LECTURE 13A

- Readings: Section 4.3, 4.4

Lecture outline

- Sum of a random number of independent random variables:
  - mean, variance, transform

Bookstore Example (1)

- George visits a number of book stores looking for the "Hair Book".
- A bookstore caries such a book with probability $\frac{1}{3}$.
- The time George spends in each book store is exponentially distributed with $\lambda = 3$.
- George will visit bookstores until he finds the book.
- We want to find the PDF, mean, variance of the time he spends in bookstores.
- Total time: $Y = X_1 + X_2 + \cdots + X_N$

Sum of a Random Number of Independent Random Variables

- N : nonnegative integer-valued r.v.
- $X_1, X_2, \cdots$: i.i.d. r.v.s, independent of N .
- Let: $Y = X_1 + \cdots + X_N$. Then:
- **Mean**: $$\begin{aligned}\mathbf{E}[Y] &= \mathbf{E}[\mathbf{E}[Y|N]] \\ &= \mathbf{E}[N\mathbf{E}[X]] \\ &= \underline{\underline{\mathbf{E}[N]\mathbf{E}[X]}}\end{aligned}$$
- **Variance**:

$$\begin{aligned}\text{Var}(Y) &= \mathbf{E}[\text{Var}(Y|N)] + \text{Var}(\mathbf{E}[Y|N]) \\ &= \underline{\underline{\mathbf{E}[N]\text{Var}(X) + (\mathbf{E}[X])^2\text{Var}(N)}}\end{aligned}$$

Bookstore Example (2)

- Number of bookstores, N :
  - **PMF** $p_N(n) = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$ (geometric, from *n*=1)
  - **Mean** $\mathbf{E}[N] = \frac{1}{\frac{1}{3}} = 3$
  - **Variance** $\mathrm{Var}(N) = \frac{1-\frac{1}{3}}{(\frac{1}{3})^2} = 6$
- Time in each bookstore, X (i.i.d., indep of N):
  - **PDF** $f_X(x) = 3e^{-3x}$ $x \geq 0$
  - **Mean** $\mathbf{E}[X] = \frac{1}{3}$
  - **Variance** $\mathrm{Var}(X) = \frac{1}{9}$
- Total time, Y :
  - **Mean** $\mathbf{E}[Y] = \mathbf{E}[N]\mathbf{E}[X] = 1$
  - **Variance** $\mathrm{Var}(Y) = \mathbf{E}[N]\mathrm{Var}(X) + (\mathbf{E}[X])^2\mathrm{Var}(N)$

    $= 1$

Review of Transforms

- Definitions: $M_X(s) = \mathbf{E}[e^{sX}] = \begin{cases} \sum_x e^{sx} p_X(x) \\ \int_{-\infty}^{\infty} e^{sx} f_X(x)dx \end{cases}$
- Moment generating properties:

$$\frac{d^n}{ds^n} M_X(s)\bigg|_{s=0} = \mathbf{E}[X^n]$$

- Transform of sum of independent r.v.s:

X, Y independent $\qquad W = X + Y$

$$M_W(s) = M_X(x) M_Y(s)$$

Transform of "Random Sum"

- N : nonnegative integer-valued r.v.
- $X_1, \cdots, X_N$: i.i.d. r.v.s, independent of N .
- If $Y = X_1 + \cdots + X_N$, we have:

$$\begin{aligned} M_Y(s) &= \mathbf{E}[e^{sY}] \\ &= \mathbf{E}\left[\mathbf{E}[e^{sY}|N]\right] \\ &= \mathbf{E}\left[\mathbf{E}[e^{s(X_1+\cdots+X_N)}|N]\right] \\ &= \mathbf{E}\left[M_X(s)^N\right] \end{aligned}$$

- Compare with: $M_N(s) = \mathbf{E}[(e^s)^N]$
- Thus, to get $M_Y(s)$, start with $M_N(s)$ and replace each occurrence of e^s by $M_X(s)$.

Bookstore Example (3)

- Number of bookstores:
  - **Transform** $M_N(s) = \dfrac{e^s/3}{1 - 2e^s/3} = \mathbf{E}\left[(e^s)^N\right]$

(compare)

