Problem Set 4 - Solutions

Question 1

The state space is $S = [-1, 1]$, with uniform probability. Indexed by $a \in [-1, 1]$, there are assets $D_a : S \to \mathbb{R}$ such that $D_a(s) = 1 + as$ for all $s \in S$. Denote by $F_a : \mathbb{R} \to [0, 1]$ the cdf of the lottery over $\mathbb{R}$ induced by D_a. The DM is a rank-dependent expected utility maximizer with preference relation $\gtrsim$ over the assets. Her probability weighting function, parametrized by $\alpha \in (-1, \infty)$, is $w : [0, \infty) \to [0, \infty)$ such that $w(p) = p^{1+\alpha}$. We wish to characterize $\gtrsim$. We will do it by computing for all $a \in [-1.1]$

$$U(D_a) = \int_{\mathbb{R}} x \, dw(F_a(x)).$$

First we obtain an expression for F_a. Observe that for all $x \in \mathbb{R}$

$$F_a(x) = Pr(D_a \leqslant x) = \frac{1}{2} \int_{-1}^{1} \mathbb{1}(1 + as \leqslant x) ds.$$

Now case by case: if $a < 0$, then

$$F_a(x) = \frac{1}{2} \int_{-1}^{1} \mathbb{1}(s \geqslant \frac{x-1}{a}) ds = \begin{cases} 0 & \text{if } x \leqslant a + 1, \\ \frac{a+1-x}{2a} & \text{if } x \in (1 + a, 1 - a], \\ 1 & \text{else.} \end{cases}$$

If $a = 0$, then

$$F_a(x) = \frac{1}{2} \int_{-1}^{1} \mathbb{1}(1 + as \leqslant x) ds = \int_{-1}^{1} \mathbb{1}(1 \leqslant x) ds = \begin{cases} 0 & \text{if } x < 1, \\ 1 & \text{else.} \end{cases}$$

If $a > 0$, then

$$F_a(x) = \frac{1}{2}\int_{-1}^{1} \mathbb{1}(1 + as \leqslant x)ds = \int_{-1}^{1} \mathbb{1}(s \leqslant \frac{x-1}{a})ds = \begin{cases} 0 & \text{if } x \leqslant 1-a, \\ \frac{x-1+a}{2a} & \text{if } x \in (1-a, 1+a], \\ 1 & \text{else.} \end{cases}$$

Now we compute $U(D_a)$. To ease notation, write $\varphi_a = w \circ F_a$. If $a < 0$, then

$$\begin{aligned} \int_{\mathbb{R}} x\, d\,\varphi_a(x) &= \int_{1+a}^{1-a} x\, d\,\varphi_a(x) \\ &= (1-a)\varphi_a(1-a) - (1+a)\varphi_a(1+a) - \int_{1+a}^{1-a} \varphi_a(x)dx \\ &= (1-a) - \int_{1+a}^{1-a} \varphi_a(x)dx \end{aligned}$$

where the first equality holds because φ_a is constant before $1 + a$ and after $1 - a$, the second inequality follows from integration by parts, and the third equality because $\varphi_a(1 - a) = 1$ and $\varphi_a(1 + a) = 0$. Finally

$$\int_{1+a}^{1-a} \varphi_a(x)dx = \int_{1+a}^{1-a} (\frac{a+1-x}{2a})^{1+\alpha}dx = [-\frac{2a}{2+\alpha}(\frac{a+1-x}{2a})^{2+\alpha}]_{1+a}^{1-a} = -\frac{2a}{2+\alpha}.$$

In conlusion

$$U(D_a) = (1-a) + \frac{2a}{2+\alpha}.$$

Moving to the other cases, Clearly $U(D_0) = 1$. The last case $a > 0$ can be treated as the case $a < 0$ to obtain

$$U(D_a) = (1+a) - \frac{2a}{2+\alpha}.$$

Summing up: for all $a \in [-1, 1]$

$$U(D_a) = (1+|a|) - \frac{2|a|}{2+\alpha},$$

where $|a|$ is the absolute value of a. Going back to the preference relation, we obtain that for all $a, a' \in [-1, 1]$

