14.454 - Macroeconomics 4
Notes on "On the Possibility of Speculation under Rational Expectations" by Jean Tirole

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4/29/2011

1 Static Rational Expectations Equilibrium

- One asset, with random value $\widetilde{p}$ realized in set E. Gains are $G^i = (\widetilde{p} - p)\, x^i$ where x^i are (net) trades in market.
- Each agent gets signal $s^i \in S^i$, and agents have common prior ν over $E \times S = \Omega$. Full support over conditional $v^i \equiv v(s\,|)$

Definition 1 (Rational Expectations Equilibrium) *A REE is a forecast function Φ such that $p = \Phi(s)$ and*

1. *$x^i\left(p, s^i, S(p)\right)$ solves optimum for agents, where $S(p) = \Phi^{-1}(p)$*
2. *Market clears for all s : $\sum x^i\left(p, s^i, S(p)\right) = 0$ for all $s \in S$*

We say that a market is purely speculative if participants initial positions (corresponding to no trade on the market i.e. $x^i = 0$ for all i) are uncorrelated with the return on the asset.

Proposition 2 *In a REE of a pure speculative market with risk-averse or risk-neutral traders, risk-averse agents do not trade; risk-neutral traders may trade, but they do not expect any gain from their trade.*

Proof. x^i solve

$$\max_{x^i} \sum_{s\in S(p),\widetilde{p}\in E} u\left(\left(\widetilde{p}-p\right)x^i\right)\nu\left(s^i,\widetilde{p}\right)$$

$x^i=0$ is always an option, so $\mathbb{E}\left(u\left(G^i\right)\mid s^i,S\left(p\right)\right)\equiv\mathbb{E}\left(u\left(\left(\widetilde{p}-p\right)x^i\right)\mid s^i,S\left(p\right)\right)\geq u\left(0\right)$. Because of Jensen's inequality, we must therefore have that

$$u\left[\mathbb{E}\left(G^i\mid s^i,S\left(p\right)\right)\right]\geq\mathbb{E}\left(u\left(G^i\right)\mid s^i,S\left(p\right)\right)\geq u\left(0\right)\Longleftrightarrow\mathbb{E}\left(G^i\mid s^i,S\left(p\right)\right)\geq 0 \tag{1}$$

Therefore, this implies that the unconditional expectation of G^i satisfies

$$\mathbb{E}\left(G^i\mid S\left(p\right)\right)=\sum_{s^i\in S(p)}\mathbb{E}\left(G^i\mid s^i,S\left(p\right)\right)\nu^i\left(s^i\right)\geq 0 \tag{2}$$

Now, market clearing implies that

$$\sum_i G^i=\sum_i\left(\widetilde{p}-p\right)x^i\left(p,s^i,S\left(p\right)\right)=\left(\widetilde{p}-p\right)\underbrace{\sum_i x^i\left(p,s^i,S\left(p\right)\right)}_{=0\text{ because of eq.}}=0$$

Then

$$\sum_i\mathbb{E}\left(G^i\mid S\left(p\right)\right)=0\underbrace{\Longrightarrow}_{\text{because of (2)}}\mathbb{E}\left(G^i\mid S\left(p\right)\right)=0$$

And again, this also implies that $\mathbb{E}\left(G^i\mid s^i,S\left(p\right)\right)=0$. But then, for all x^i :

$$u\left(0\right)\geq\mathbb{E}\left\{u\left(G^i\mid s^i,S\left(p\right)\right)\right\}=\mathbb{E}\left\{u\left(\left(\widetilde{p}-p\right)x^i\mid s^i,S\left(p\right)\right)\right\}$$

which implies that $x^i=0$ is optimal (i.e. no trade!!). ■

The idea is that in equilibrium, prices are fully revealing about the gains of trade of all agents (so even if agents have private information, they do learn "a lot" about the beliefs of all other agents by observing prices)

How can we then have trade in equilibrium?

