# 6.302 Feedback Systems

Recitation 24: Describing Functions
Prof. Joel L. Dawson

We've spent almost all of our time on linear system theory and its consequences. We are lucky that, although all systems in nature are nonlinear, linear system theory gets us so far.

What are we to do when forced to confront nonlinearity head on? Some of the options:

① Linearize..and hope. ☺

  1.5) Linearize about a number of different points, and change our control strategy for each point.

② Use nonlinear analysis explicitly (nonlinear system theory is intensely formal. Insight, or general applicability, are hard to come by.)

③ Describing functions

④ Basic reasoning

In this recitation, we're going to talk about ③. But never forget ④: the concepts of control theory are bigger than any mathematical framework.

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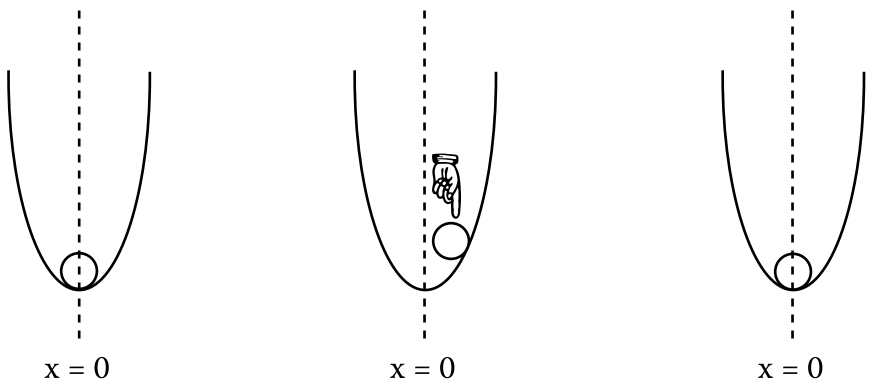
Before we go on, there's a concept that has crept into the class that observes careful treatment. With feedback systems, the word "stable" has a very definite meaning: all the poles of the system are in the left-half plane. But in other contexts, the word "stable" has a different meaning. A system is said to be in a stable "state" if the system tends to restore itself to that state in response to a perturbation. This is a general concept that shows up all over the place in engineering.

<u>EXAMPLE</u>:



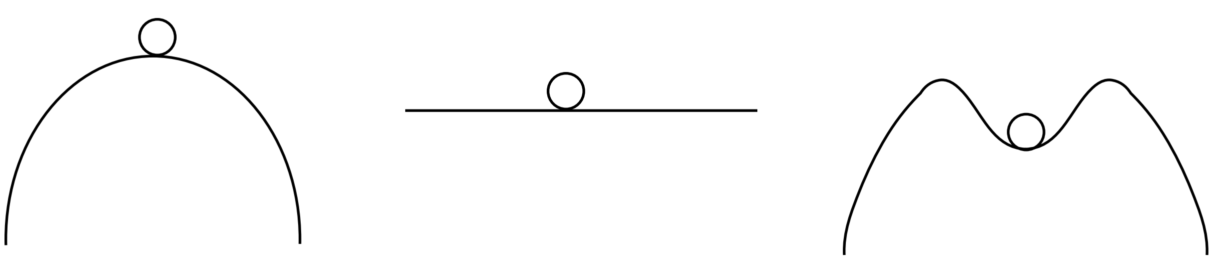

state: ball @ x = 0

perturbation: take my finger and move the ball

restorative force: remove disturbance, and system returns to x = 0

⇒ this x = 0 is a stable equilibrium.

<u>OTHER EXAMPLES</u>:



"unstable"

"marginally stable"

"stable for small perturbations"

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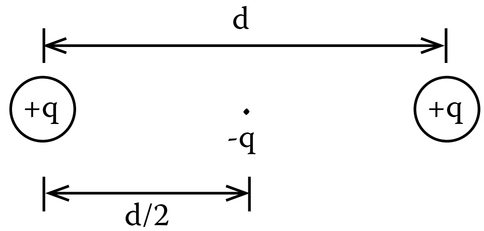
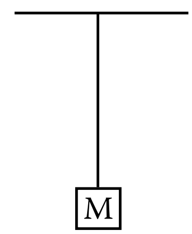
**CLASS EXERCISE:** Describe the stability of each of the following equilibria.

①




②

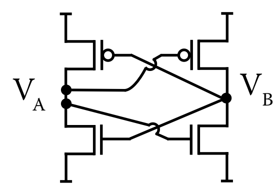



$\theta = 0$ stable or unstable?

③ $$\frac{dx}{dt} = -\sin(x - x_0)$$

Is $x_0$ a stable or unstable equilibrium?

