LECTURE 11

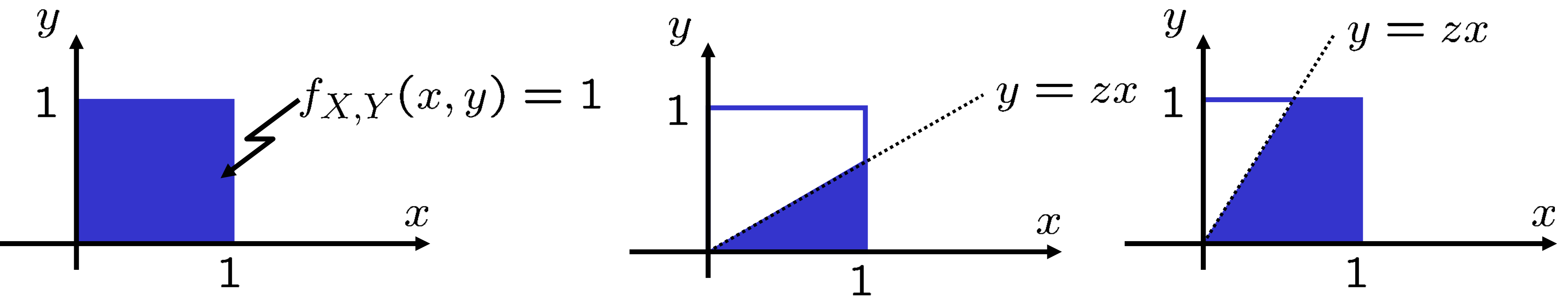
- Readings: Review Chapter 3
Section 4.2

Lecture outline

- More on continuous r.v.s
- More on derived distributions

Derived Distributions: Original Example

- Let X, Y be independent, uniform:



- Find the CDF of $Z = g(X,Y) = Y/X$

$$F_Z(z) = \quad z/2 \qquad 0 \leq z \leq 1$$

$$F_Z(z) = \quad 1 - 1/2z \qquad z \geq 1$$

- Differentiate to obtain PDF.

Example: Difference of Exp r.v.s (1)

- Romeo and Juliet are to meet. Romeo is late by X minutes. Juliet is late by Y.

- Model: X, Y independent.

$$f_X(x) = \lambda e^{-\lambda x},\ x \geq 0$$

$$f_Y(y) = \lambda e^{-\lambda y},\ y \geq 0$$

- Let $Z = X - Y$. Find $f_Z(z)$.

Example: Difference of Exp r.v.s (2)

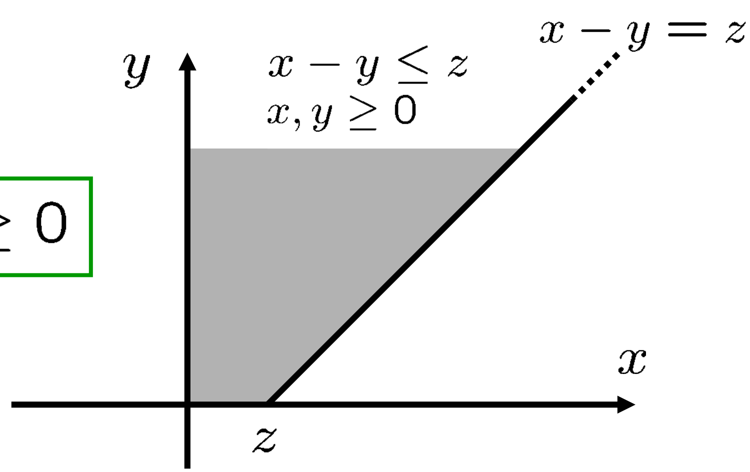
- We have: $f_{X,Y}(y) = \lambda^2 e^{-\lambda(x+y)} \quad x, y \geq 0$

  $Z = X - Y$

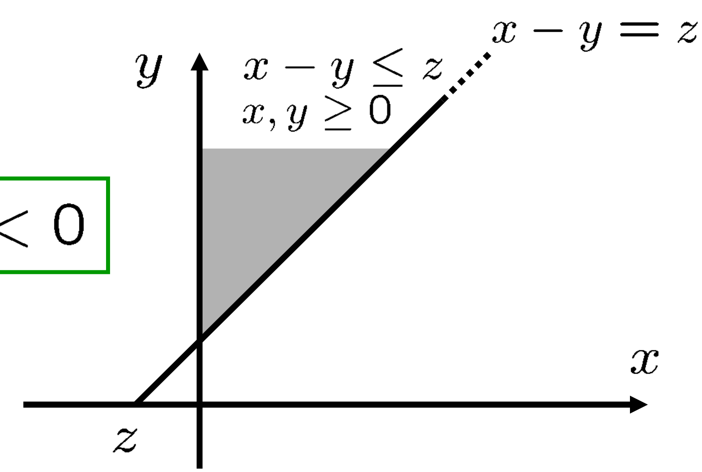
- Compute $F_Z(z) = \mathrm{P}(X - Y \leq z)$

