Problem 1

Recall that VNM utility functions $u : \{a, b, c\} \times \{L, M, R\} \to \mathbb{R}$ and $\tilde{u} : \{a, b, c\} \times \{L, M, R\} \to \mathbb{R}$ represent the same preferences over P if and only if there exist $a \in \mathbb{R}_+$ and $b \in \mathbb{R}$ such that $\tilde{u} = au + b$.

(a) Player 1's preferences are not the same. For if they were, considering outcomes (a, L) and (a, M) implies that $12 = 2a + b$ and $5 = a + b$, which implies that $a = 7$ and $b = -2$. But, for outcome (a, R), $-3 \neq -3(7) - 2$.

Player 2's preferencs are the same, because $\tilde{u} = \frac{1}{3}u - \frac{1}{3}$ (where $\tilde{u}$ is the VNM utility function on the right).

(b) Player 1's.preferences are the same, because $\tilde{u} = u$.

Player 2's preferences are not the same. For if they were, considering outcomes (a, L) and (a, M) implies that $5 = 2a + b$ and $1 = 0a + b$, which implies that $a = 2$ and $b = 1$. But, for outcome (b, M), $4 \neq 2(2) + 1$.

Problem 2

(a) Since $P = \Delta(C) = \{(p_x, p_y, p_x) : p_x, p_y, p_x \geq 0, p_x + p_y + p_z = 1\}$, it follows that $I_1 = I_2 = (1, 0, 0)$. Thus, the VNM utility function $u : C \to \mathbb{R}$ given by $u_x = 1$, $u_y = 0$, and $u_z = 0$ represents a preference relation with indifference sets I_1 and I_2.

(b) The VNM utility function $u : C \to \mathbb{R}$ given by $u_x = 1$, $u_y = -2$, and $u_z = 0$ represents a preferences relation with indifference sets I_1 and I_2.

(c) Continuity is violated: Suppose that every lottery in I_2 is strictly prefered to every lottery in I_1 (the opposite case in analogous). Let $p = (1, 0, 0)$ and let $q_n = \left(\frac{1}{2} + \frac{1}{n}, \frac{1}{2} - \frac{1}{n}, 0\right)$. Then $q_n \succsim p$ for all $n \in \mathbb{N}$, but $p \succ q$ for $q = \left(\frac{1}{2}, \frac{1}{2}, 0\right) = \lim_{n\to\infty} q_n$. Independence is also violated.

(d) Independence is violated: $(0, 0, 1)$ and $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$ are in I_2, so $(0, 0, 1) \sim \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. Independence implies that $\frac{1}{2}(0, 0, 1) + \frac{1}{2}\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right) \sim (0, 0, 1)$, or equivalently that $\frac{1}{2}(0, 0, 1) +$

$\frac{1}{2}\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right) \in I_2$. But $\frac{1}{2}(0,0,1) + \frac{1}{2}\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right) = \left(\frac{1}{4},\frac{1}{8},\frac{5}{8}\right)$, and $\frac{1}{8} \neq \left(\frac{1}{4}\right)^2$, so $\frac{1}{2}(0,0,1) + \frac{1}{2}\left(\frac{1}{2},\frac{1}{4},\frac{1}{4}\right) \notin I_2$.

Problem 3

Let $C = \{x,y,z\}$ and consider the lexicographic preference relation $\succsim$ given by $p \succsim q$ if and only if either $p_x > q_x$ or $[p_x = q_x$ and $p_y \geq q_y]$. I check that this is indeed a preference relation (i.e., satisfies Completeness and Transitivity) and that it satisfies Independence but violates Continuity.

Completeness: For all $p,q \in P$, either $p_x > q_x$, $q_x > p_x$, or $[p_x = q_x$ and either $p_y \geq q_y$ or $q_y \geq p_y]$. Hence, $p \succsim q$ or $q \succsim p$.

Transitivity: If $p \succsim q \succsim r$ for $p,q,r \in P$, then $p_x \geq q_x$ and $q_x \geq r_x$, with $p_x = q_x = r_x$ only if $p_y \geq q_y$ and $q_y \geq r_y$. Therefore, either $p_x > r_x$ or $[p_x = r_x$ and $p_y \geq r_y]$. Hence, $p \succsim r$.

Independence: For all $p,q,r \in P$ and $\alpha \in (0,1]$,

$$\begin{array}{rcl}
\alpha p + (1-\alpha) r & \succsim & \alpha q + (1-\alpha) r \\
& \Longleftrightarrow & \\
\alpha p_x + (1-\alpha) r_x & > & \alpha q_x + (1-\alpha) r_x \text{ or} \\
{[}\alpha p_x + (1-\alpha) r_x & = & \alpha q_x + (1-\alpha) r_x \text{ and } \alpha p_y + (1-\alpha) r_y \geq \alpha q_y + (1-\alpha) r_y] \\
& \Longleftrightarrow & \\
p_x & > & q_x \text{ or } [p_x = q_x \text{ and } p_y \geq q_y] \\
& \Longleftrightarrow & \\
p & \succsim & q.
\end{array}$$

Violation of Continuity: Let $p = \left(\frac{1}{2},0,\frac{1}{2}\right)$ and let $q_n = \left(\frac{1}{2}-\frac{1}{n},\frac{1}{2}+\frac{1}{n},0\right)$. Then $p \succsim q_n$ for all $n \in \mathbb{N}$, but $q \succ p$ for $q = \left(\frac{1}{2},\frac{1}{2},0\right) = \lim_{n\to\infty} q_n$.