Mathematics for Computer Science

MIT 6.042J/18.062J

Set Theory: Russell Paradox

Self application

Self application is notoriously doubtful:

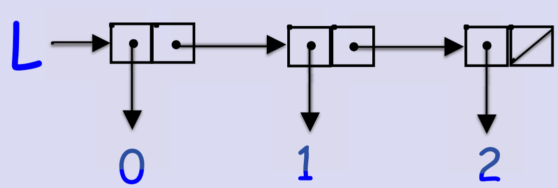
"This statement is false."

is it true or false?

Self membership

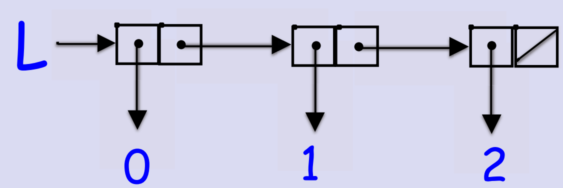
The list

L = (0 1 2)



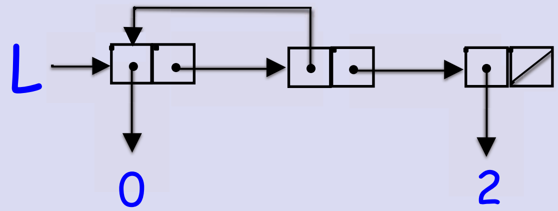
Self membership

(setcar (second L) L)



Self membership

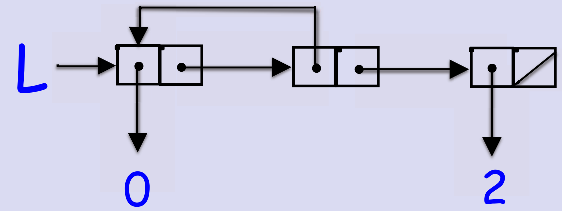
(setcar (second L) L)



Self membership

Lists are member of themselves:

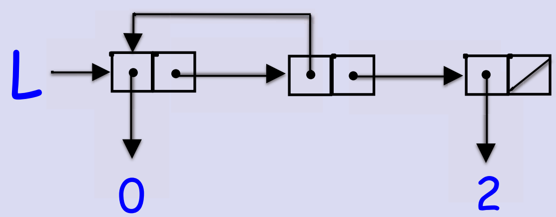
L = (0 L 2)



Self membership

Lists are member of themselves:

L = (0 (0 (0…2) 2) 2)



Self application

compose procedures

```
(define (compose f g)
  (define (h x)
    (f (g x)))
  h)
```

Self application

compose procedures

((compose square add1) 3)

$\Longrightarrow$ 16 (= $(3 + 1)^2$)

((compose square square) 3)

$\Longrightarrow$ 81 (= $(3^2)^2$)

Self application

compose procedures

(define (comp2 f)
(compose f f))

((comp2 square) 3)

$\Longrightarrow$ 81

Self application

apply procedure to itself:

(((comp2 comp2) add1) 3)

$\Longrightarrow$ 7

(((comp2 comp2) square) 3)

$\Longrightarrow$ 43046721 (= 3^{16})

Russell's Paradox

Let $W ::= \{s \in \text{Sets} \mid s \notin s\}$

so $[s \in W \text{ IFF } s \notin s]$

Now let s be W, and reach a contradiction:

$[W \in W \text{ IFF } W \notin W]$

Disaster: Math is broken!

I am the Pope,

Pigs fly,

and verified programs crash...

...but paradox is buggy

Assumes that W is a set!

$$\left[s \in W \text{ IFF } s \notin s \right]$$

for all sets s

...can only substitute W for s if W is a set

...but paradox is buggy

Assumes that W is a set!

We can avoid the paradox, if we deny that W is a set!

...which raises the key question: just which well-defined collections are sets?

Zermelo-Frankel Set Theory

No simple answer, but the axioms of Zermelo-Frankel along with the Choice axiom (ZFC) do a pretty good job.