  Integration region varies for two cases:

$z \geq 0$



$z < 0$



Example: Difference of Exp r.v.s (3)

- Thus, for $z < 0$:

$$F_Z(z) = \mathbf{P}(X - Y \leq z)$$

$$= \int_0^\infty \left(\int_{x-z}^\infty f_{X,Y}(x,y) dy \right) dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \left(\int_{x-z}^\infty \lambda e^{-\lambda y} dy \right) dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} e^{-\lambda(x-z)} dx \qquad = \frac{1}{2} e^{\lambda z}$$

- Fact: $Z \sim -Z$ (same distribution). So, for $z \geq 0$:

$$F_Z(z) = \mathbf{P}(Z \leq z) = \mathbf{P}(-Z \geq -z) = \mathbf{P}(Z \geq -z)$$

$$= 1 - F_Z(-z) \quad = 1 - \frac{1}{2} e^{-\lambda z}$$

Example: Difference of Exp r.v.s (4)

- We thus have:

$$F_Z(z) = \begin{cases} 1 - \frac{1}{2}e^{-\lambda z}, & \text{if } z \geq 0 \\ \frac{1}{2}e^{\lambda z}, & \text{if } z < 0 \end{cases}$$

- Differentiate:

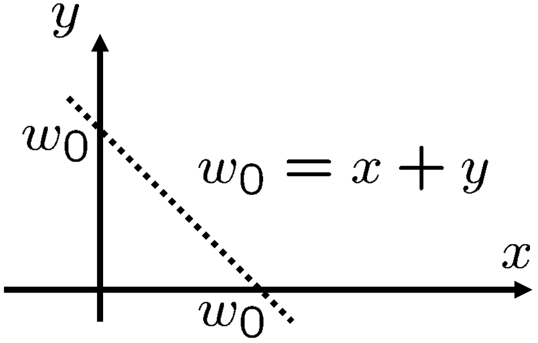
$$f_Z(z) = \begin{cases} \frac{\lambda}{2}e^{-\lambda z}, & \text{if } z \geq 0 \\ \frac{\lambda}{2}e^{\lambda z}, & \text{if } z < 0 \end{cases}$$

- Rewrite, to obtain a two-sided exponential PDF:

$$f_Z(z) = \frac{\lambda}{2}e^{-\lambda |z|}$$

The distribution of $X + Y$.

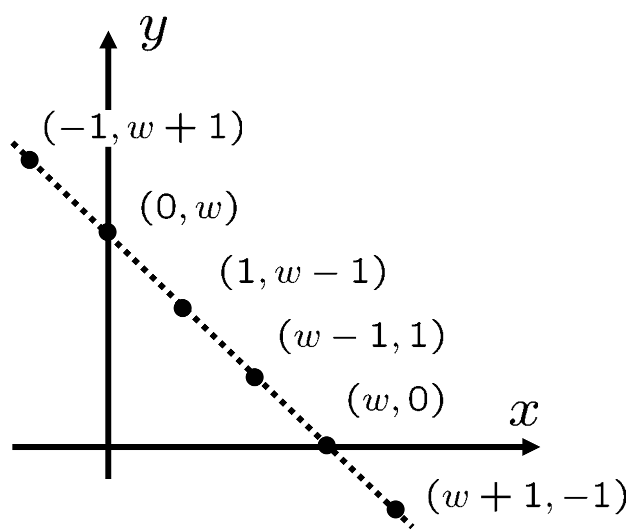
- Let X, Y be two r.v.s, and let $W = X + Y$
- Points where the value $W = w_0$ is some constant lie on the following line:



- Idea
  - Discrete case: add probabilities of all points on this line.
  - Continuous case: integrate the joint density on this line.

$X + Y$: Independent Discrete Integers

- Let X, Y be integer-valued, independent.
- Then $W = X + Y$ is also integer-valued.
- Picture:



- Thus:

