


6	9	13	7
12		10	5
3	1	4	14
15	8	11	2

Mathematics for Computer Science  
MIT 6.042J/18.062J


# State Machines

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## State machines

step by step processes  
(may step in response  
to **input** —not today)

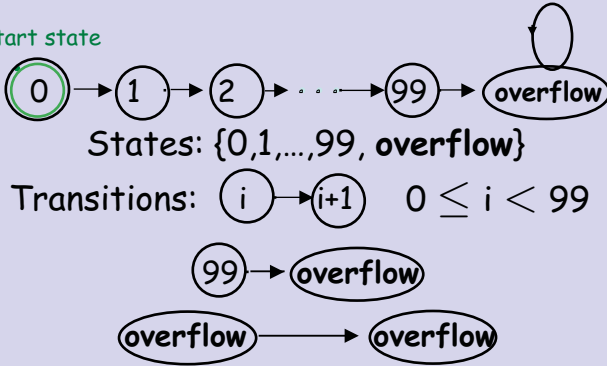
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6	9	13	7
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## State machines

The **state graph** of a 99-bounded counter:


start state



States:  $\{0, 1, \dots, 99, \text{overflow}\}$

Transitions:  $i \rightarrow i+1 \quad 0 \leq i < 99$

$99 \rightarrow \text{overflow}$   
 $\text{overflow} \rightarrow \text{overflow}$

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## Die Hard

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## Die Hard

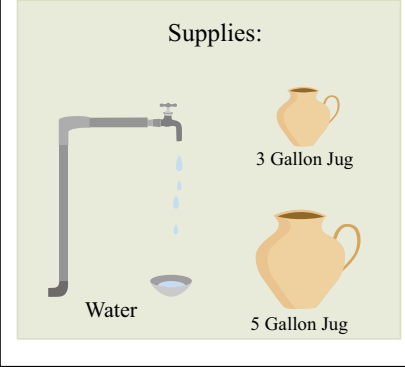
**Simon says:** On the fountain, there should be 2 jugs, do you see them? A 5-gallon and a 3-gallon. Fill one of the jugs with exactly 4 gallons of water and place it on the scale and the timer will stop. You must be precise; one ounce more or less will result in detonation. If you're still alive in 5 minutes, we'll speak.

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6	9	13	7
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## Die Hard

Supplies:



Water

3 Gallon Jug

5 Gallon Jug


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6	9	13	7
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## Die Hard

Transferring water:



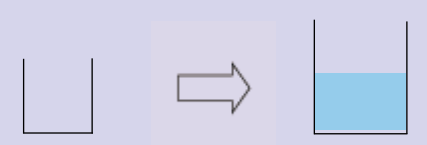
3 Gallon Jug                  5 Gallon Jug

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6	9	13	7
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## Die Hard

Transferring water:




3 Gallon Jug                  5 Gallon Jug

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6	9	13	7
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### Die hard state machine

State:  
amount of water in jugs:  $(b,l)$   
 $0 \leq b \leq 5, 0 \leq l \leq 3$   
Start State:  $(0,0)$


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### State machines

#### Die Hard Transitions:


1. Fill little jug:  $(b, l) \rightarrow (b, 3)$  for  $l < 3$
2. Fill big jug:  $(b, l) \rightarrow (5, l)$  for  $b < 5$
3. Empty little jug:  $(b, l) \rightarrow (b, 0)$  for  $l > 0$
4. Empty big jug:  $(b, l) \rightarrow (0, l)$  for  $b > 0$

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### State machines

5. Pour big jug into little jug
  - (i) If no overflow, then  $(b,l) \rightarrow (0,b+l)$   
 $b+l \leq 3$
  - (ii) otherwise  $(b,l) \rightarrow (b-(3-l),3)$
6. Pour little jug into big jug.  
Likewise


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### Die Hard

#### Simon's challenge:

Disarm the bomb by putting precisely 4 gallons of water on the scale, or it will blow up.  
(You can figure out how)

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## Die Hard


# Work it out now!

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## How to do it

Start with empty jugs: (0,0)  
Fill the big jug: (5,0)




3 Gallon Jug                  5 Gallon Jug

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6	9	13	7
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## How to do it

Pour from big to little: (2,3)




3 Gallon Jug                  5 Gallon Jug

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6	9	13	7
12		10	5
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## How to do it

Empty the little: (2,0)




3 Gallon Jug                  5 Gallon Jug

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6	9	13	7
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### How to do it

Pour from big to little: (0,2)




