Mathematics for Computer Science
MIT 6.042J/18.062J

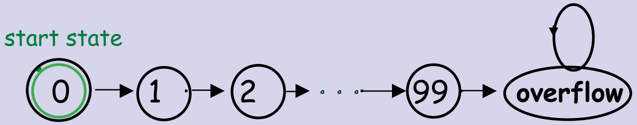
State Machines

State machines

step by step processes (may step in response to input —not today)

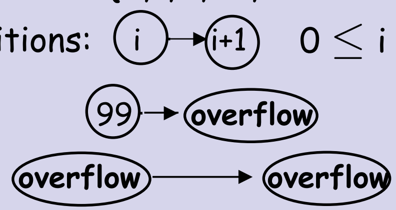
State machines

The state graph of a 99-bounded counter:



States: {0,1,...,99, **overflow**}

Transitions:



$0 \leq i < 99$

Die Hard

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Die Hard

Simon says: On the fountain, there should be 2 jugs, do you see them? A 5-gallon and a 3-gallon. Fill one of the jugs with exactly 4 gallons of water and place it on the scale and the timer will stop. You must be precise; one ounce more or less will result in detonation. If you're still alive in 5 minutes, we'll speak.

Die Hard

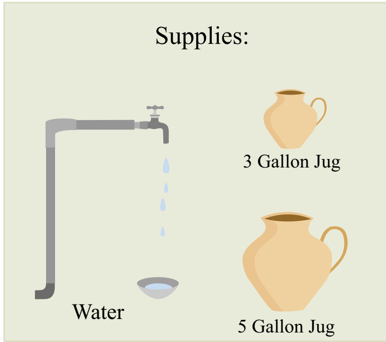
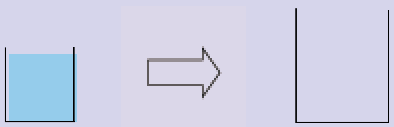


Image by MIT OpenCourseWare.

Die Hard

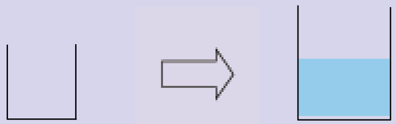
Transferring water:



3 Gallon Jug 5 Gallon Jug

Die Hard

Transferring water:



3 Gallon Jug 5 Gallon Jug

Die hard state machine

State:

amount of water in jugs: (b,l)

$0 \leq b \leq 5, 0 \leq l \leq 3$

Start State: (0,0)

State machines

Die Hard Transitions:

1. Fill little jug: (b, l) → (b, 3) for $l < 3$
2. Fill big jug: (b, l) → (5, l) for $b < 5$
3. Empty little jug: (b, l) → (b, 0) for $l > 0$
4. Empty big jug: (b, l) → (0, l) for $b > 0$

State machines

5. Pour big jug into little jug

(i) If no overflow, then (b,l)→(0,b+l)

$b+l \leq 3$

(ii) otherwise (b,l) →(b−(3−l),3)

6. Pour little jug into big jug. Likewise

Die Hard

Simon's challenge:
Disarm the bomb by putting precisely 4 gallons of water on the scale, or it will blow up.

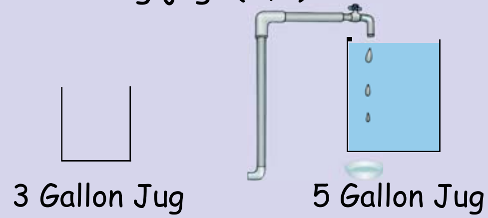
(You can figure out how)

Die Hard

Work it out now!

How to do it

Start with empty jugs: (0,0)
Fill the big jug: (5,0)



How to do it

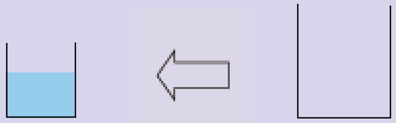
Pour from big to little: (2,3)

How to do it

Empty the little: (2,0)

How to do it

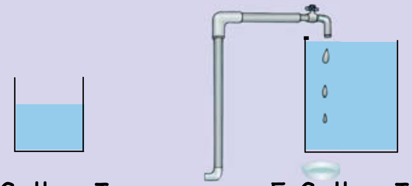
Pour from big to little: (0,2)



3 Gallon Jug 5 Gallon Jug

How to do it

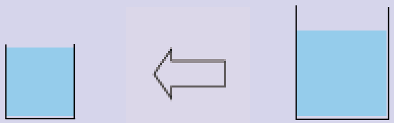
Fill the big jug: (5,2)



3 Gallon Jug 5 Gallon Jug

How to do it

Pour from big to little: (4,3)

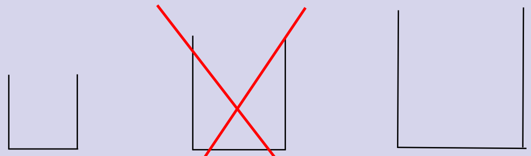


3 Gallon Jug 5 Gallon Jug

Done!