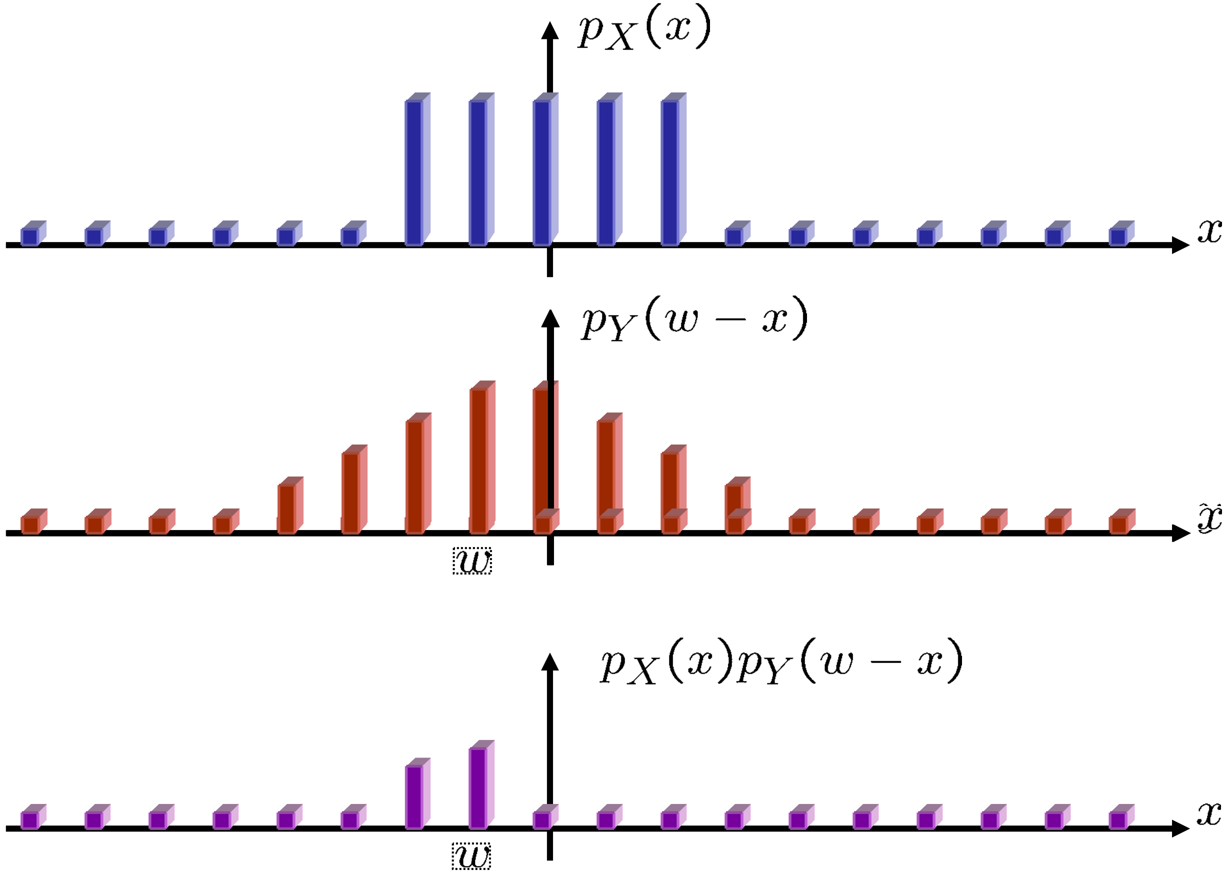
$$p_W(w) = \mathbf{P}(X + Y = w)$$

$$= \sum_x \mathbf{P}(X = x)\mathbf{P}(Y = w - x)$$

$$= \sum_x p_X(x) p_Y(w - x)$$

Obtaining $p_W(w)$ by convolution (1)

$$p_W(w) = \sum_x p_X(x) p_Y(w-x)$$

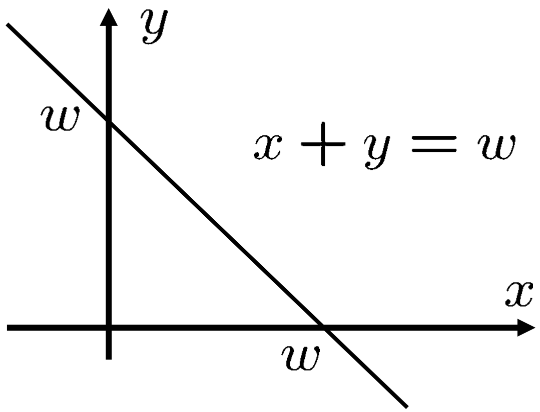


$X + Y$: Independent Continuous

- Let X, Y be independent, continuous r.v.s:

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$$

- Then the density of $W = X + Y$ is given by:

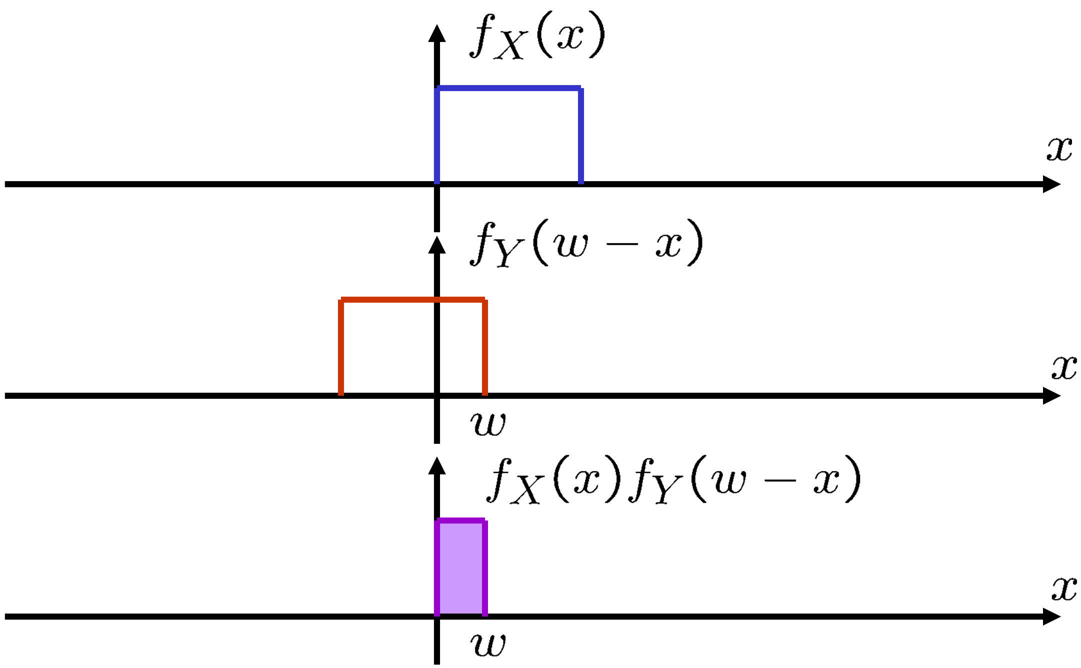


$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x) dx$$

$X + Y$ Example: Independent Uniform

- Let X, Y be independent, uniform on [0, 1]:
- Find the density of $W = X + Y$.
- Convolution idea applies:

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w - x) dx$$



Two Independent Normals

- Let X, Y be independent, normal r.v.s:

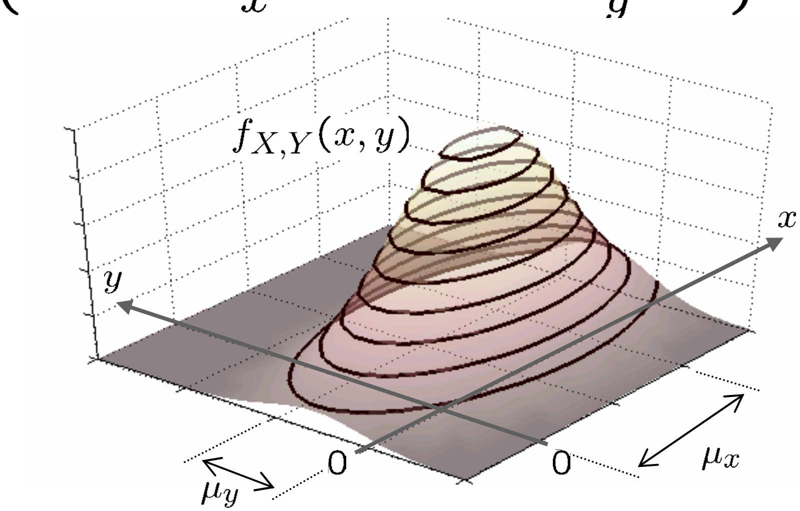
$$X \sim N(\mu_x, \sigma_x^2) \qquad Y \sim N(\mu_y, \sigma_y^2)$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} \right\}$$

- PDF is constant on ellipses:

$$-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2} = c^2$$



- Circles, when $\sigma_x = \sigma_y$

Sum of Two Independent Normals

- Let X, Y be independent, zero-mean normals:

$$X \sim N(0, \sigma_x^2) \qquad Y \sim N(0, \sigma_y^2)$$

- Find the density of $W = X + Y$.

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x) dx$$

$$= \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} e^{-x^2/2\sigma_x^2} e^{-(w-x)^2/2\sigma_y^2} dx$$

$$= ce^{-\gamma w^2}$$

- Conclusion: W is normal. $\mu_w = 0$

$$\sigma_w^2 = \sigma_x^2 + \sigma_y^2$$

Continuous Bayes' Rule

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_X(x) f_{Y|X}(y|x)}{f_Y(y)}$$

- Common case: $Y = X + N$

  Signal (X) — Additive Noise (N)

  Independent

- Then:

$$f_{Y|X}(y|x) = f_N(y - x)$$

- Remarkable fact:
  If X, N are normal, then $f_{X|Y}(x|y)$ is a normal PDF, for any given y.