④




Is $V_A = V_B$ a stable equilibrium?

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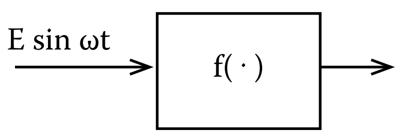
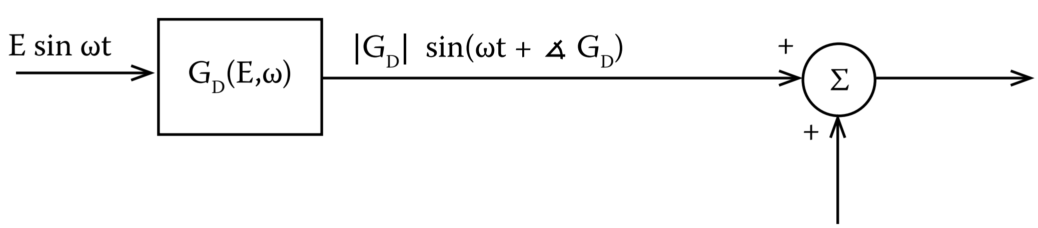
With describing function analysis, we make the following approximation, which I present here in three steps:

①




$$\left|G_D(E,\omega)\right| \sin[\omega t + \measuredangle G_D(E,\omega)] + \sum_{n=2}^{\infty} A_n \sin n\omega t + \sum_{n=2}^{\infty} B_n \cos n\omega t$$

②




$$\Sigma A_n \sin n\omega t + \Sigma B_n \cos n\omega t$$

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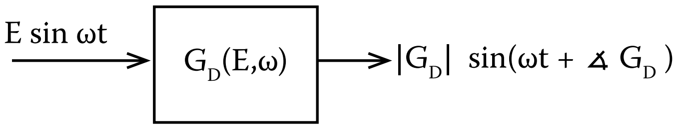
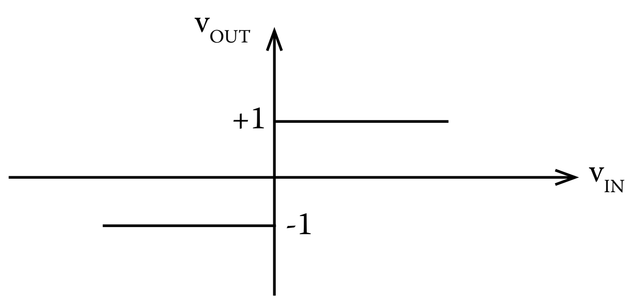
Now, assume harmonics are small:

③

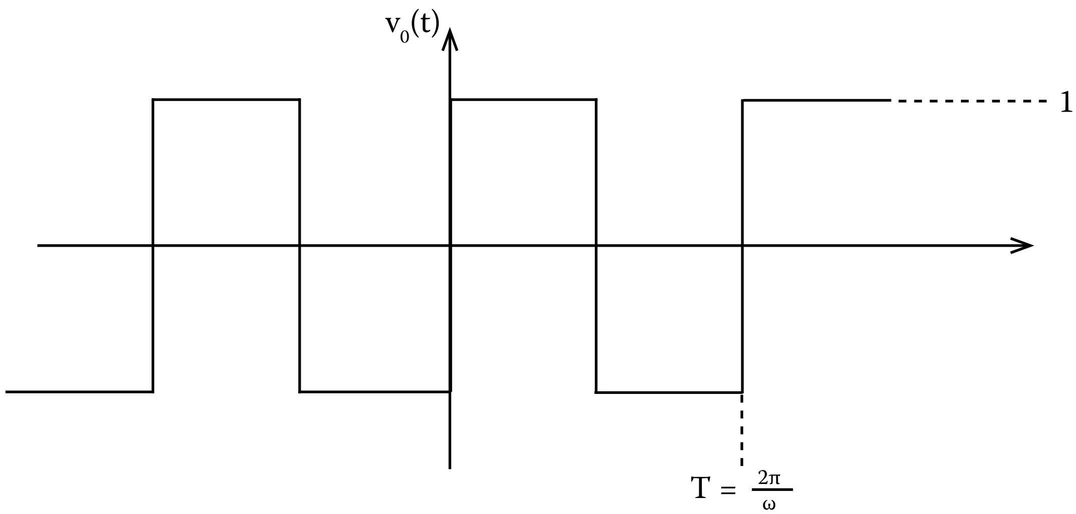



$\boxed{\text{We only keep track of the fundamental.}}$

Let's use this to look carefully at the describing function for a comparator:




In response to sine wave input Esinωt (period, $T = \frac{2\pi}{\omega}$), we get a square wave that looks as follows:




We put in a sine wave, and the nonlinear block returned that sine wave plus a bunch of harmonics. For describing function analysis, we only care about what happened to the fundamental.