Die Hard once and for all

What if have a 9 gallon jug instead?



3 Gallon Jug ~~5 Gallon Jug~~ 9 Gallon Jug

Can you do it? Can you prove it?

Preserved Invariants

Die hard once and for all

preserved invariant:

P(state) ::= "3 divides the number of gallons in each jug."

$P((b,l)) ::= (3 \mid b \text{ AND } 3 \mid l)$

Preserved Invariants

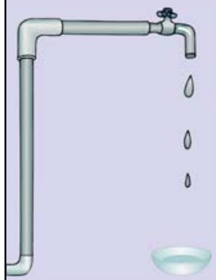
Die hard once and for all

preserved invariant:

$(b,l) \rightarrow (b-(3-l),3)$

$P((b,l)) ::= (3 \mid b \text{ AND } 3 \mid l)$

Die Hard Once & For All



Corollary: No state (4,x) is reachable, so Bruce Dies!

Floyd's Invariant Principle

(induction for state machines)

Preserved Invariant, P(state):

if P(q) and (q) → (r) , then P(r)

Conclusion: if P(start), then P(r)

for all reachable states r,

including final state (if any)

The Diagonal Robot

the robot is on a grid

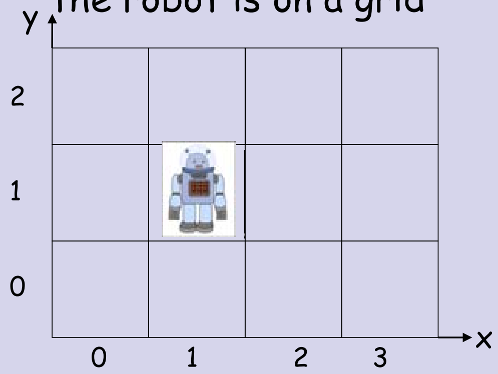


Image by MIT OpenCourseWare.

The Diagonal Robot

it can move diagonally

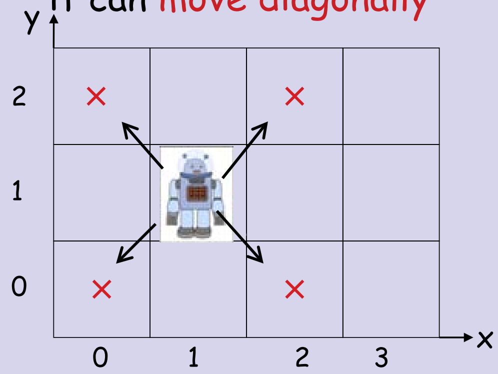


Image by MIT OpenCourseWare.

The Diagonal Robot

can it get from (0,0) to (1,0)?

Image by MIT OpenCourseWare.

Robot Preserved Invariant

NO! preserved invariant:

$P((x, y))$ ::= $x + y$ is even

move adds ±1 to **both** x & y, preserving parity of x+y.

Also, $P((0, 0))$ is true.

Robot Preserved Invariant

So in all positions (x,y) reachable from (0,0),

x + y stays even

But 1 + 0 = 1 is odd, so

(1,0) is not reachable

The Fifteen Puzzle Explained!

--by similar reasoning details in problem 2

Fast Exponentiation

compute a^b using registers **X,Y,Z,R**

```
X:= a;  Y:= 1;  Z:= b;
REPEAT:
if Z=0, then return Y
R:= remdr(Z,2); Z:= quotnt(Z,2)
if R=1,then Y:= X·Y
X:= X^2
```


Fast Exponentiation

State Machine:

States ::= $\mathbb{R} \times \mathbb{R} \times \mathbb{N}$

start ::= (a,1,b)

transitions ::= $(X,Y,Z) \rightarrow$

$(X^2, Y, \text{quotnt}(Z,2))$ if $Z>0$ is even

$(X^2, X \cdot Y, \text{quotnt}(Z,2))$ if $Z>0$ is odd

Fast Exponentiation

Preserved Invariant: $YX^Z = a^b$

$(X,Y,Z) \rightarrow$ [Z>0 is odd]

$(X^2, X\cdot Y, (Z-1)/2)$

$(X\cdot Y)(X^2)^{(Z-1)/2} = (X\cdot Y)X^{Z-1}$

$= YX^Z = a^b$

O.K.

Partial Correctness

preserved invariant: $YX^Z = a^b$

at start $1\cdot a^b = a^b$ O.K.

at end Z=0, so return

$Y = YX^0 = a^b$ O.K.

Fast Termination

at each transition

Z:= quotient(Z,2)

Z= b at start, so Z= 0

in $\leq \log_2(b)$ transitions

O.K.

Robert W Floyd (1934–2001)

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