$$D_a \gtrsim D_{a'} \quad \Leftrightarrow \quad \text{sgn}(\alpha)|a|| \geqslant \text{sgn}(\alpha)|a'|,$$

where sgn is the signum function (i.e., $\text{sgn}(\alpha) = -1$ if $\alpha < 0$, $\text{sgn}(0) = 0$, and $\text{sgn}(\alpha) = 1$ else). Comment: The absolute value $|\alpha|$ parametrizes the variance of the lottery, while $\text{sgn}(\alpha)$ indicates whether the DM is "optimistic" ($\alpha > 0$), "pessimistic" ($\alpha < 0$) or risk-neutral ($\alpha = 0$). If the DM is optimistic, she prefers lotteries with bigger variance; if she is pessimistic, the converse is true.

Question 2

If F is (the cdf of) a lottery over $\mathbb{R}$ and x_0 is initial wealth, then

$$U(F|x_0) = \int_{x \geqslant x_0} x - x_0 \, dF(x) + \lambda \int_{x < x_0} x - x_0 \, dF(x).$$

Moreover the lottery $\frac{3}{5}(x_0 + 1) + \frac{2}{5}(x_0 - 1)$ is indifferent to the lottery x_0:

$$\frac{3}{5}(1) + \lambda \frac{2}{5}(-1) = 0 \quad \Rightarrow \quad \lambda = \frac{3}{2}.$$

As a result the DM we are considering are different only in terms of initial wealth (i.e., reference point). There we wish to find the pair $(x_0, G) \in \mathbb{R}$ which minimizes G subject to

$$U(\frac{1}{2}(x_0 + G) + \frac{1}{2}(x_0 - L)|x_0) \geqslant U(x_0|x_0) = 0.$$

By monotonicity the constraint is satisfied only if $G \geqslant 0$. Therefore we can rewrite the constraint as

$$\frac{1}{2}G + \frac{3}{4}(-L) \geqslant 0 \quad \Rightarrow \quad G = \frac{3}{2}L,$$

and the implication gives the optimal choice of G, while x_0 is undetermined.

Question 3

Part (a)+(c)

The indifference condition is

$$\frac{1}{2}u(W + x) + \frac{1}{2}u(W - x) = u(W - P(x, W)).$$

Using $u(z) = \sqrt{z}$ and rearranging, we get

$$P(x, W) = W - \frac{1}{4}(\sqrt{W + x} + \sqrt{W - x})^2.$$

The profit margin is

$$\frac{P(x, W)}{x} = \frac{W}{x} - \frac{1}{4}(\sqrt{\frac{W}{x} + 1} + \sqrt{\frac{W}{x} - 1})^2.$$

We maximize wrt $t = \frac{W}{x}$. Note first that t is at least $\bar{W}/\bar{x} \geqslant 1$, while its range is unbounded from above. Differentiating

$$\frac{\partial}{\partial t}\{t - \frac{1}{4}(\sqrt{t+1} + \sqrt{t-1})^2\} = \frac{1-t}{2\sqrt{t^2-1}} < 0,$$

for $t > \bar{W}/\bar{x}$. Hence the profit margin is maximized at $W = \bar{W}$ and $x = \bar{x}$. Comment: Ann's coefficient of absolute risk aversion is $1/2z$, which is decreasing. Hence the profit margin must be maximized for the lowest value of initial wealth W. On the other hand increasing x raise the variance of the risk, and therefore Ann is willing to pay more to get rid of it.

Part (b)+(c)

First we compute Ann's value of the lottery $\frac{1}{2}(W+x) + \frac{1}{2}(W-x)$ with cdf $F(z)$. Her probability weighting function is $w(p) = p$ for all $p \in [0, \infty)$: thre is no distortion, and therefore $G(z|W) = F(z)$. Her reference-dependent utility function

$$u(z|W) = v(z-W) = \begin{cases} \sqrt{z-W} & \text{if } z \geqslant W, \\ -2\sqrt{W-z} & \text{else.} \end{cases}$$

Hence the value of the lottery $\frac{1}{2}(W+x) + \frac{1}{2}(W-x)$ is

$$\frac{1}{2}\sqrt{x} + \frac{1}{2}(-2\sqrt{x}) = -\frac{1}{2}\sqrt{x}.$$

The indifference condition therefore is

$$-\frac{1}{2}\sqrt{x} = -2\sqrt{P(x,W)} \quad \Rightarrow \quad P(x,W) = \frac{x}{16}.$$

In this case profit margin $P(x,W)/x$ is independent of x and W. Comment: initial wealth does not matter, since it is reference point. Moreover, raising x does not help, since Ann is risk-averse towards gain but risk-seeking towards losses, and therefore the two effects on the profit margin cancel out.