1. Risk lovers
2. Non-identical priors
3. Introduce non-rational agents
4. Dynamic setting (buy because in the future there might be gains)

2 Dynamic Speculation with Myopic agents

- Signals are about dividend streams d_t (which is a random process). Dividends at time t are declared immediately prior to trading at time t, and paid to traders who hold the stock at $(t-1)$. Here, E is the set of all paths for dividends of the asset that have positive probability.
- Signals $s_t^i \in F_t^i$, a partition over the set S^i, where F_t^i is a filtration (i.e. $F_t^i \subset F_{t+1}^i$, so as time goes by, we get finer and finer partitions). Let $s_t = \left(..., s_t^i, ...\right)$ be the vector of signals. Assume that all signals have positive probability.
- Finite set of I risk neutral agents, with discount rate β.
- Aggregate supply of asset is $\overline{x}$ (fixed and inelastic to eq. prices p_t)

Note that with information at time t, we can predict prices at $p_{t+\tau}$ according to the law of motion of the filtration.

Definition 3 (Active traders) *An agent is an active trader at t iff* $x_t^i \neq x_{t-1}^i$ *or* $x_t^i = x_{t-1}^i$ *and* $0 < x_t^i = x_{t-1}^i < \overline{x}$

Definition 4 (Short sales) *A short sale is when* $x^i < 0$. *If short sales are prohibited, this is a restriction* $x^i \geq 0$ *for all* i

Definition 5 (Myopic REE) *A myopic* ***REE*** *is a sequence of self-fulfilling forecast functions* $s_t \longrightarrow p_t = \Phi_t\left(s_t\right)$, *such that there exists a sequence of associated stock holdings* $\left\{x^i\left(s_t^i, p_t\right) \equiv x^i\left(s_t^i, S\left(p_t\right), p_t\right)\right\}$ *where* $S_t\left(p_t\right) \equiv \Phi_t^{-1}\left(s_t\right) \subset \Omega$ *such that:*

1. ***Market clearing:*** $\forall\left(t, s_t\right)$ *we have that* $\sum_i x^i\left(s_t^i, p_t\right) = \overline{x}$
2. *Short-run optimizing behavior:*

   (i) : *If short sales are allowed, then*

$$\forall\left(t, s_t, i\right) : p_t = \beta\mathbb{E}\left[\left(d_{t+1} + p_{t+1}\right) \mid s_t^i, S_t\left(p_t\right)\right] \tag{3}$$

   (ii) : *If short sales are not allowed, then*

   - *If* $p_t = \beta\mathbb{E}\left[\left(d_{t+1} + p_{t+1}\right) \mid s_t^i, S_t\left(p_t\right)\right] \Longrightarrow x^i\left(s_t^i, p_t\right) \in [0, \overline{x}]$
   - *If* $p_t > \beta\mathbb{E}\left[\left(d_{t+1} + p_{t+1}\right) \mid s_t^i, S_t\left(p_t\right)\right] \Longrightarrow x^i\left(s_t^i, p_t\right) = 0$
   - *If* $p_t < \beta\mathbb{E}\left[\left(d_{t+1} + p_{t+1}\right) \mid s_t^i, S_t\left(p_t\right)\right] \Longrightarrow x^i\left(s_t^i, p_t\right) = \overline{x}$

Condition (3) is basically a no-arbitrage condition: the price today has to be equal to the discounted dividends plus the capital gain tomorrow. The following proposition states that even if short sales are not allowed, we still must have (3) being satisfied

Proposition 6 *Even if short sales are prohibited, for any trader i active at time t, we must have that* (3) *is satisfied*