3 Gallon Jug      5 Gallon Jug

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6	9	13	7
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### How to do it

Fill the big jug: (5,2)




3 Gallon Jug      5 Gallon Jug

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6	9	13	7
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### How to do it

Pour from big to little: (4,3)



3 Gallon Jug      5 Gallon Jug

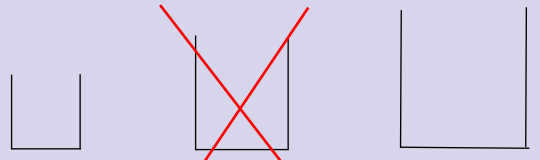
**Done!**

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6	9	13	7
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### Die Hard **once and for all**

What if have a **9** gallon jug instead?



3 Gallon Jug    ~~5 Gallon Jug~~    9 Gallon Jug

Can you do it? Can you prove it?

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4	9	13	7
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### Preserved Invariants

Die hard once and for all  
 preserved invariant:  
 $P(\text{state}) ::= \text{"3 divides the number of gallons in each jug."}$   
 $P((b,l)) ::= (3 | b \text{ AND } 3 | l)$

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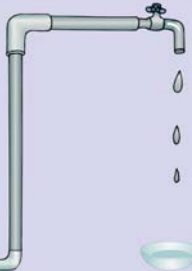
### Preserved Invariants

Die hard once and for all  
 preserved invariant:  
 $(b,l) \rightarrow (b-(3-l),3)$   
 $P((b,l)) ::= (3 | b \text{ AND } 3 | l)$

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4	9	13	7
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### Die Hard Once & For All



Corollary: No state  $(4,x)$  is reachable, so  
**Bruce Dies!**

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### Floyd's Invariant Principle

(induction for state machines)  
 Preserved Invariant,  $P(\text{state})$ :  
 if  $P(q)$  and  $q \rightarrow r$ , then  $P(r)$   
 Conclusion: if  $P(\text{start})$ , then  $P(r)$   
 for all reachable states  $r$ ,  
 including final state (if any)

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### The Diagonal Robot

the robot is on a grid

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### The Diagonal Robot

it can **move diagonally**

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### The Diagonal Robot

can it get from (0,0) to (1,0)?

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### Robot Preserved Invariant


**NO!** preserved invariant:  
 $P((x, y)) ::= x + y$  is even  
 move adds  $\pm 1$  to **both**  $x$  &  $y$ ,  
 preserving parity of  $x+y$ .  
 Also,  $P((0, 0))$  is true.

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### Robot Preserved Invariant


So in all positions  $(x,y)$  reachable from  $(0,0)$ ,  
 $x + y$  stays **even**  
 But  $1 + 0 = 1$  is odd, so  
 $(1,0)$  is **not reachable**

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### The Fifteen Puzzle Explained!

--by similar reasoning  
 details in problem 2

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### Fast Exponentiation

compute  $a^b$  using registers  $X, Y, Z, R$

```

X := a; Y := 1; Z := b;
REPEAT:
  if Z=0, then return Y
  R := remdr(Z, 2); Z := quotnt(Z, 2)
  if R=1, then Y := X·Y
  X := X2


```

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### Fast Exponentiation

**State Machine:**  
 States ::=  $\mathbb{R} \times \mathbb{R} \times \mathbb{N}$   
 start ::=  $(a, 1, b)$   
 transitions ::=  $(X, Y, Z) \rightarrow$   
 $(X^2, Y, \text{quotnt}(Z, 2))$  if  $Z > 0$  is even  
 $(X^2, X \cdot Y, \text{quotnt}(Z, 2))$  if  $Z > 0$  is odd

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### Fast Exponentiation

Preserved Invariant:  $YX^Z = a^b$   
 $(X, Y, Z) \rightarrow [Z > 0 \text{ is odd}]$   
 $(X^2, X \cdot Y, (Z-1)/2)$



$$(X \cdot Y)(X^2)^{(Z-1)/2} = (X \cdot Y)X^{Z-1}$$

$$= YX^Z = a^b$$


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### Partial Correctness


preserved invariant:  $YX^Z = a^b$   
 at start  $1 \cdot a^b = a^b$    
 at end  $Z=0$ , so return  
 $Y = YX^0 = a^b$  

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### Fast Termination

at each transition  
 $Z := \text{quotient}(Z, 2)$   
 $Z = b$  at start, so  $Z = 0$   
 in  $\leq \log_2(b)$  transitions



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### Robert W Floyd (1934–2001)

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