- Time in each bookstore:
  - **Transform** $M_X(s) = \dfrac{3}{3 - s}$

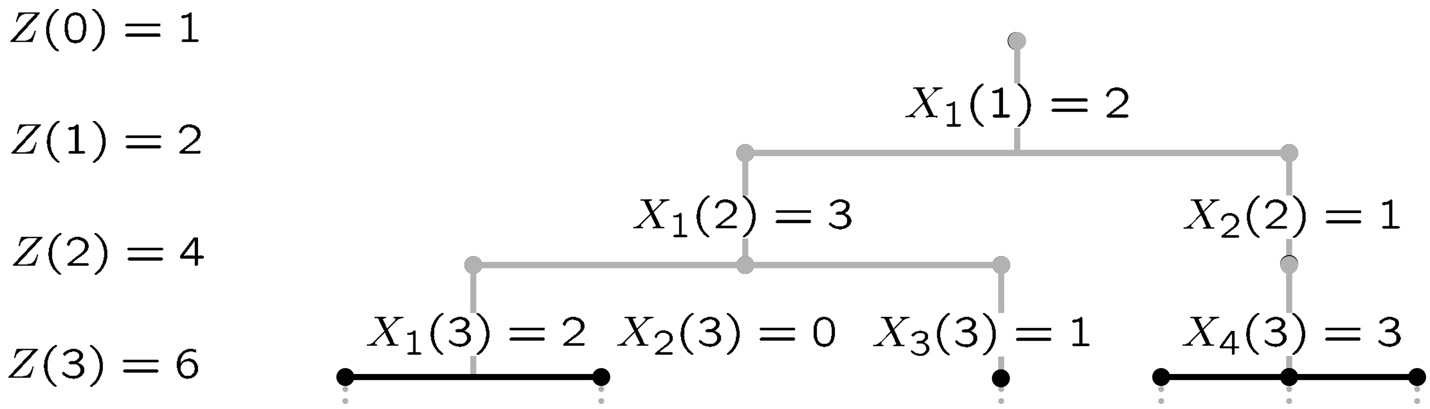
- Total time:
  - **Transform** $M_Y(s) = \mathbf{E}\left[M_X(s)^N\right]$

$$= \frac{\left(\frac{3}{3-s}\right)/3}{1 - 2\left(\frac{3}{3-s}\right)/3} = \frac{1}{1 - s}$$

  - **PDF:** $f_Y(y) = e^{-y} \quad y \geq 0$ (exponential, with $\lambda = 1$)

Motivational Example

- **Branching Process**:
  - Evolution, growth of a population of cells, increase of neutrons in a reactor, spread of an epidemic…



$$Z(t) = X_1(t) + X_2(t) + \cdots + X_{Z(t-1)}(t)$$

- $Z(0) = 1$ and $X_i(t)$ i.i.d. geometric, incl. zero.
- We need: mean, variance, PMF of $Z(t)$.

Branching Process: Mean

- Recall: $Z(t) = X_1(t) + \cdots + X_{Z(t-1)}(t)$
- For time step t : $N = Z(t-1)$

  $Y = Z(t) = X_1 + \cdots + X_N$
- $X_i(t)$ i.i.d.: $p_X(x) = p(1-p)^x \quad x = 0, 1, \cdots$

  $E[X] = \mu = \frac{1-p}{p}$ $\quad$ $\text{Var}(X) = \sigma^2 = \frac{1-p}{p^2}$
- **Mean** (using previous slide):

$$\mathbf{E}[Z(t)] = \mathbf{E}[Z(t-1)]\mu$$

- Solve recursively, e.g.: $\mathbf{E}[Z(t)] = \mu^t$

Branching Process: Variance

$$\text{Var}(Z(t)) = \mathbf{E}[Z(t-1)]\sigma^2 + \mu^2\text{Var}(Z(t-1))$$

$$= \mu^2\text{Var}(Z(t-1)) + \mu^{t-1}\sigma^2$$

$$= \begin{cases} t\sigma^2 & \mu = 1 \\ \frac{\sigma^2\mu^{t-1}(\mu^t - 1)}{\mu - 1} & \mu \neq 1 \end{cases}$$

Branching Process: Transforms

$$Z(t) = X_1(t) + \cdots + X_{Z(t-1)}(t)$$

- Recall, for time step t :

$$N = Z(t-1) \qquad Y = Z(t) = X_1 + \cdots + X_N$$

- Thus, to get $M_{Z(t)}(s)$, start with $M_{Z(t-1)}(s)$ and replace each occurrence of e^s by $M_X(s)$, where:

$$p_X(x) = p(1-p)^x \quad (x = 0, 1, \cdots) \iff M_X(s) = \frac{p}{1-(1-p)e^s}$$

$$M_{Z(0)}(s) = e^s \qquad p_{Z(0)}(z) = 1 \text{ if } z = 1$$

$$M_{Z(1)}(s) = \frac{p}{1-(1-p)e^s} \qquad p_{Z(1)}(z) = p_X(z)$$

$$M_{Z(2)}(s) = \frac{p}{1-(1-p)\frac{p}{1-(1-p)e^s}} = \frac{p[1-(1-p)e^s]}{1-p(1-p)-(1-p)e^s}$$

Challenge

- For $p = .5$
- Show that

$$P(Z(n) = 0) = \frac{n}{n+1}$$