Proof. Let $g^i \equiv -p_t \Delta x_t^i$ be the change in $i's$ cash flow at t and ${}_t g_{t+1}^i \equiv (p_{t+1} + d_{t+1}) \Delta x_t^i$ the cash flow change in $t+1$, where $\Delta x_t^i \equiv x_i^i - x_{t+1}^i$ is the net purchases at time t. See that if agent i is active, then $g^i \neq 0$ and ${}_t g_{t+1}^i \neq 0$ as well. From market clearing, we have $\sum_i g_t^i = \sum_i {}_t g_{t+1}^i = 0$,which implies

$$\sum_i \left(g_t^i + \beta_t g_{t+1}^i\right) = 0 \Longrightarrow \sum_i \mathbb{E}\left(g_t^i + \beta_t g_{t+1}^i \mid S_t\left(p_t\right)\right) = 0$$

If agent i maximizes, then we must have that $\mathbb{E}\left(g_t^i + \beta_t g_{t+1}^i \mid s_t^i, S_t\left(p_t\right)\right) \geq 0 \Longrightarrow \mathbb{E}\left(g_t^i + \beta_t g_{t+1}^i \mid S_t\left(p_t\right)\right) \geq 0$ as well. From here on, same reasoning as in the previous proposition. ■

Bubbles, as we have seen in class, are assets for which the price is above it's "fundamental value", understood as the NPV of the dividend stream that that asset generates. This is usually the case in all the macro models you have seen so far (until this course, at least).

2.1 Bubbles in Finite Horizon economies

Definition 7 (Market Fundamentals and Bubbles) *Given a REE and information $\left(s_t^i, S_t\left(p_t\right)\right)$ for agent i at time t, we define the **market fundamental** $F\left(s_t^i, S_t\left(p_t\right)\right)$ as the expected NPV of the stream of dividends for that agent. Namely*

$$F\left(s_t^i, p_t\right) \equiv \mathbb{E}\left(\sum_{\tau=1}^{\infty} \beta^\tau d_{t+\tau} \mid s_t^i, S_t\left(p_t\right)\right) \tag{4}$$

Based on this definition, we define a price bubble $B\left(s_t^i, S_t\left(p_t\right)\right)$ as the difference between the equilibrium price and the market fundamental:

$$B\left(s_t^i, p_t\right) \equiv p_t - F\left(s_t^i, p_t\right) \tag{5}$$

See that in principle, the market fundamental can be different for different agents, since the expected value is calculated based on

The next proposition shows that, for finite horizon economies, there are no bubbles, independent of whether short sales are allowed or not. This is also true in the Rational Bubbles literature (as in Santos and Woodford (1997))

Proposition 8 *In a stock market with finite horizon $\overline{T}$, whether short sales are allowed or not, the price bubbles are all equal to zero for the traders active in the market. Thus a market fundamental can be uniquely defined as the common market fundamental of all active traders, and is equal to the price:*

$$\forall (t,i) \;\; \textit{active at } t : p_t = \mathbb{E}\left(\sum_{\tau=1}^{\overline{T}} \beta^{\tau} d_{t+\tau} \mid s_t^i, S_t(p_t)\right) \Longrightarrow B\left(s_t^i, p_t\right) = 0$$

Proof. (From paper). The price of stock at $\overline{T}$ has to be zero, because agents just eat the dividends and after that, assets are worthless for any agent. Consider a trader i who is active at $\left(\overline{T}-1\right)$. The previous proposition implies that $p_{\overline{T}-1} = \beta \mathbb{E}\left[d_{\overline{T}} \mid s_t^i, S_t(p_t)\right]$, which implies that an active trader is indifferent between selling and holding the stock until the end period $\overline{T}$. Following by induction, we prove the desired result. ■

2.2 Bubbles in Infinite Horizon Economies

In infinite horizon economies and myopic behavior of traders, bubbles may actually emerge (compare with OLG!!).

