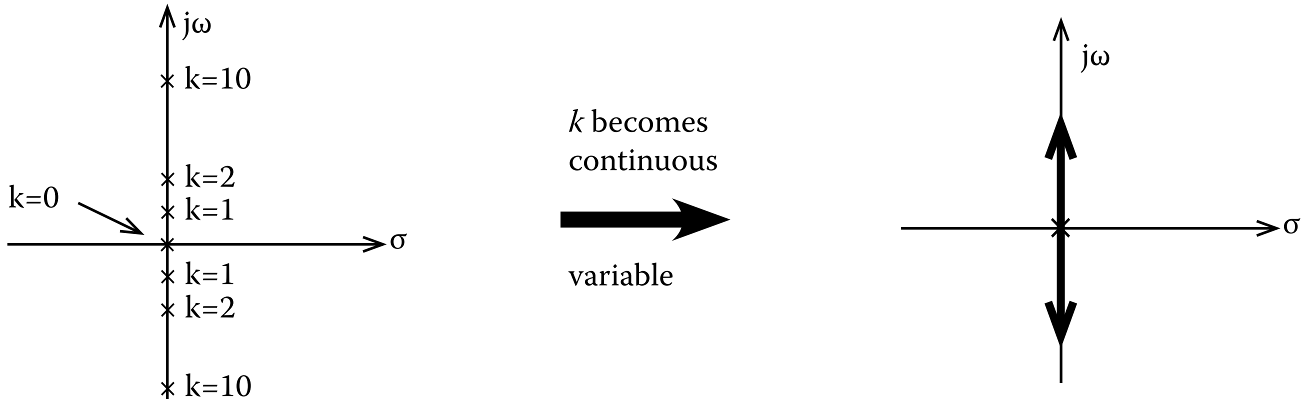
Where are the closed-loop poles?

$$\frac{Y(s)}{x(s)} = \frac{k/s^2}{1 + k/s^2} = \frac{k}{s^2 + k} \quad \rightarrow \quad s^2 = -k, \quad s = \pm j\sqrt{k}$$

Evidently, the answer depends on *k*. We can build a table:

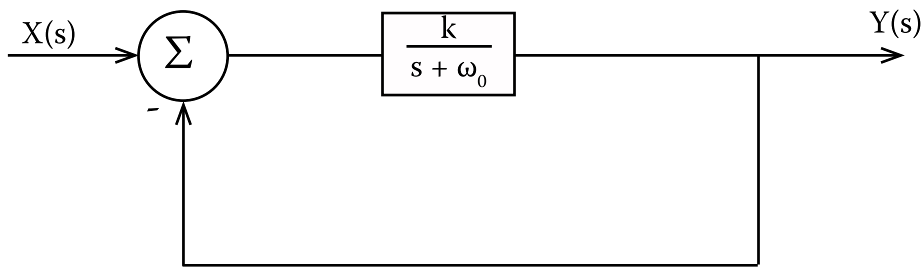
k	Pole Locations
0	0 , 0 (two poles @ origin)
1	$\pm j$
2	$\pm j\sqrt{2}$
10	$\pm j\sqrt{10}$

How would this look on a <u>parameterized</u> pole-zero diagram?



CLASS EXERCISE

Draw the root locus plot for the following system:



Now, on to the rules of root locus plotting. In the last two examples, we could do a plot just based on algebra. But what about when the order of the characteristic equation is too high for that? We use the rules, which all derive from

$$1 + kL_0(s) = 0$$
$$kL_0(s) = -1$$

This equation implied two things:

① $|\, kL_0(s) \,| = 1 \Rightarrow$ magnitude condition

② $\measuredangle kL_0(s) = -180^\circ$

or, for positive k

$\measuredangle L_0(s) = -180^\circ \Rightarrow$ angle condition

An astonishing number of rules can be derived from these simple conditions.

RULE #1: The number of branches, which are the paths of the closed-loop poles, is equal to the number of open-loop poles.

Characteristic equation:

$$P(s) = 1 + kL_0(s) = 0$$

Let $L_0(s) = \dfrac{n(s)}{d(s)}$

$$P(s) = 1 + k\,\frac{n(s)}{d(s)} = 0$$

$$\hookrightarrow d(s) + kn(s) = 0$$

$L_0(s)$ is a physical system → has at least as many poles as zeros → $d(s)$ is of order greater than or equal to $n(s)$ → the order of $d(s)$ is the order of $d(s) + kn(s)$.

In other words, closing a loop around something doesn't alter the number of poles.

RULE #2: The branches start at the open-loop poles, and end at open-loop zeros. (In addition to the z open-loop zeros in the loop transmission, there are p-z open-loop zeros at infinity.)