Using Fourier Analysis, we can extract the fundamental component from the output wave form.

$$v_0(t) = \cancel{a_0}^{\,0} + \sum_{n=1}^{\infty} A_n \sin n\omega t + \sum_{n=1}^{\infty} B_n \cos n\omega t$$

($a_0$ is zero by inspection: the output waveform clearly has no DC component.)

We're intersted in $A_1$ and $B_1$:

$$A_1 = \frac{2}{T}\int_0^T v_0(t)\sin\omega t\,dt = \frac{2}{T}\int_0^{T/2}\sin\omega t\,dt - \frac{2}{T}\int_{T/2}^{T}\sin\omega t\,dt$$

$$= \frac{2}{T}\left[-\frac{1}{\omega}\cos\omega t\right]_0^{T/2} - \frac{2}{T}\left[-\frac{1}{\omega}\cos\omega t\right]_{T/2}^{T}$$

$$\vdots$$

$$= \frac{4}{\pi}$$

$$B_1 = \frac{2}{T}\int_0^T v_0(t)\cos\omega t\,dt = \cdots = 0$$

Since $B_1 = 0$, there is no phase shift.

$$\boxed{G_D(E) = \left(\frac{\sqrt{A_1^2 + B_1^2}}{\text{input amplitude}}\right) = \frac{4}{\pi E}}$$

Notice curious behavior: as the input amplitude gets larger, the <u>gain</u> gets smaller! This is useful in building constant-amplitude oscillators. $\Rightarrow$

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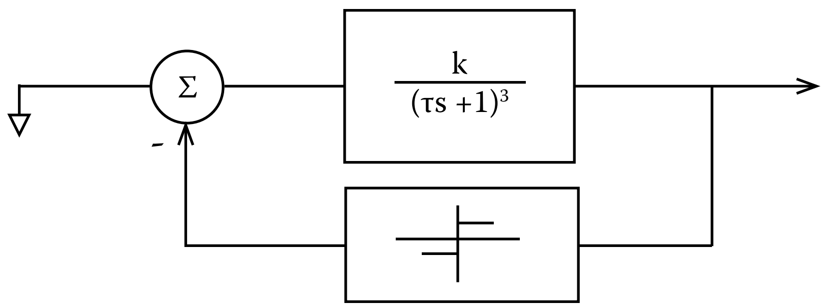
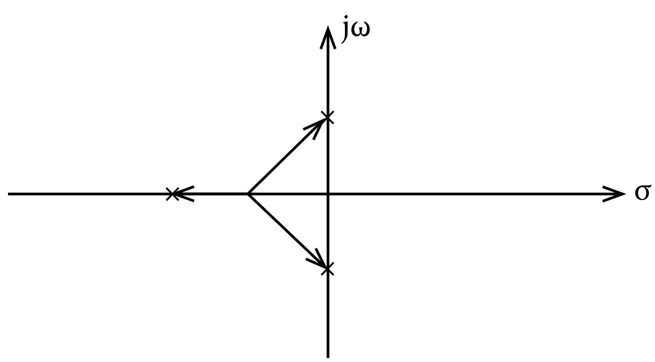
Consider the 3-pole oscillator we looked at in lecture.

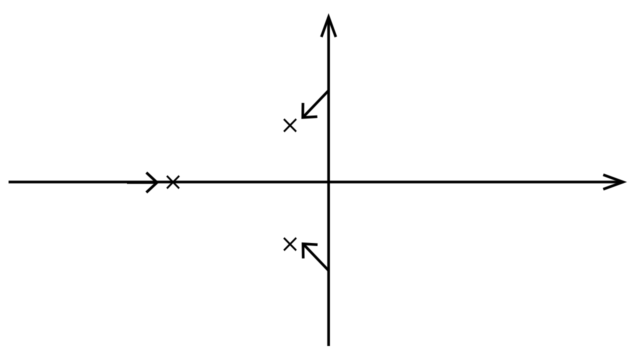



Ideally, we have two closed-loop poles sitting on jω axis:




We pick k to satisfy this condition. But what happens when we perturb the system and the amplitude gets bigger than intended? Does the system tend to restore itself?

Using our describing function, $G_D(E) = \frac{4}{\pi E}$ , notice that as E gets bigger, the gain gets smaller. So in response to a perturbation that makes the amplitude bigger, the root locus changes as follows:



Poles move into LHP and amplitude decreases. If we perturb system so amplitude gets smaller, poles move into RHP.

System fights to keep itself in a constant-amplitude oscillation.