Proposition 9 *Suppose $\overline{T} = \infty$. The following statements are true:*

(**a**) : *If short sales are allowed, then price bubbles are (discounted) martingales:*

$$B\left(s_t^i, p_t\right) = \gamma^T \mathbb{E}\left(B\left(s_{t+T}^i, p_{t+T}\right) \mid s_t^i, S_t(p_t)\right) \tag{6}$$

(**b**) : *If short sales are prohibited, the price bubble of trader i endowed with information $\left(s_t^i, S_t(p_t)\right)$ satisfies (6) iff conditionally on his information at t, trader i is active in each period $t, t+1, ..., t+T-1$*

Proof. Simply use forward induction using (3) :

$$p_t = \mathbb{E}\left[\beta d_{t+1} + \beta p_{t+1} \mid s_t^i, S_t\left(p_t\right)\right] =$$

$$\mathbb{E}\left[\beta d_{t+1} + \beta \mathbb{E}\left[\beta d_{t+2} + \beta p_{t+2} \mid s_{t+1}^i, S_t\left(p_{t+1}\right)\right] \mid s_t^i, S_t\left(p_t\right)\right] \underset{\text{using iterated expectations}}{=}$$

$$\mathbb{E}\left[\beta d_{t+1} + \beta^2 d_{t+2} + \beta^2 p_{t+2} \mid\mid s_t^i, S_t\left(p_t\right)\right]$$

and using induction T steps

$$p_t = \mathbb{E}\left(\sum_{\tau=1}^{T} \beta^t d_{t+\tau} + \beta^T p_T \mid s_t^i, S_t\left(p_t\right)\right) =$$

$$\mathbb{E}\left(\sum_{\tau=1}^{T} \beta^t d_{t+\tau} \mid s_t^i, S_t\left(p_t\right)\right) + \beta^T \mathbb{E}\left(\mathbb{E}\left(\sum_{\tau=1}^{\infty} \beta^\tau d_{t+T+\tau} \mid s_{t+T}^i, S_{t+T}\left(p_{t+T}\right)\right) \mid s_t^i, S_t\left(p_t\right)\right)$$

$$\underset{\text{using iterated expectations}}{=} \mathbb{E}\left(\sum_{\tau=1}^{T} \beta^t d_{t+\tau} \mid s_t^i, S_t\left(p_t\right)\right) + \beta^T \underbrace{\mathbb{E}\left(\sum_{\tau=1}^{\infty} \beta^\tau d_{t+T+\tau} \mid s_t^i, S_t\left(p_t\right)\right)}_{\equiv F\left(s_{t+T}^i, p_{t+T}\right)}$$

$$+\beta^T \mathbb{E}\left(B\left(s_{t+T}^i, p_{t+T}\right) \mid s_t^i, S_t\left(p_t\right)\right) =$$

$$\mathbb{E}\left(\sum_{\tau=1}^{T} \beta^t d_{t+\tau} \mid s_t^i, S_t\left(p_t\right)\right) + \beta^T \mathbb{E}\left(F\left(s_{t+T}^i, p_{t+T}\right) \mid s_t^i, S_t\left(p_t\right)\right) + \beta^T \mathbb{E}\left(B\left(s_{t+T}^i, p_{t+T}\right) \mid s_t^i, S_t\left(p_t\right)\right) =$$

$$\underbrace{\mathbb{E}\left(\sum_{\tau=1}^{\infty} \beta^t d_{t+\tau} \mid s_t^i, S_t\left(p_t\right)\right)}_{\equiv F\left(s_t^i, p_t\right)} + \beta^T \mathbb{E}\left(B\left(s_{t+T}^i, p_{t+T}\right) \mid s_t^i, S_t\left(p_t\right)\right) = F\left(s_t^i, p_t\right) + \beta^T \mathbb{E}\left(B\left(s_{t+T}^i, p_{t+T}\right) \mid s_t^i, S_t\left(p_t\right)\right)$$