To prove this rule (#2), we rely on the magnitude condition:

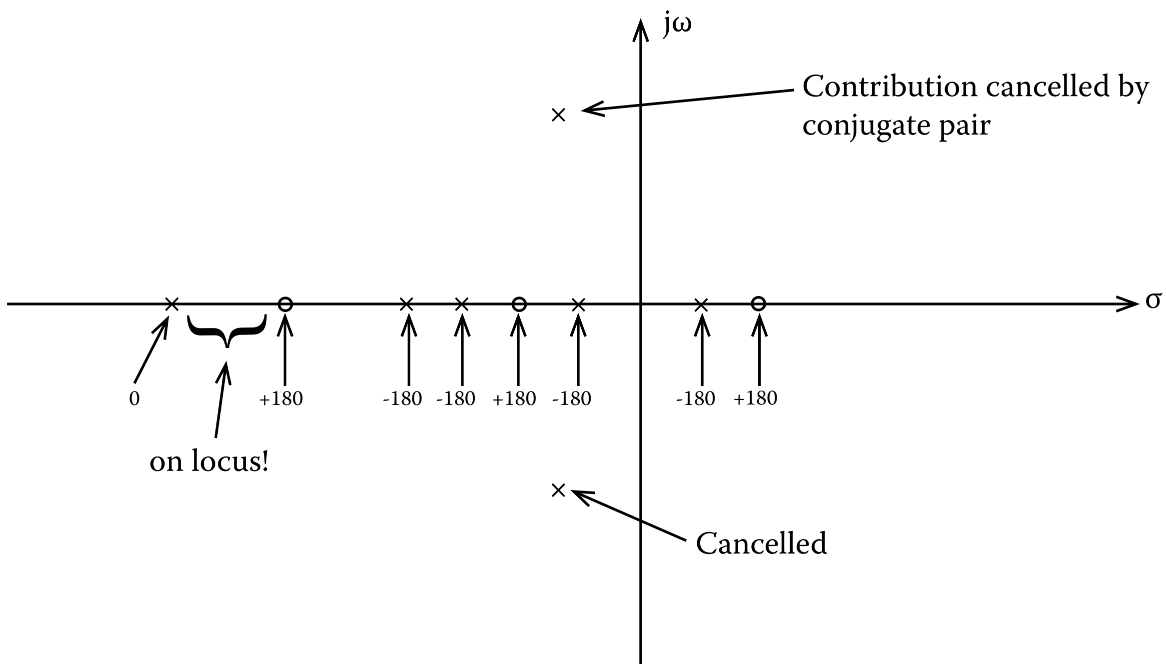
$$|\, kL_0(s) \,| = 1$$

We say "branches start out" when we're talking about small values of *k*, and talk of "branches ending" when *k* is LARGE. The thought experiment here is that we're beginning with *k* small, cranking it higher and higher, and watching where the poles go.

- For LARGE *k*, $L_0(s)$ must be small. For infinite *k*, $L_0(s)$ must be zero.
- For small *k*, $L_0(s)$ must be LARGE. For zero *k*, $L_0(s)$ must be infinite.

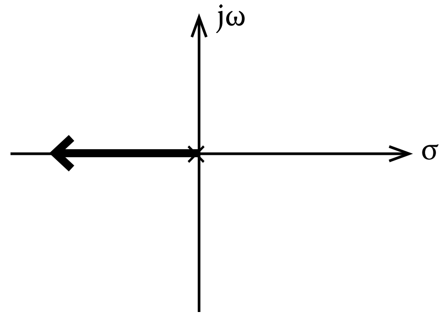
RULE #3: Branches of the root locus lie on the real axis to the left of an odd number of real poles and zeros. Complex-conjugate pairs of poles do not count, since on the real axis they contribute no net angle.

Angle condition: $\measuredangle L_0(s) = (2n + 1) \cdot 180^\circ$



Examples using what we have so far:

$$\boxed{L(s) = \frac{k}{s}}$$

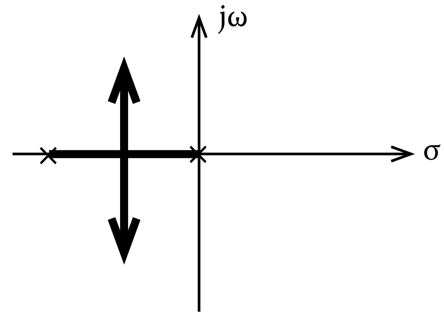


Rule 1: → 1 branch

Rule 2: → Goes off to infinity

Rule 3: → Locus on real axis to the left of one pole

$$\boxed{L(s) = \frac{k}{s(s+1)}}$$



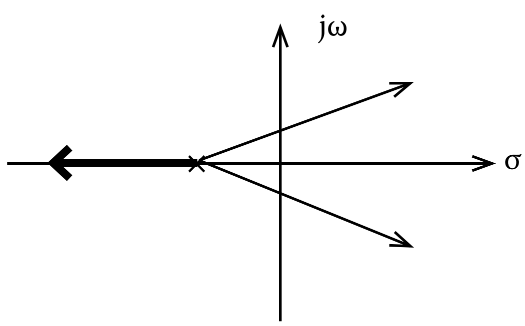
Rule 1: → 2 branches

Rule 2: → 2 poles go off to infinity

Rule 3: → Locus on real axis lies to the left of one pole

Rule 4: → (We don't know this one yet.)

$$\boxed{L(s) = \frac{k}{(s+1)^3}}$$



Rule 1: → 3 branches

Rule 2: → 3 poles go off to infinity

Rule 3: → Locus on real axis lies to the left of three poles

+ more rules

RULE #4: If a branch on the real axis lies between a pair of poles, the root locus must break away from the real axis somewhere between the poles. Similarly, if a branch on the real axis lies between a pair of zeros, there must be an entry point between that pair of zeros.