Question 4

Part (a)

Denote Ann's demand by $d(p)$. Given p, Ann chooses $d \in \mathbb{R}$ to maximize

$$\begin{aligned} U(d) &:= \min_{\mu \in [\underline{\mu}, \bar{\mu}]} E[u((y-p)d|\mu] = \\ &= -\max_{\mu \in [\underline{\mu}, \bar{\mu}]} \exp(-\alpha((\mu - p)d - \frac{1}{2}\alpha d^2 \sigma^2) \\ &= -\begin{cases} \exp(-\alpha((\underline{\mu} - p)d - \frac{1}{2}\alpha d^2\sigma^2) & \text{if } d \geqslant 0, \\ \exp(-\alpha((\bar{\mu} - p)d - \frac{1}{2}\alpha d^2\sigma^2) & \text{else.} \end{cases} \end{aligned}$$

Therefore $d \in \mathbb{R}$ is chosen to maximize

$$V(d) := \begin{cases} (\underline{\mu} - p)d - \frac{1}{2}\alpha d^2\sigma^2 & \text{if } d \geqslant 0, \\ (\bar{\mu} - p)d - \frac{1}{2}\alpha d^2\sigma^2 & \text{else.} \end{cases}$$

We solve the optimization case-by-case. If $p \geqslant \bar{\mu}$, any $d > 0$ gives $V(d) < 0$, and therefore is dominated by $V(0) = 0$. So looking for a solution in $d \in (-\infty, 0]$, we take the first order condition and get

$$d(p) = \frac{\bar{\mu} - p}{\alpha\sigma^2} \in (-\infty, 0].$$

Now assume that $p \in (\underline{\mu}, \bar{\mu})$. Now $V(d) < 0$ for all $d \neq 0$, and therefore $d(p) = 0$. If $p \leqslant \underline{\mu}$, any $d < 0$ gives $V(d) < 0$, and therefore is dominated by $V(0) = 0$. So looking for a solution in $d \in [0, \infty)$, we take the first order condition and get

$$d(p) = \frac{\underline{\mu} - p}{\alpha\sigma^2} \in [0, \infty).$$

Summing up:

$$d(p) = \begin{cases} \frac{\underline{\mu} - p}{\alpha\sigma^2} & \text{if } p \leqslant \underline{\mu}, \\ 0 & \text{if } p \in (\underline{\mu}, \bar{\mu}). \\ \frac{\bar{\mu} - p}{\alpha\sigma^2} & \text{else.} \end{cases}$$

Part (b)+(c)

If $Y = 0$, the market clearing price any $p \in [\underline{\mu}, \bar{\mu}]$. If $Y > 0$, the market clearing prince is

$$p = \underline{\mu} - \frac{\alpha\sigma^2 Y}{n} \leqslant \underline{\mu}.$$

Finally, if $Y < 0$, the market clearing prince is

$$p = \bar{\mu} - \frac{\alpha \sigma^2 Y}{n} \geqslant \bar{\mu}.$$

Comment: with maxmin agents, only extreme beliefs matter. To make the agents willing to buy, the price has to be below the worst case scenario $\underline{\mu}$. On the other hand, to make the agents willing to sell, the price has to be above the best case scenario $\bar{\mu}$. Prices are therefore more extreme in this case (wrt expected utility).

Part (c)

Fix $\mu \in [\underline{\mu}, \bar{\mu}]$. Given p, Ann chooses $d \in \mathbb{R}$ to maximize the certainty equivalent

$$(\mu - p)d - \frac{1}{2}\alpha d^2 \sigma^2.$$

Therefore $d(p) = \frac{\mu - p}{\alpha \sigma^2}$. The market clearing price is

$$p = \mu - \frac{\alpha \sigma^2 Y}{n}.$$