So, in conclusion

$$p_t = F\left(s_t^i, p_t\right) + \beta^T \mathbb{E}\left(B\left(s_{t+T}^i, p_{t+T}\right) \mid s_t^i, S_t\left(p_t\right)\right) \Longleftrightarrow$$

$$B\left(s_t^i, p_t\right) \equiv p_t - F\left(s_t^i, p_t\right) = \beta^T \mathbb{E}\left(B\left(s_{t+T}^i, p_{t+T}\right) \mid s_t^i, S_t\left(p_t\right)\right)$$

as we wanted to show. ■

Here, bubbles may exist. See Example in page 1174 of original paper

3 Fully Dynamic REE

However, the possibility of bubbles may come just from the myopic behavior of agents (i.e. they only maximize period t utility and nothing else). What if each agents maximizes the expected NPV, instead of just one period gains?

Definition 10 (Fully Dynamic REE) *A fully dynamic REE is a sequence of self-fulfilling forecast functions $p_t = \Phi_t(s_t)$ such that there exists a sequence of (contingent) stock holding strategies $x^i(s_t^i, p_t \mid h^t)$ (where h^t is history of observed prices and signals) satisfying*

(**i**) : ***Market clearing:*** $\forall (t, s_t) : \sum_i x^i(s_t^i, p_t \mid h^t) = \overline{x}$

(**ii**) : ***Maximizing behavior:*** *At each (s_t, p_t), $x^i(s_t^i, p_t \mid h^t)$ solves*

$$\max \mathbb{E}_t \left\{ \sum_{\tau=t}^{\infty} \beta^\tau d_{t+\tau} x_{t+\tau-1}^i + \sum_{\tau=t}^{\infty} \beta^\tau p_{t+\tau} \left(x_{t+\tau-1}^i - x_{t+\tau}^i \right) \mid s_t^i, S_t(p_t) \right\} \tag{7}$$

The most important result of the paper is that when we take a more general (and perhaps realistic) assumption about objectives of traders, we get that bubbles cannot survive (at least they cannot in a purely speculative environment)

Proposition 11 (No Bubbles) *Whether short sales are allowed or not, price bubbles do not exist in a fully dynamic REE:*

$$\forall (t, s_t, i) : B(s_t^i, p_t) = 0$$

Proof. Following the author, we will prove the result when short sales are prohibited. Define $G_t^i \equiv \sum_{\tau=t}^{\infty} \beta^{\tau-t+1} d_{t+\tau} x_{t+\tau-1}^i + \sum_{\tau=t}^{\infty} \beta^t p_{t+\tau} \left(x_{t+\tau-1}^i - x_{t+\tau}^i \right)$ as a particular realization of the discounted sum of dividends plus capital gains in $i's$ optimal strategy. In equilibrium, using the market clearing condition $\sum_i x^i = \overline{x}$, we have that

$$\sum_i G_t^i = \sum_i \sum_{\tau=t}^{\infty} \beta^\tau d_{t+\tau} x_{t+\tau-1}^i + \sum_i \sum_{\tau=t}^{\infty} \beta^\tau p_{t+\tau} \left(x_{t+\tau-1}^i - x_{t+\tau}^i \right) =$$

$$\sum_{\tau=t}^{\infty} \beta^\tau d_{t+\tau} \left(\underbrace{\sum_i x_{t+\tau-1}^i}_{=\overline{x}} \right) + \sum_{\tau=t}^{\infty} \beta^t p_{t+\tau} \left(\underbrace{\sum_i x_{t+\tau-1}^i - \sum_i x_{t+\tau}^i}_{=\overline{x}-\overline{x}} \right) =$$

$$\left(\sum_{\tau=t}^{\infty}\beta^{\tau}d_{t+\tau}\right)\overline{x}=f_t\overline{x} \tag{8}$$

where we define f_t as the "realized market fundamental" $\sum_{t=\tau}^{\infty}\beta^{\tau}d_{t+\tau}$ of investing one unit in the stock. Define

$$F\left(p_t\right)=\mathbb{E}\left(f_t \mid S_t\left(p_t\right)\right) \tag{9}$$

