2D Inversion

We begin with the "primitive equations" which make only the hydrostatic, beta-plane assumptions. We write them in a coordinate system $(x,\ y,\ \xi)$ where ξ is a function of pressure.

From 12.802 "coordinates" notes, we can write the momentum, mass, and thermodynamic equations as

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{k}} \times \mathbf{u} = -\nabla\varphi \tag{p.1}$$

$$\frac{\partial}{\partial\xi}\varphi = g\frac{\rho_c}{\rho} \tag{p.2}$$

$$\nabla \cdot \mathbf{u} + \frac{1}{\rho_c}\frac{\partial}{\partial\xi}(\rho_c\omega) = 0 \tag{p.3}$$

$$\frac{\partial}{\partial t}\eta + \mathbf{u} \cdot \nabla\eta + \omega\frac{\partial}{\partial\xi}\eta = 0 \tag{p.4}$$

where φ is the geopotential height of a surface of constant ξ (coincident with a surface of constant pressure). The entropy is η, and ρ_c is the density associated with the coordinate change

$$\frac{\partial p}{\partial\xi} = -g\rho_c$$

The PV is

$$q = \frac{1}{\rho_c}(\nabla_3 \times \mathbf{u} + f\hat{\mathbf{k}}) \cdot \nabla_3\eta \tag{pv}$$

Ocean form:

In the Boussinesq approximation, $\rho = \rho_0(1+\sigma)$, $\rho_c = \rho_0$ and we simply use $\eta \propto -\sigma$. Then

$$\frac{\partial}{\partial \xi}\phi = -g\sigma \qquad (oc - hyd)$$

and

$$q = -(\nabla_3 \times \mathbf{u} + f\hat{\mathbf{k}}) \cdot \nabla_3 \sigma \qquad (oc - pv)$$

Atmospheric form:

Here we choose ξ so that $\rho_c/\rho = \theta/\theta_0$. Using the relationship between $\theta,\ \rho,\ p$

$$\frac{\theta}{\theta_0} = \left(\frac{\rho}{\rho_0}\right)^{-1} \left(\frac{p}{p_0}\right)^{1/\gamma} q$$

we have

$$\xi = H_s \frac{\gamma - 1}{\gamma}\left[1 - \left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}\right]$$

Then

$$\frac{\partial}{\partial \xi}\phi = g\frac{\theta}{\theta_0} \qquad (atm - hyd)$$

$$q = \frac{1}{\theta_0}(\nabla_3 \times \mathbf{u} + f\hat{\mathbf{k}}) \cdot \nabla_3 \theta \qquad (atm - pv)$$

2D forms

Now we assume the fields are independent of x. Then

$$q = (f - \frac{\partial u}{\partial y})\frac{\partial b}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial b}{\partial y}$$

(where we've used the buoyancy b for either $-\sigma$ or θ/θ_0 and z for ξ). Using the geostrophic equation $fu = -\frac{\partial \phi'}{\partial y}$ and hydrostatic equation $b' = \frac{\partial \phi'}{\partial z}$, where $b = \int^z N^2 + b'$, gives

$$q = fN^2 + f\frac{\partial^2 \phi'}{\partial z^2} + \frac{N^2}{f}\frac{\partial^2 \phi'}{\partial y^2} + \frac{1}{f}\frac{\partial^2 \phi'}{\partial y^2}\frac{\partial^2 \phi'}{\partial z^2} - \frac{1}{f}\left(\frac{\partial^2 \phi'}{\partial y \partial z}\right)^2$$

This is the form we wish to invert for ϕ' and therefore u and b'.

As a basically elliptic equation, we need to specify conditions on the four boundaries of the domain. On the top and bottom, we specify $b' = \frac{\partial \phi'}{\partial z}$; on the left and right, we have a number of choices – and they do matter.