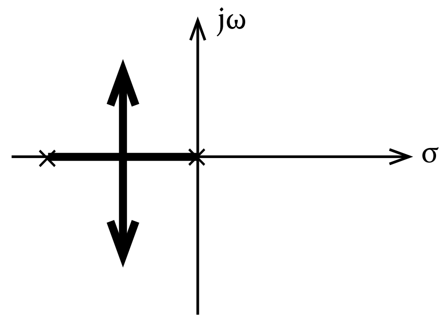
Back to characteristic equation:

$$1 + k\,\frac{n(s)}{d(s)} = 0$$

$$\underbrace{d(s) + kn(s)}_{\text{polynomial with real coefficients}} = 0$$

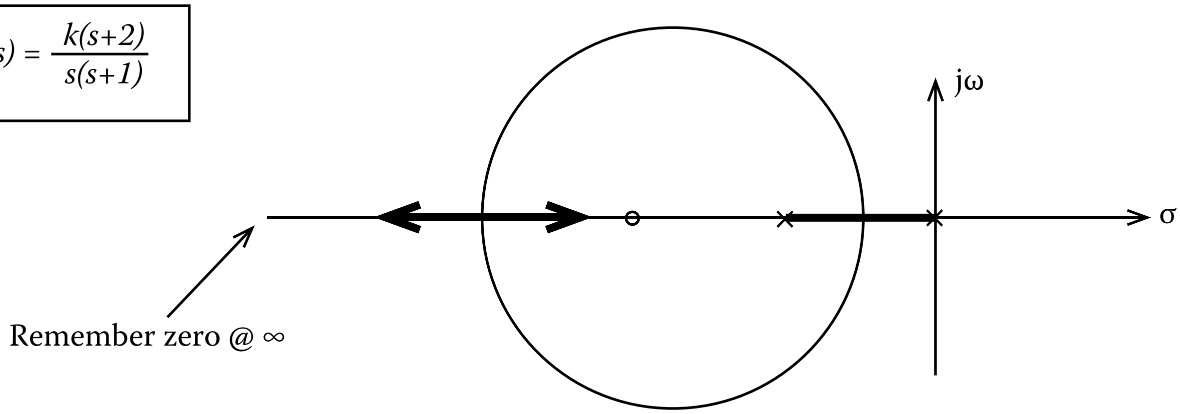
⇒ Poles occur only in conjugate pairs
⇒ Root locus symmetric about real axis

EXAMPLE: $\boxed{L(s) = \frac{k}{s(s+1)}}$



(Note: Poles could not simply pass through one another and remain on real axis.)

$\boxed{L(s) = \frac{k(s+2)}{s(s+1)}}$



RULE #5: As k gets very large, P-Z branches go off to infinity (rule 2). These branches approach asymptotes as angles to the real axis of

$$\alpha n = \frac{(2n+1)\,180^\circ}{P - Z}$$

Where $n = 0....(P\text{-}Z\text{-}1)$ and the centroid of these asymptotes is on the rear axis at

$$\sigma_a = \frac{\Sigma p_i - \Sigma z_i}{P - Z}$$

(offered w/out proof)

RULE #6: The departure angles of the branches from an m^{th} order pole on the rear axis are

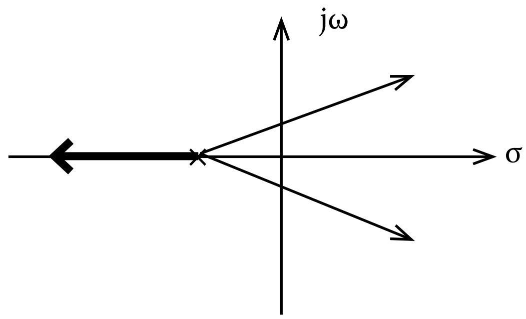
$$\delta_n = \frac{(2n+1)\,180^\circ}{m}$$

If the m^{th} order pole is to the left of an even number of poles and zeros. If to the left of an odd number, the departure angles are

$$\delta_n = \frac{2n180^\circ}{m}$$

KEY: In the vicinity of these poles, the angle contributed by all other singularities looks constant.

$$\boxed{L(s) = \frac{k}{(s+1)^3}}$$



EXAMPE: $L(s) = \dfrac{ks}{(s+1)^3}$

Rule 1: → 3 branches

Rule 2: → 2 branches go off to infinity, one heads toward zero

Rule 3: → Real axis between poles and zero is on the locus

Rule 4: → N/A

Rule 5: → -asymptotes are at 90°, 270°

-centroid at $\frac{-3-0}{2} = -1.5$

Rule 6: → Departure angles determined by $\delta_n = \frac{2n180^\circ}{m} = \{0, 120^\circ, 240^\circ\}$