The proof will rely on two useful Lemmas:

Lemma 12 *For all s_t we have that $F\left(p_t\right)\geq p_t$*

Since trader i optimizes, at the optimum in particular she has to have greater expected utility than the strategy in which she sells x_t^i and leaves the market at time t, i.e. $\mathbb{E}\left(G_t^i \mid s_t^i, S_t\left(p_t\right)\right)\geq x_t^i p_t$. Thus

$$\mathbb{E}\left(G_t^i \mid S_t\left(p_t\right)\right)=\sum_{s_t\in S_t(p_t)}\mathbb{E}\left(G_t^i \mid s_t^i, S_t\left(p_t\right)\right)\nu^i\left(s_t^i \mid S_t\left(p_t\right)\right)\geq p_t\left[\sum_{s_t\in S_t(p_t)}x_t^i\left(s^i,p_t\right)\nu^i\left(s_t^i \mid S_t\left(p_t\right)\right)\right]$$

where S_t^i is the projection of S_t on F_t^i. This then implies that

$$\sum_i\mathbb{E}\left(G_t^i \mid S_t\left(p_t\right)\right)\geq p_t\sum_i\left[\sum_{s_t\in S_t(p_t)}x_t^i\left(s^i,p_t\right)\nu^i\left(s_t^i \mid S_t\left(p_t\right)\right)\right]=$$

$$p_t\sum_{s_t\in S_t(p_t)}\nu^i\left(s_t^i \mid S_t\left(p_t\right)\right)\underbrace{\sum_i x_t^i\left(s^i,p_t\right)}_{=\overline{x}}=p_t\overline{x}\underbrace{\sum_{s_t\in S_t(p_t)}\nu^i\left(s_t^i \mid S_t\left(p_t\right)\right)}_{=1}=p_t\overline{x}$$

and using (8) we then get that

$$F\left(p_t\right)\overline{x}=\mathbb{E}\left(f_t\overline{x} \mid S_t\left(p_t\right)\right)\underset{\text{using (8)}}{=}\sum_i\mathbb{E}\left(G_t^i \mid S_t\left(p_t\right)\right)\geq p_t\overline{x}$$

proving the desired result

The other Lemma is that no trader expects a gain from trades at no point in time, and under no realizations of the signals:

Lemma 13 $\forall\left(t,s_t,i\right)$ *we have that*

$$\mathbb{E}\left(G_t^i \mid s_t^i, S_t\left(p_t\right)\right)=\mathbb{E}\left(G_t^i\left(x_{t-1}^i\right) \mid s_t^i, S_t\left(p_t\right)\right) \tag{10}$$

where $G_t^i\left(x_{t-1}^i\right)$ is identical to G_t^i except that involves a strategy that at time t, the agent does not trade (i.e. she keeps x_{t-1}^i)

See proof of Lemma in paper (not a long proof).

Suppose now that $F\left(s_t^{i_0}, S_t\left(p_t\right)\right) > p_t$ for some agent i_0 at period t. Then i_0 could buy and make a strictly positive expected profit, contradicting Lemma 2. This, for all i such that $x_{t-1}^i \neq \overline{x}$ we must have that $F\left(s_t^i, S_t\left(p_t\right)\right) = p_t$. Now, if any agent i holds the whole stock at the beginning of the period, his market fundamental cannot be lower than p_t. Thus $F\left(s_t^i, S_t\left(p_t\right)\right) \geq p_t$. But then $F\left(s_t^i, S_t\left(p_t\right)\right) = p_t$ as we wanted to show. ∎

4 Implications of "No Bubbles" Proposition

- Hard to get purely speculative bubbles, when there is no hedging motive or a reason for the existence of rational bubbles (as in the case of dynamic inefficiency)
- In order to get purely speculative bubbles, we either need some players to be irrational (as in Abreu and Brunnermeier), myopic or not be completely speculative (i.e. they must have some motive for insurance).